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New periodic recurrences with applications

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ABSTRACT

We develop two methods for constructing several new and explicit m-periodic difference equations. Then we apply our results to two different problems. Firstly we show that two simple natural conditions appearing in the literature are not necessary conditions for the global periodicity of the difference equations. Secondly we present the first explicit non-linear analytic potential differential system having a global isochronous center.

1. Introduction and main results

In general, a real difference equation (or recurrence) of order n writes as

$$x_{j+n} = f(x_j, x_{j+1}, \ldots, x_{j+n-1}) \tag{1}$$

where $x_j \in \mathbb{R}$ for all $j \in \mathbb{N}$, and f is a map from an open subset of $\mathbb{R}^n$ into $\mathbb{R}$. This recurrence can be studied through the dynamical system generated by the map $F : U \to U$, where U is an open subset of $\mathbb{R}^n$ and F is

$$F(x_1, \ldots, x_n) = \big(x_2, \ldots, x_n, f(x_1, \ldots, x_n)\big). \tag{2}$$

Recurrence (1) or the dynamical system generated by (2) is m-periodic if $F^m = \mathrm{Id}$ and $m \geqslant n$ is the smallest natural with this property. In terms of the recurrence (1) this property reads as $x_{j+m} = x_j$ for all $j \geqslant 0$. Several examples of periodic recurrences are given for instance in [1,4,8,9,14,15,19].

The first part of the paper is devoted to give several new examples of periodic recurrences (1), defined in the whole $\mathbb{R}^n$. In our constructions we follow the next two simple ideas:

- The difference equation (1) is transformed into the new difference equation
$$y_{j+n} = \varphi^{-1}\big(f\big(\varphi(y_j), \varphi(y_{j+1}), \ldots, \varphi(y_{j+n-1})\big)\big), \tag{3}$$
with the change of variables $x_n = \varphi(y_n)$, where φ is any invertible map. Hence if (1) is m-periodic the same holds for (3). Moreover if f, φ and φ^{-1} are all elementary functions, the same happens with the function defining the new recurrence (3).
- Let G be a given m-periodic recurrence of order n and let Φ be a homeomorphism of $V \subset \mathbb{R}^n$ into $U \subset \mathbb{R}^n$. Then $F = \Phi \circ G \circ \Phi^{-1}$ gives a periodic map on U which is not necessarily a periodic recurrence. We investigate the structure of the map Φ in order that F still gives a recurrence on $\mathbb{R}^n$ and we study in more detail the case $n = 2$.

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