Journal of Difference Equations and Applications
Vol. 18, No. 9, September 2012, 1457–1465

On Coxeter recurrences

Anna Cima*, Armengol Gasull[1] and Francesc Mañosas[2]

Department de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

(*Received 7 January 2011; final version received 11 February 2011*)

An interesting family of recurrences of order $n \geq 2$, which are globally $(n+3)$-periodic was introduced by Coxeter in 1971. We prove a surprising property of this family: 'all' the possible geometrical behaviours that linear real $(n+3)$-periodic recurrences can have are present inside the Coxeter recurrences.

2000 Mathematics Subject Classification: 39A05; 39A20; 39B12

Keywords: globally periodic difference equations; recurrences; Coxeter difference equations; linearization

1. Introduction and main result

Although recently globally periodic recurrences have attracted the interest of many researchers, see for instance [1–5,7,8] and the references there in, an interesting family introduced by Coxeter in 1971, see [6], is rarely referenced.

For each natural number $n \geq 2$, Coxeter proved that the recurrences

$$x_{j+n} = 1 - \frac{x_{j+n-1}}{1 - \frac{x_{j+n-2}}{1-((x_{j+n-3})/(1-\cdots(x_{j+1}/(1-x_j))))}} := f_n(x_j, x_{j+1}, \ldots, x_{j+n-1}) \tag{1}$$

are globally $(n+3)$-periodic, that is for any admissible set of initial conditions, $x_{j+n+3} = x_j$, for all $j \geq 0$. For instance, for $n = 2, 3$, the recurrences are

$$x_{j+2} = 1 - \frac{x_{j+1}}{1-x_j}, \quad \text{and} \quad x_{j+3} = 1 - \frac{x_{j+2}}{1 - \frac{x_{j+1}}{1-x_j}} = \frac{1 - x_j - x_{j+1} - x_{j+2} + x_j x_{j+2}}{1 - x_j - x_{j+1}},$$

respectively. It is easy to see that for $n = 2$, in the new variables $u_j = x_j - 1$, it corresponds to the well-known 5-periodic Lyness recurrence

$$u_{j+2} = \frac{1 + u_{j+1}}{u_j}.$$

*Corresponding author. Email: cima@mat.uab.cat

ISSN 1023-6198 print/ISSN 1563-5120 online

http://dx.doi.org/10.1080/10236198.2011.564576
http://www.tandfonline.com