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On Poncelet's maps

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ABSTRACT

Given two ellipses, one surrounding the other one, Poncelet introduced a map P from the exterior one to itself by using the tangent lines to the interior ellipse. This procedure can be extended to any two smooth, nested and convex ovals and we call these types of maps, Poncelet's maps. We recall what he proved around 1814 in the dynamical systems language: In the two ellipses' case and when the rotation number of P is rational there exists an $n \in \mathbb{N}$ such that $P^n = \mathrm{Id}$, or in other words, Poncelet's map is conjugate to a rational rotation. In this paper we study general Poncelet's maps and give several examples of algebraic ovals where the corresponding Poncelet's map has a rational rotation number and is not conjugate to a rotation. Finally, we also provide a new proof of Poncelet's result based on dynamical and computational tools.

1. Introduction and main results

Let γ and Γ be two $\mathcal{C}^r$, $r \geq 1$, simple, closed and nested curves, each one of them being the boundary of a convex set. Furthermore we assume for instance that Γ surrounds γ.

Given any $p \in \Gamma$ there are exactly two points q_1, q_2 in γ such that the lines pq_1, pq_2 are tangent to γ. We define *Poncelet's map*, $P : \Gamma \to \Gamma$, associated to the pair γ, Γ as

$$P(p) = P_{\gamma,\Gamma}(p) = \overline{pq_1} \cap \Gamma,$$

where $p \in \Gamma$, $\overline{pq_1} \cap \Gamma$ is the first point in the set $\{\overline{pq_1} \cap \Gamma, \overline{pq_2} \cap \Gamma\}$ that we find when, starting from p, we follow Γ counterclockwise, see Fig. 1. Notice that $P^{-1}(p) = \overline{pq_2} \cap \Gamma$.

The implicit function theorem together with the geometrical interpretation of the construction of P imply that it is a $\mathcal{C}^r$ diffeomorphism from Γ into itself. So P can be seen as a $\mathcal{C}^r$ diffeomorphism of the circle and has associated a *rotation number*

$$\rho = \rho(P) = \rho(\gamma, \Gamma) \in (0, 1/2).$$

See for instance [1,2] for the definition of rotation number. Notice that usually a rotation number is in $(0, 1)$. Our choice of q_1 for Poncelet's map implies that indeed $\rho < 1/2$. A well known result of Denjoy states that if Φ is any diffeomorphism of the circle of class at least $\mathcal{C}^2$ and such that $\rho(\Phi) \notin \mathbb{Q}$ then Φ is conjugate to a rotation of angle $2\pi\rho(\Phi)$, see [3, p. 107] or [4, p. 45] for instance. So this is the situation for Poncelet's map P when $\rho(P) \notin \mathbb{Q}$ and $r \geq 2$.

With the above notation the celebrated *Poncelet's Theorem* asserts that if γ and Γ are ellipses, with arbitrary relative positions, and $\rho = \rho(\gamma, \Gamma) \in \mathbb{Q}$ then Poncelet's map is also conjugate to the rotation of angle $2\pi\rho$ in $\mathbb{S}^1$. In geometrical

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