On Poncelet's Maps

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Abstract

Given two ellipses, one surrounding the other one, Poncelet introduced a map P from the exterior one to itself by using the tangent lines to the interior ellipse. This procedure can be extended to any two smooth, nested and convex ovals and we call this type of maps Poncelet's maps. We recall what he proved around 1814 in the dynamical systems language: In the two ellipses case and when the rotation number of P is rational there exists a $n \in \mathbb{N}$ such that $P^n = \text{Id}$, or in other words, the Poncelet's map is conjugated to a rational rotation. In this paper we study general Poncelet's maps and give several examples of algebraic ovals where the corresponding Poncelet's map has a rational rotation number and it is not conjugated to a rotation. Finally, we also provide a new proof of Poncelet's result based on dynamical and computational tools.

1 Introduction and Main Results

Let γ and Γ be two $\mathcal{C}^r$, $r \geq 1$, simple, closed and nested curves, each one of them being the boundary of a convex set. Furthermore we assume for instance that Γ surrounds γ.

Given any $p \in \Gamma$ there are exactly two points q_1, q_2 in γ such that the lines $p\,q_1$, $p\,q_2$ are tangent to γ. We define the *Poncelet's map*, $P : \Gamma \to \Gamma$, associated to the pair γ, Γ as

$$P(p) = P_{\gamma,\Gamma}(p) = \overline{pq_1} \cap \Gamma,$$

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where $p \in \Gamma$, $\overline{pq_1} \cap \Gamma$ is the first point in the set $\{\overline{pq_1} \cap \Gamma, \overline{pq_2} \cap \Gamma\}$ that we find when, starting from p, we follow Γ counterclockwise, see Figure 1. Notice that $P^{-1}(p) = \overline{pq_2} \cap \Gamma$.
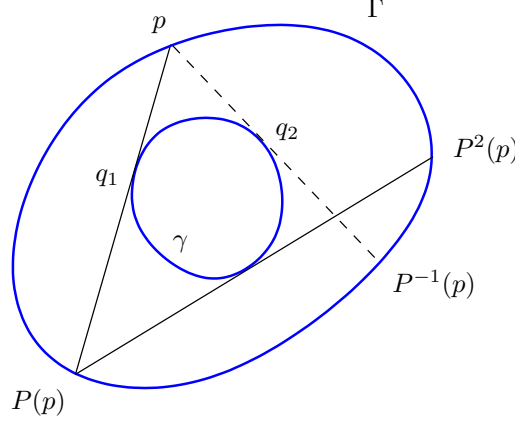
FIGURE 1. The Poncelet's map.

The implicit function theorem together with the geometrical interpretation of the construction of P imply that it is a $\mathcal{C}^r$ diffeomorphism from Γ into itself. So P can be seen as a $\mathcal{C}^r$ diffeomorphism of the circle and has associated a *rotation number*

$$\rho = \rho(P) = \rho(\gamma, \Gamma) \in (0, 1/2).$$

See for instance [1, 2] for the definition of rotation number. Notice that usually a rotation number is in $(0, 1)$. Our choice of q_1 for the Poncelet's map implies that indeed $\rho < 1/2$. It is also well known that if Φ is any diffeomorphism of the circle of class at least $\mathcal{C}^2$ and such that $\rho(\Phi) \notin \mathbb{Q}$ then Φ is conjugated to a rotation of angle $\rho(\Phi)$. So this is the situation for the Poncelet's map P when $\rho(P) \notin \mathbb{Q}$ and $r \geq 2$.

With the above notation the celebrated *Poncelet's Theorem* asserts that if γ and Γ are ellipses, with arbitrary relative positions, and $\rho = \rho(\gamma, \Gamma) \in \mathbb{Q}$ then the Poncelet's map is also conjugated to the rotation of angle ρ in $\mathbb{S}^1$. In geometrical terms, if starting at some point $p \in \Gamma$ the Poncelet's process of drawing tangent lines to γ closes after n steps then the same holds for any other starting point in Γ. There are several proofs of this nice result in [10, Sec. 4.3] and a different one, based on a beautiful approach of Bertrand and Jacobi through differential equations and elliptic integrals in [9, pp. 191-194]. In Section 4 we give another proof based on dynamical and computational tools, by using the results of [4]. The problem of determining explicit conditions over the coefficients of the two ellipses to ensure that the Poncelet's map is conjugated to a rational rotation was solved by Cayley. An excellent exposition of this result is given in [7].

A monograph devoted to Poncelet's theorem and related results has recently appeared, see [6].

It is clear that in Poncelet's Theorem it is not restrictive to assume that $\gamma = \{x^2 + y^2 = 1\}$. The first question that we face in this paper is the following: Is the Poncelet's result also true when we consider $\gamma = \{x^2 + y^2 = 1\}$ and Γ given by an oval of an algebraic curve of higher degree? We prove:

Theorem 1. *Fix $\gamma = \{x^2 + y^2 = 1\}$. Then for any $m \in \mathbb{N}$, $m > 2$, there is an algebraic curve of degree m, containing a convex oval Γ, such that the Poncelet's map associated to γ and Γ has rational rotation number and it is not conjugated to a rotation.*

This result will be a consequence of a more general result proved in Section 2, see Proposition 2. Moreover the full dynamics of the introduced Poncelet's maps $P : \Gamma \to \Gamma$ will be described in that section.

From Theorem 1 it is clear that, in general, Poncelet's maps with rational rotation numbers are not conjugated to rotations. It is natural to wonder about this question when both ovals are level sets of the same polynomial map $V : \mathbb{R}^2 \to \mathbb{R}$. As one of the simplest cases we consider the homogeneous map $V(x, y) = x^4 + y^4$, giving

$$\gamma = \Gamma_1 = \{x^4 + y^4 = 1\} \quad \text{and} \quad \Gamma = \Gamma_k = \{x^4 + y^4 = k\},$$

for $k \in \mathbb{R}, k > 1$. As we will see in Section 3 even in this situation the conjugacy with a rotation is not true.

The Poncelet's maps also provide a natural way of defining an integrable function from an open set of $\mathbb{R}^2$ into itself as follows: We foliate the open unbounded connected component $\mathcal{U}$ of $\mathbb{R}^2 \setminus \Gamma_1$ as

$$\mathcal{U} := \bigcup_{k>1} \{x^4 + y^4 = k\}.$$

Then the Poncelet's construction gives a new diffeomorphism, also of class $\mathcal{C}^r$, that is defined from $\mathcal{U} \subset \mathbb{R}^2$ into itself, which simply consists in applying to each point p the corresponding Poncelet's map, associated to the level set of V passing trough p. For sake of simplicity we also call it P. Notice that this map is trivially integrable by means of $V(x, y) = x^4 + y^4$, that is $V(P(x, y)) = V(x, y)$ for all $(x, y) \in \mathcal{U}$.

As we will see in Subsection 3.4, this extended Poncelet's map P will be useful to give properties of a functional equation that helps to study integrable planar maps.

Finally, in the above context it is natural to introduce the rotation function $\rho(k) := \rho(\Gamma_1, \Gamma_k)$, as the rotation number of P associated to γ and Γ_k, and to study some of its properties.

In the case of two concentric circles

$$\gamma = \widetilde{\Gamma}_1 = \{x^2 + y^2 = 1\} \quad \text{and} \quad \Gamma = \widetilde{\Gamma}_k = \{x^2 + y^2 = k\},$$

it is easy to prove that the rotation function $\widetilde{\rho}(k) = \rho(\widetilde{\Gamma}_1, \widetilde{\Gamma}_k)$ is the smooth monotonous function $\widetilde{\rho}(k) = \arctan(k-1)/\pi$. On the other hand, in Section 3 we will show that the function $\rho(k) := \rho(\Gamma_1, \Gamma_k)$, is much more complicated. Indeed all what we prove seems to indicate that it has the usual shape of the rotation function of generic one-parameter families of diffeomorphisms: the devil's staircase, see for instance [3, 8].

2 A Circle and an Oval

We prove a preliminary result that implies Theorem 1.

Proposition 2. *Consider*

$$\gamma = \{x^{2n} + y^{2n} = 1\} \quad and \quad \Gamma = \{x^{2m} + y^{2m} = 2\} \quad with \quad n, m \in \mathbb{N},$$

and let P be the Poncelet's map associated to them. Then $\rho_{n,m}(P) = 1/4$. Moreover, the map is conjugated a rotation if and only if $n = m = 1$.

Proof. It is easy to check that for any n and m, the Poncelet map P has the periodic orbit of period 4, given by the points $\mathcal{O} = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$. Hence $\rho_{n,m}(P) = 1/4$.

When $n = m = 1$, both ovals are coniques and Poncelet's Theorem proves one implication of our result. Let us prove the converse. Assume that P is conjugated to a rotation. Then, since $\rho(P) = 1/4$, for each $p \in \Gamma$, $P^4(p) = p$. Consider the particular point $p = p_1 := (0, \sqrt[2m]{2}) \in \Gamma$. By using the symmetries of the problem, the 4-periodicity of P forces that $p_2 := P(p_1) = (-\sqrt[2m]{2}, 0)$. Thus the line trough p_1 and p_2 has to be tangent to the oval $x^{2n} + y^{2n} = 1$ at some point $\bar{p} = (\bar{x}, \bar{y})$. Putting all the conditions together gives that $\bar{p}$ has to be solution of the system

$$\begin{cases} y = x + \sqrt[2m]{2}, \\ x^{2n} + y^{2n} = 1, \\ (2nx^{2n-1}, 2ny^{2n-1}) \cdot (1,1) = x^{2n-1} + y^{2n-1} = 0. \end{cases}$$

It has a real solution only when

$$2n = \frac{2m}{2m-1} = 1 + \frac{1}{2m-1}, \tag{1}$$

and in this case it is $\bar{p} = (-\sqrt[2m]{2}/2, \sqrt[2m]{2}/2)$. It is clear from (1) that the only solution with natural values of this equation is $n = m = 1$, as we wanted to prove. ∎

Proof of Theorem 1. Clearly the proof when m is even is a corollary of the Proposition 2. The proof when $m \geq 3$ is odd follows by noticing that the sets $\{x^{2m} + y^{2m} - 2 = 0\}$ and $\{(x + 10)\left(x^{2m} + y^{2m} - 2\right) = 0\}$ coincide in $\{x > -10\}$ and so in both cases the Poncelet's maps coincide. ∎

For the simplest case studied in Proposition 2, $n = 1$, we prove the following result that characterizes the dynamics of P.

Proposition 3. *Consider*

$$\gamma = \{x^2 + y^2 = 1\} \quad and \quad \Gamma = \{x^{2m} + y^{2m} = 2\} \quad with \quad m \in \mathbb{N}, m > 1.$$

and let P be the Poncelet's map associated to them. Then $\rho(P) = 1/4$, the orbit $\mathcal{O} = \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$ is a 4-periodic orbit of P and it is the α and the ω limit of all the orbits of P.

Proof. Set $\widetilde{\Gamma} := \{x^2 + y^2 = 2\}$, and let P and $\widetilde{P}$ be the the Poncelet maps associated to Γ and γ, and to $\widetilde{\Gamma}$ and γ, respectively. As usual given a point $p_1 \in \Gamma$ (resp. $q_1 \in \widetilde{\Gamma}$) we write $p_{i+1} := P(p_i) \in \Gamma$ (resp. $q_{i+1} := \widetilde{P}(q_i) \in \widetilde{\Gamma}$), for $i \geq 1$. Our goal will be to compare both maps. In fact, we will use the map $\widetilde{P}$ as a kind of Lyapunov function for the map P.
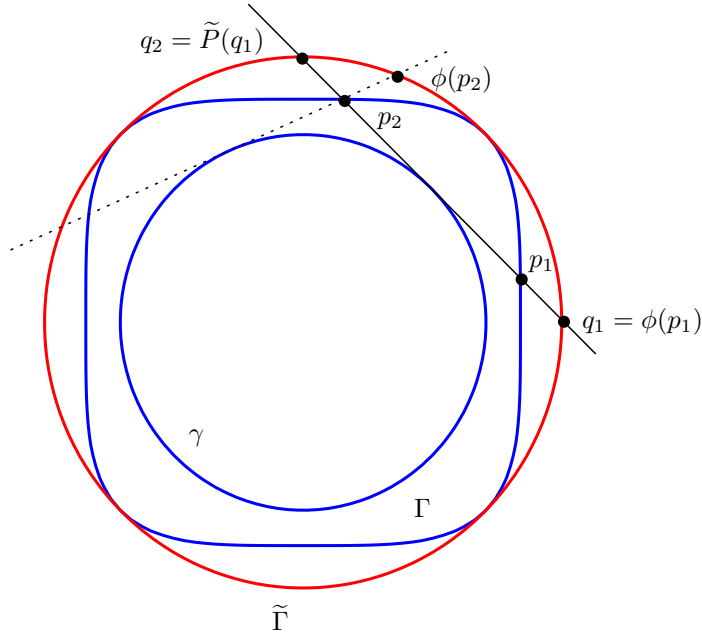
FIGURE 2. A comparison between the Poncelet's maps P and $\widetilde{P}$.

Recall that Poncelet's Theorem establishes that $\widetilde{P}$ is conjugated to a rotation. It is easy to check that the orbit $\mathcal{O} = \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$ is a 4-periodic orbit for $\widetilde{P}$. So, $\rho(\widetilde{P}) = 1/4$ and then $\widetilde{P}^4(q) = q$ for all $q \in \widetilde{\Gamma}$.

Notice that $\mathcal{O}$ is also a 4-periodic orbit of P. Hence $\rho(P) = 1/4$.

Let us introduce some notation. Fixed γ it is easy to construct a bijection between Γ and $\widetilde{\Gamma}$ as follows: Given $p_1 \in \Gamma$ we define $\phi(p_1)$ as the point of intersection between $\widetilde{\Gamma}$ and the half-straight line starting at $p_2 = P(p_1)$ and passing trough p_1, see Figure 2. Notice that by construction $q_1 := \phi(p_1)$, p_1, p_2 and $q_2 = \widetilde{P}(q_1)$ are aligned. Given a point $r \neq 0$, we denote by $\mathrm{Arg}(r)$ the argument modulus 2π of r thought as a point of $\mathbb{C} \setminus \{0\}$.

Take any $p_1 \in \Gamma \cap \{(x,y) \ : \ x > 0, y > 0 \text{ and } y \neq x\}$, and $q_1 \in \widetilde{\Gamma}$ given by $q_1 = \phi(p_1)$. By construction p_1, q_1, p_2 and q_2 are aligned. Notice also that $\mathrm{Arg}(q_2) > \mathrm{Arg}(\phi(p_2))$ where both angles are in $(0, \pi]$. This can be understood as a "delay" of P with respect to $\widetilde{P}$. This delay is propagated through the next three iterates giving that P^4 does not complete a tour around Γ. Hence the lifting of the map P^4 is below the identity except at the four points corresponding to the 4-periodic orbit of P. Then the result follows. ∎

Remark 4. *From the proof of Proposition 2 it is not difficult to construct $\mathcal{C}^1$ ovals for which the dynamic of the Poncelet's map is different from the one described in Proposition 3. By using the compatibility condition (1), it is easy to prove that, for instance when $n = 2$ and $m = 2/3$ in the statement of Proposition 2, i.e. when*

$$\gamma = \{x^4 + y^4 = 1\} \quad and \quad \Gamma = \{x^{4/3} + y^{4/3} = 2\},$$

the Poncelet's map P is such that $\rho(P) = 1/4$. It has the two 4-periodic orbits $\mathcal{O} = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$, and $\mathcal{O}^* = \{(0,a), (-a,0), (0,-a), (a,0)\}$ where $a = 2^{3/4}$ and it is not conjugated to a rotation.

3 Two Ovals

This section is devoted to study in more detail the Poncelet's maps associated to the ovals $\gamma = \{x^4 + y^4 = 1\}$ and $\Gamma_k = \{x^4 + y^4 = k\}$, $k > 1$.

3.1 How to find periodic orbits

Let us impose which condition has to satisfy an orbit to be n periodic for a Poncelet's map associated to the two ovals $\gamma = \{g(x,y) = 0\}$ and $\Gamma = \{G(x,y) = 0\}$. Take n counterclockwise ordered points on γ, $p_1, p_2, \ldots, p_n$. Draw the n tangent lines to γ, corresponding to these points, and denote them by $\ell_1, \ell_2, \ldots, \ell_n$, respectively. To generate a n-periodic orbit of P the following assertions must hold:

(i) The lines ℓ_i and ℓ_{i+1} for $i = 1, 2, \ldots, n$, where $\ell_{n+1} := \ell_1$, intersect. We denote these intersections by: $q_{i,i+1} = \ell_i \cap \ell_{i+1}$.

(ii) The following n equalities are satisfied:

$$G(q_{1,2}) = G(q_{2,3}) = \cdots = G(q_{n-1,n}) = G(q_{n,n+1}) = 0.$$

Notice that the above set of conditions gives a non-linear system with n unknowns (corresponding to the n points on γ) and n equations.

On the other hand when we consider the same problem between $\gamma = \{g(x,y) = 0\}$ and $\Gamma_k = \{G(x,y) = k\}$ and we take k also as a free unknown, the problem of searching n-periodic orbits is equivalent to impose instead of item (ii), the following $n-1$ equalities:

$$G(q_{1,2}) = G(q_{2,3}) = \cdots = G(q_{n-1,n}) = G(q_{n,n+1}).$$

So in this case we get again a non-linear system with n unknowns but now with only $n-1$ equations. Hence, it is natural to believe that either it has no solution or it has a continuum of them. Notice that this continuum can be interpreted as a continuum of values $k \in J_k \subset \mathbb{R}$ for which P has a periodic orbit of period n. Observe also that each one of these continua gives rise to an interval J_k where the rotation function $\rho(k)$ associated to the Poncelet's map between $\gamma = \{g(x,y) = 0\}$ and $\Gamma_k = \{G(x,y) = k\}$ has a constant value j/n, for some $j \in \mathbb{N}$. Each of these intervals will give rise to one stair in the devil's staircase which seems to be associated to P.

In next subsections we study in more detail the case $g(x,y) = x^4 + y^4 - 1$ and $G(x,y) = x^4 + y^4$, and we will give a geometrical interpretation of the localization of the starting and the ending points of some of the stairs of $\rho(k)$.

3.2 Some symmetric periodic orbits

Notice that the ovals $\Gamma_k = \{x^4 + y^4 = k\}, k \geq 1$ have several symmetries. These are given by the two axes and the diagonals $\{(x,y) \in \mathbb{R}^2 : y = \pm x\}$. In this section we search some periodic orbits of the Poncelet's maps P, associated to Γ_1 and Γ_k that share some of these symmetries. For these type of periodic orbits the order of the system described in the previous subsection, that has to be solved to find the periodic orbits, can be reduced.

As an example we find a value k for which the corresponding Poncelet's map has rotation number $1/3$ due to the existence of a symmetric 3-periodic orbit with respect to the $Oy-$axis. First we take $\{(x_1, y_1), (0, 1)\} \in \gamma$. The corresponding tangent lines to γ are $\ell_1(x,y) = x_1^3 x + y_1^3 y - 1 = 0$ and $\ell_2(x,y) = y - 1 = 0$, see the left picture in Figure 3. We have that $\ell_1 \cap \ell_2 = ((1 - y_1^3)/x_1^3, 1)$. To give rise to a 3-periodic orbit with the searched symmetry, the third point has to be $(x_3, y_3) := (-x_1, y_1)$. Moreover we get that $\ell_1 \cap \ell_3 = \ell_1 \cap \{x = 0\} = (0, 1/y_1^3)$. Hence the conditions that imply that the three intersection points between

the tangent lines belong to the same Γ_k reduce to the single equation

$$\left(\frac{1 - y_1^3}{x_1^3}\right)^4 + 1 = \left(\frac{1}{y_1^3}\right)^4,$$

or equivalently to impose that y_1 satisfies the equation

$$R(y) := y^{12}(1 - y^4)^3 + y^{12}(1 - y^3)^4 - (1 - y^4)^3 = 0,$$

where we have used that $x_1^4 + y_1^4 = 1$. Some calculations give that

$$\begin{aligned} R(y) &= -4y^{21} + 3y^{20} + 6y^{18} - 3y^{16} - 4y^{15} + 3y^{12} - 3y^8 + 3y^4 - 1 = \\ &\quad -(y-1)^4(y^2 + y + 1)R_{15}(y), \end{aligned}$$

where R_{15} is a polynomial of degree 15 which has only one real root, $y = y_1^* \simeq -0.779644$. The value of k corresponding to this solution is $k = k^* := 1/(y_1^*)^{12} \simeq 19.8264$. This result is reflected in Table 1.

| $k \simeq 19.8264$ | $k \simeq 20.1961$ | $k \simeq 20.5087$ |
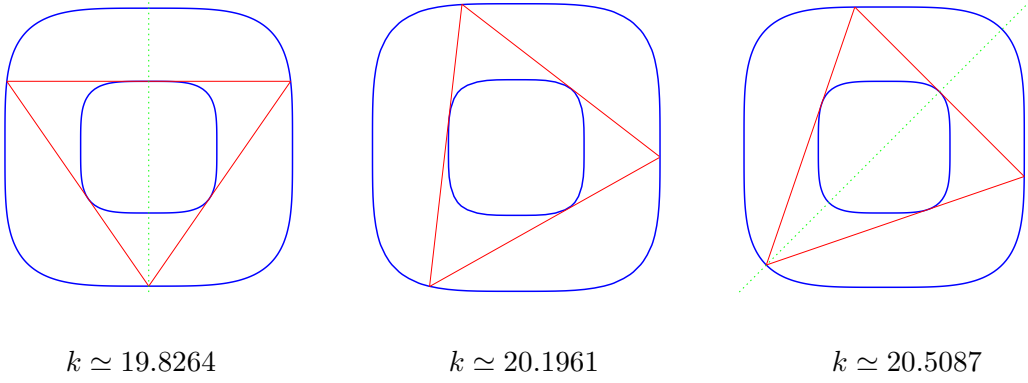
FIGURE 3. Three Poncelet's maps with rotation number $1/3$ associated to $\gamma = \{x^4 + y^4 = 1\}$ and $\Gamma = \{x^4 + y^4 = k\}$. Notice that the middle one is not symmetric.

| $k \simeq 1.5588$ | $k \simeq 1.5596$ |
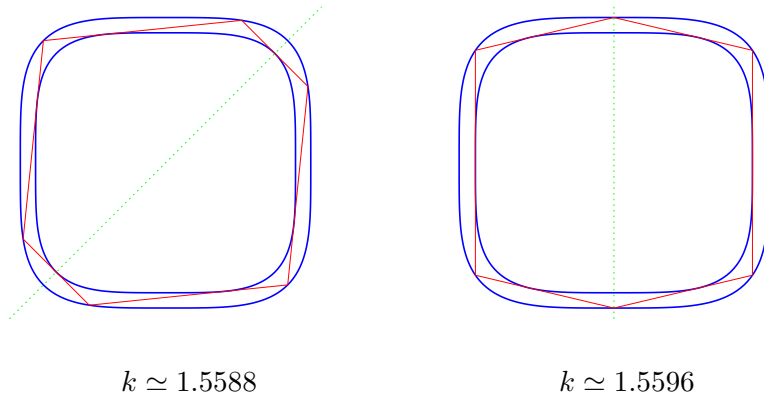
FIGURE 4. Two Poncelet's maps with rotation number $1/6$ associated to $\gamma = \{x^4 + y^4 = 1\}$ and $\Gamma = \{x^4 + y^4 = k\}$.

We have done similar computations to find values of k for which the corresponding Poncelet's maps have symmetric periodic orbits (with respect to either the axes or the diagonals) and have rotation numbers $1/3$, $1/4$ and $1/6$, see Figures 3 and 4. All these results, together with the fact that

$$\lim_{k \to 1} \rho(k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \rho(k) = \frac{1}{2},$$

which simply follows from the geometrical interpretation of P, are collected in Table 1.

k:	1	$\simeq 1.5588$	$\simeq 1.5596$	2	8	$\simeq 19.8264$	$\simeq 20.5087$	∞
$\rho(k)$	0	1/6	1/6	1/4	1/4	1/3	1/3	1/2
Symmetry :	–	diagonals	axes	all	all	axes	diagonals	–

Table 1: Some values of k with rational rotation number and symmetric period orbits.

Remark 5. *Notice that, once we have a n-periodic orbit for a given Poncelet's map, then the six sets obtained by applying to it either a rotation of $0, \pi/2, \pi$ or $3\pi/2$ radians or one of the symmetries with respect to the diagonals $\{y = \pm x\}$, are also n-periodic orbits. Moreover, unless the original periodic orbit has some of the symmetries considered in this subsection, these six n-periodic orbits are all different.*

As a corollary of the above remark we get, for instance, that for $k = k^*$ the Poncelet map has four 3-periodic orbits. Similarly, for $k \simeq 20.5087$ the corresponding Poncelet's map has four different periodic orbits, as well. Moreover in both cases it is not difficult to check that P is not conjugated to the rotation of angle $2\pi/3$.

3.3 On the stairs of the rotation function

Let Φ_λ be a smooth one-parameter family of diffeomorphism of the circle. Let $\rho(\lambda)$ be the rotation number of Φ_λ. Fix a natural number m. Recall that the existence of an open interval I_m where $\rho(\lambda) \equiv j/m$, for some natural j, coprime with m, is a consequence of the existence of a hyperbolic m-periodic orbit for Φ_λ for some $\lambda \in I_m$. For this reason it is said that for generic one-parameter families of diffeomorphims the graph of the rotation function has a devil's staircase shape.

The existence of open intervals on which the rotation function $\rho(k)$ for Poncelet's maps P associated to $\gamma = \{x^4 + y^4 = 1\}$ and $\Gamma_k = \{x^4 + y^4 = k\}$ is constant can be interpreted by using the above facts. Here we discuss how the existence of this type of intervals can also be interpreted more geometrically.

Consider for instance the two values of k for which $\rho(k) = 1/3$ obtained in Subsection 3.2, $k^* \simeq 19.8264$ and $\tilde{k} \simeq 20.5087$. By using the method described in Subsection 3.1 we have done a numerical study about the existence of other values of k having as well a 3-periodic orbit. We have obtained that for any $k \in [k^*, \tilde{k}]$ such an orbit exists, see for instance the middle picture in Figure 3. Moreover for any $k \in (k^*, \tilde{k})$ the orbit that we have found has none of the symmetries described in this section. By using Remark 5, we know that for each of these values of k, the Poncelet's map has six different periodic orbits. On the other hand the boundaries of the interval correspond with values of k for which some of these six 3-periodic orbits collide giving rise to some symmetric 3-period orbit, which indeed has to be a multiple 3-periodic orbit and so no hyperbolic for P.

We have also checked that a similar phenomenon occurs when $k \in [2, 8]$. On this interval $\rho(k) \equiv 1/4$.

An open problem is to prove if the situation described for $1/3$ and $1/4$ holds for any rational number in $(0, 1/2)$ and also study the same question for other families of convex ovals.

3.4 A consequence of the behavior of the rotation function

In [4] it is proved the following result:

Theorem 6. *Let $\mathcal{U} \subset \mathbb{R}^2$ be an open set and let $\Phi : \mathcal{U} \to \mathcal{U}$ be a diffeomorphism such that:*

(a) *It has a smooth regular first integral $V : \mathcal{U} \to \mathbb{R}$, having its level sets $\Gamma_k =: \{p \in \mathcal{U} : V(p) = k\}$ as simple closed curves.*

(b) *There exists a smooth function $\mu : \mathcal{U} \to \mathbb{R}^+$ such that for any $p \in \mathcal{U}$,*

$$\mu(\Phi(p)) = \det(D\Phi(p))\,\mu(p). \tag{2}$$

Then the map Φ restricted to each Γ_k is conjugated to a rotation with rotation number $\tau(k)/T(k)$, where $T(k)$ is the period of Γ_k as a periodic orbit of the planar differential equation

$$\dot{p} = \mu(p) \left(-\frac{\partial V(p)}{\partial p_2}, \frac{\partial V(p)}{\partial p_1} \right)$$

and $\tau(k)$ is the time needed by the flow of this equation for going from any $q \in \Gamma_k$ to $\Phi(q) \in \Gamma_k$.

Notice that it provides a way to check whether integrable planar maps Φ of the circle are conjugated to rotations or not. It consists in studying the existence of solutions μ of the functional equation (2).

A natural problem in this context is to study under which conditions over an integrable map Φ, the functional equation (2) has non-trivial solutions. Let us see that the Poncelet's map $\Phi = P$ constructed in Section 1 associated to the curves $\{x^4 + y^4 = 1\}$ and $\{x^4 + y^4 = k\}, k > 1$, and defined in the set $\mathcal{U} = \{2 < x^4 + y^4 < 8\}$, provides an example of map F for which equation (2) has no solution. Notice that P is clearly integrable, with first integral $V(x, y) = x^4 + y^4$. If associated to P it would exist a function μ satisfying the functional equation (2) then, by the results of the previous subsection and Theorem 6, $P^4 = \text{Id}$ on $\mathcal{U}$, result that is trivially false.

4 A new proof of Poncelet's Theorem

This section is devoted to give a new proof of Poncelet's Theorem based on Theorem 6. To do this it is more convenient to take coordinates in such a way that the outer ellipse is given by the circle $\Gamma = \{x^2 + y^2 = 1\}$ and the inner one is given by the set $\gamma = \{g(x, y) := Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\}$. Moreover it is not restrictive to assume that g is positive on the exterior of the ellipse. Following our point of view we will consider the Poncelet's map P defined from the region $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : |(x, y)| > d_0\}$ into itself, where d_0 is the maximum distance between γ and the origin. Hence $\Gamma \subset \mathcal{U}$. Some straightforward computations, done with an algebraic manipulator, give that the Poncelet's map writes as:

$$P(x, y) = \left(\frac{-N_1 N_2 - 4N_3\sqrt{\Delta}}{M}, \frac{-N_1 N_3 + 4N_2\sqrt{\Delta}}{M} \right) \tag{3}$$

where

$$\Delta = \left(AE^2 - BDE + CD^2 + F(B^2 - 4AC) \right) \left[Ax^2 + Bxy + Cy^2 + Dx + Ey + F \right],$$

$$N_1 = 4AF + 4CF - D^2 - E^2 + 2(2CD - BE)x + 2(2AE - DB)y$$
$$+ \left(4AC - B^2 \right) \left[x^2 + y^2 \right],$$

$$N_2 = \left(4CF + D^2 - 4AF - E^2 \right) x + 2(DE - 2BF)y$$
$$+ \left[2(2CD - BE) + \left(4AC - B^2 \right) x \right] \left[x^2 + y^2 \right],$$

$$N_3 = 2(DE - 2BF)x + \left(E^2 - D^2 - 4CF + 4AF \right) y$$
$$+ \left[2(2AE - BD) + \left(4AC - B^2 \right) y \right] \left[x^2 + y^2 \right],$$

and

$$
\begin{aligned}
M = \ & E^4 + D^4 + 16F^2A^2 - 16EFBD + 16F^2C^2 + 16F^2B^2 + 2D^2E^2 - 8FAD^2 \\
& + 8FAE^2 - 8FE^2C + 8FCD^2 - 32F^2AC + \left(-12D^2EB - 16AEFB\right. \\
& \left. -32FCAD + 4BE^3 + 16DAE^2 + 8CD^3 - 8CDE^2 - 16FEBC\right. \\
& \left. +32FC^2D + 16DB^2F\right)x + \left(-32FACE - 8AD^2E + 16CD^2E + 16B^2EF\right. \\
& \left. +32FA^2E + 4BD^3 + 8E^3A - 16FADB - 16BCDF - 12BDE^2\right)y \\
& + \left(16C^2D^2 + 6E^2B^2 + 8FAB^2 - 8FB^2C - 32FA^2C + 32FC^2A\right. \\
& \left. +2B^2D^2 + 16A^2E^2 - 16DBEC - 8AE^2C + 8ACD^2 - 16ABDE\right)x^2 \\
& + \left(32ACDE - 8DB^2E - 64BCAF + 16B^3F\right)yx + \left(-16DBEC + 16A^2E^2\right. \\
& \left. + 2E^2B^2 - 8FAB^2 - 32FC^2A + 6B^2D^2 + 16C^2D^2 + 8AE^2C - 16ABDE\right. \\
& \left. +8FB^2C + 32FA^2C - 8ACD^2\right)y^2 \\
& +4(4AC - B^2)\left[(2CD - BE)x + (2AE - BD)y\right]\left[x^2 + y^2\right] \\
& + \left(4AC - B^2\right)^2\left[x^2 + y^2\right]^2.
\end{aligned}
$$

In Figure 5 we show some points of two orbits generated by this map corresponding to different ellipses. Recall that by construction the map P given in (3) is a diffeomorphism on $\mathcal{U}$ and $V(x,y) = x^2 + y^2$ is a first integral for it.

Then in order to apply Theorem 6 we only need to find a smooth function, defined on $\mathcal{U}$, such that the functional equation (2) with $\Phi = P$ holds. It can be seen, again by using an algebraic manipulator, that a function satisfying this equality is given by

$$
\mu(x,y) = \sqrt{(x^2 + y^2)g(x,y)} = \sqrt{(x^2 + y^2)\left(Ax^2 + Bxy + Cy^2 + Dx + Ey + F\right)}.
$$

Hence Poncelet's Theorem follows. Notice that our proof also works when the rotation number of the Poncelet's map is irrational.

We end this section with some comments of how we have obtained the above function μ. By using the Change of Variables Theorem it is not difficult to check that the existence of a positive function μ satisfying equality (2) implies that the absolute continuous measure

$$
\nu(\mathcal{B}) := \iint_{\mathcal{B}} \frac{1}{\mu(x,y)}\,dxdy,
$$

is an invariant measure for P, that is $\nu(P^{-1}(\mathcal{B})) = \nu(\mathcal{B})$ for any measurable set $\mathcal{B} \subset \mathcal{U}$. On the other hand, one of the proofs given in [10] –the one of [5]– geometrically constructs an invariant measure on the outer circumference to prove Poncelet's Theorem. Inspired on this construction we have thought how to extend this measure to the whole open set $\mathcal{U}$ and as a consequence we have got a suitable μ.
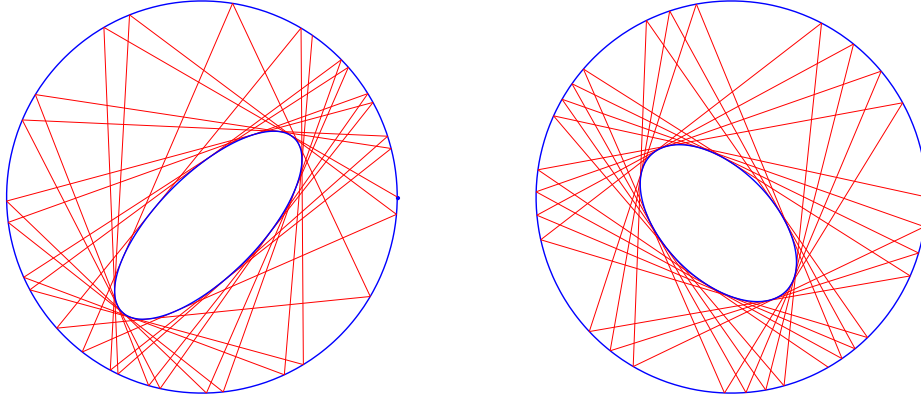
FIGURE 5. Thirty points of two Poncelet's orbits.
The dot corresponds to the initial condition.

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