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On 2- and 3-periodic Lyness difference equations

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We describe the sequences $\{x_n\}_n$ given by the non-autonomous second-order Lyness difference equations $x_{n+2} = (a_n + x_{n+1})/x_n$, where $\{a_n\}_n$ is either a 2-periodic or a 3-periodic sequence of positive values and the initial conditions x_1, x_2 are also positive. We also show an interesting phenomenon of the discrete dynamical systems associated with some of these difference equations: the existence of one oscillation of their associated rotation number functions. This behaviour does not appear for the autonomous Lyness difference equations.

Keywords: difference equations with periodic coefficients; circle maps; rotation number

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1. Introduction and main result

This paper fully describes the sequences given by the non-autonomous second-order Lyness difference equations

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \tag{1}$$

where $\{a_n\}_n$ is a k-periodic sequence taking positive values, $k = 2, 3$ and the initial conditions x_1, x_2 are as well positive. This question is proposed in ([5], Section 5). Recall that non-autonomous recurrences appear, for instance, as population models with a variable structure affected by some seasonality [11,12], where k is the number of seasons. Some dynamical issues of similar type of equations have been studied in several recent papers [1,9,10,14–16,18].

Recall that when $k = 1$, that is $a_n = a > 0$, for all $n \in \mathbb{N}$, then (1) is the famous Lyness recurrence which is well understood, see for instance [2,19]. The cases $k = 2, 3$ have been already studied and some partial results are established. For both cases, it is known that the solutions are persistent near a given k-periodic solution, which is stable. This is proved by using some known invariants, see [15,17,18]. Recall that in our context, it is said that a solution $\{x_n\}_n$ is persistent if there exist two real positive constants c and C, which depend on the initial conditions, such that for all $n \geq 1, 0 < c < x_n < C < \infty$.

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