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Examples and counterexamples for Markus-Yamabe and LaSalle global asymptotic stability problems

Anna Cima, Armengol Gasull and Francesc Mañosas

Universitat Autònoma de Barcelona, Spain

cima@mat.uab.cat, gasull@mat.uab.cat, manyosas@mat.uab.cat

Abstract

We revisit the known counterexamples and the state of the art of the Markus-Yamabe and LaSalle's problems on global asymptotic stability of discrete dynamical systems. We also provide new counterexamples, associated to difference equations, for some of these problems.

1 Introduction

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ map and consider the discrete dynamical system

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k). \tag{1}$$

Let $A = (a_{ij})$ be a real $n \times n$ matrix. We denote by $\sigma(A)$ the spectrum of A, i.e., the set of eigenvalues of A and by $|A| = (|a_{ij}|)$. We also denote by $DF(\mathbf{x}) = \left(\frac{\partial F_i(\mathbf{x})}{\partial x_j}\right)$ the Jacobian matrix of F at $\mathbf{x} \in \mathbb{R}^n$. When $F(\mathbf{0}) = \mathbf{0}$, we can write $F(\mathbf{x})$ in the form $F(\mathbf{x}) = A(\mathbf{x})\mathbf{x}$, where $A(\mathbf{x})$ is an $n \times n$ matrix function. Note that this $A(\mathbf{x})$ is not unique.

LaSalle in [12] gave some possible generalizations of the sufficient conditions for global asymptotic stability (GAS) for $n = 1$. Concretely, the conditions are the following:

(I) $|\lambda| < 1$ for all $\lambda \in \sigma(A(\mathbf{x}))$ and for all $\mathbf{x} \in \mathbb{R}^n$,

(II) $|\lambda| < 1$ for all $\lambda \in \sigma(|A(\mathbf{x})|)$ and for all $\mathbf{x} \in \mathbb{R}^n$,

(III) $|\lambda| < 1$ for all $\lambda \in \sigma(DF(\mathbf{x}))$ and for all $\mathbf{x} \in \mathbb{R}^n$,

(IV) $|\lambda| < 1$ for all $\lambda \in \sigma(|DF(\mathbf{x})|)$ and for all $\mathbf{x} \in \mathbb{R}^n$.

In [6] it is proved that none of the conditions I and II implies GAS, even for $n = 2$. In particular in both cases there are polynomial maps satisfying them and such that the origin of (1) is not GAS.

Conditions III and IV are also known as Markus-Yamabe type conditions because they are similar to a condition proposed for ordinary differential equations, see [4, 10] and the references therein. In [5] it is proved that condition III implies GAS for planar polynomial maps and that there