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Simple examples of planar involutions with non-global Montgomery–Bochner linearizations

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ABSTRACT

We give two planar polynomial involutions, one preserving and the other one reversing orientation, for which the Montgomery–Bochner linearization is not a global diffeomorphism.

1. Introduction and results

A map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be *periodic* if there exists $p \in \mathbb{N}$ such that $F^p = \mathrm{Id}$, where $F^m = F \circ F^{m-1}$. If p is the minimum positive integer satisfying $F^p = \mathrm{Id}$ then we will say that the map is p-periodic. As usual, 2-periodic maps are called *involutions*.

When $n = 2$, from the results of Kerékjártó (1920), it is known that if the map is p-periodic and continuous then it has a fixed point and it is globally conjugate to a linear p-periodic map, see [1].

In general, if $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is of class $\mathcal{C}^k$, $k \geq 1$, p-periodic and has a fixed point $\mathbf{y}$, then F is always locally $\mathcal{C}^k$ conjugate, in a neighborhood of $\mathbf{y}$, to the linear map $(DF)_{\mathbf{y}}$. This result is known as Montgomery–Bochner theorem [2]. Moreover its proof is constructive and the conjugation is given by the map

$$\psi(\mathbf{x}) = \sum_{i=0}^{p-1} (DF)_{\mathbf{y}}^{-i} F^i(\mathbf{x}), \tag{1}$$

which satisfies $\psi \circ F = (DF)_{\mathbf{y}} \circ \psi$ and is a diffeomorphism in a neighborhood of $\mathbf{y}$ because $(D\psi)_{\mathbf{y}} = p\,\mathrm{Id}$. We call the map ψ the *Montgomery–Bochner linearization*. Note that ψ has the same regularity that F.

In [3] it was proved that in several cases the Montgomery–Bochner linearization is in fact a global linearization. Moreover in [4] the authors give sufficient conditions to ensure that the Montgomery–Bochner linearization is a global diffeomorphism for some planar involutions. That paper also builds some $\mathcal{C}^1$-involutions for which the Montgomery–Bochner linearization fails to be a global diffeomorphism. The examples presented there rather than being explicit are either described in terms of geometrical properties or constructed gluing some suitable maps.

The goal of this note is to present two simple planar polynomial involutions, one preserving and the other one reversing orientation, for which the corresponding Montgomery–Bochner linearization ψ given in (1) is not a global diffeomorphism.

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