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GLOBAL LINEARIZATION OF PERIODIC DIFFERENCE EQUATIONS

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Abstract. We deal with m-periodic, n-th order difference equations and study whether they can be globally linearized. We give an affirmative answer when $m = n + 1$ and for most of the known examples appearing in the literature. Our main tool is a refinement of the Montgomery-Bochner Theorem.

1. **Introduction.** In this paper we investigate the linearization of periodic difference equations defined on an open subset of $\mathbb{R}$. A difference equation or a recurrence of order n on $\mathbb{R}$ of class $\mathcal{C}^k$ ($k \in \mathbb{N} \cup \{0\} \cup \{\infty\} \cup \{\omega\}$) is an equation of the form:

$$x_{j+n} = f(x_j, x_{j+1}, \dots, x_{j+n-1}) \tag{1}$$

where $x_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, and f is a $\mathcal{C}^k$ map from an open subset of $\mathbb{R}^n$ into $\mathbb{R}$.

The study of the dynamics of the recurrence (1) is given by the dynamics of the associated map $F : U \to U$, where U is an open subset of $\mathbb{R}^n$, not necessarily connected, and F is given by

$$F(x_1, \dots, x_n) = (x_2, \dots, x_n, f(x_1, \dots, x_n)). \tag{2}$$

We will say that equation (1) is m-periodic if $F^m = \mathrm{Id}$ and m is the smallest natural with this property. Clearly, in this case, $m \geq n$ and the only possibility for $m = n$ is when $f(x_1, \dots, x_n) = x_1$.

Recall that it is said that a map $F : U \to U$, $\mathcal{C}^k$-*linearizes* on an open set $U \subset \mathbb{R}^n$ if there exists a $\mathcal{C}^k$-homeomorphism, $\psi : U \to \psi(U) \subset \mathbb{R}^n$, for which $G := \psi \circ F \circ \psi^{-1}$ is the restriction of a linear map to $\psi(U)$. The map ψ is called a *linearization of F on U*. When the map F is of the form (2) and the linearized map G is as well of the form (2) then we will say that the associated recurrence (1) linearizes in the corresponding domain.

Notice that real periodic difference equations are a particular case of periodic maps on subsets of $\mathbb{R}^n$. These maps have been largely studied. In order to have a better understanding of our goal when we restrict our interest to maps of the form (2), first we give a brief summary of some of the most relevant results on general periodic maps on $\mathbb{R}^n$.

It is a well-known result that every periodic $\mathcal{C}^k$ map on $\mathbb{R}$ is either the identity, or 2-periodic and that in this later case it $\mathcal{C}^k$-linearizes (notice that in this situation this is equivalent to say that it is $\mathcal{C}^k$-conjugated to $-\mathrm{Id}$), see for instance [18]. From

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