An explicit bound of the number of vanishing double moments forcing composition

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Abstract

We give two new characterizations of pairs of polynomials or trigonometric polynomials that form a composition pair. One of them proves that the cancellation of a given number of double moments implies that they form a composition pair. This number only depends on the maximum degree of both polynomials. This is the first time that composition is characterized in terms of the cancellation of an explicit number of double moments. Our results allow to recognize the composition centers for polynomial and trigonometric Abel differential equations.

2000 Mathematics Subject Classification: 39C25, 37C27.
Keywords: Periodic orbits, centers, trigonometric Abel equation, double moments, Composition pair, parameterizable algebraic curve, topological degree.

1 Introduction and main results

Abel equations of the form

$$\dot{r} = \frac{dr}{ds} = A(s)\, r^3 + B(s)\, r^2, \tag{1}$$

with A and B either polynomials or trigonometric polynomials are a subject of increasing interest; see [6, 7, 8, 9, 10, 11, 16, 26]. One of the main reasons is their relation with the center-focus problem and the second part of the Hilbert Sixteenth problem. Both questions deal with the number of periodic orbits of planar polynomial systems; see [2, 4, 15, 17, 21, 25]. In particular, given $a < b$, the center-focus problem in this setting reduces to find conditions on A and B such that all the solutions $r = r(s, r_0)$, with initial condition $r(a, r_0) = r_0$ and $|r_0|$ small enough, satisfy $r(a, r_0) = r(b, r_0)$. When this happens it is said that the Abel equation has a *center* at the origin, $r = 0$. The case where A and B are trigonometric polynomials and $a = 0$ and $b = 2\pi$ is the motivating problem and is the only one that we will consider in this trigonometric setting. On the other hand, when A and B are polynomials the values a and b can be arbitrarily taken.

A sufficient condition for (1) to have a center at the origin is introduced in [4]. When there exist $\mathcal{C}^1$-functions A_1, B_1 and u, with $u(a) = u(b)$, such that

$$\widetilde{A}(s) := \int_a^s A(z)\,dz = A_1(u(s)) \quad \text{and} \quad \widetilde{B}(s) := \int_a^s B(z)\,dz = B_1(u(s))$$

it is said that $\widetilde{A}$ and $\widetilde{B}$ *form a composition pair*. It is well-known that the corresponding Abel equation has a center. From now on, when there is no confusion we will write $\int_a^s A(z)\,dz$ simply as $\int_a^s A$.

Recently we have proved the following result, where $\mathbb{N}$ denotes the set of all non-negative integer numbers and by $\mathbb{N}^+$ the positive ones.

Theorem 1. *([14]) Let A and B real polynomials or trigonometric polynomials and $a < b$. Moreover in the later case $a = 0, b = 2\pi$. The following statements are equivalent:*

(i) For all $i, j \in \mathbb{N}$ it holds that $\int_a^b B = \int_a^b \widetilde{A}^i \widetilde{B}^j A = 0$.

(ii) For all $i, j \in \mathbb{N}$ it holds that $\int_a^b A = \int_a^b \widetilde{A}^i \widetilde{B}^j B = 0$.

(iii) The functions $\widetilde{A}$ and $\widetilde{B}$ form a composition pair.

(iv) For all $i, j \in \mathbb{N}$ it holds that $\int_a^b \widetilde{A}^i \widetilde{B}^j A = \int_a^b \widetilde{A}^i \widetilde{B}^j B = 0$.

In [14] it is also shown the equivalence among these conditions and a type of persistence by perturbations of the center of the associated Abel equation. We notice that in the polynomial case a stronger result than the equivalence between (i) (or (ii)) with (iv) was already proved in [23]. More concretely, instead of item (i) it is proved that it suffices for $\widetilde{A}$ and $\widetilde{B}$ to form a composition pair that the given integrals vanish for all $j \geq 0$ and all $0 \leq i \leq \mu_a + \mu_b$, where μ_a (resp. μ_b) is the multiplicity of a (resp. b) as a zero of $\widetilde{B}$.

The quantities $\int_a^b \widetilde{A}^i \widetilde{B}^j A$ and $\int_a^b \widetilde{A}^i \widetilde{B}^j B$, $i, j \in \mathbb{N}$ are called *double moments* of A and B. The above result shows that they provide a way, computing infinitely many double moments, of characterizing when a couple of functions A and B form a composition pair. It is worth to comment that these moments have been introduced in [5, 13, 26] because it has been shown that the cancellation of all the usual *moments*: $\int_a^b \widetilde{B}^i A$ and $\int_a^b \widetilde{A}^i B$, $i \in \mathbb{N}$, is not enough for characterizing when A and B form a composition pair; see [13, 22].

Notice that, given a and b and fixing the degrees of A and B, the double moments are polynomial expressions in the coefficients of A and B. In view of Theorem 1, the composition pairs are characterized as the common zeros of these infinitely many polynomials. Using the

Hilbert's basis Theorem we know that finitely many of them suffice to characterize when $\widetilde{A}$ and $\widetilde{B}$ form a composition pair. Unfortunately Hilbert's result is not constructive and in general an explicit bound of the number of needed polynomials is not known.

The aim of this paper is to give this bound for our particular problem. Concretely, we will provide an explicit bound of the number of double moments that have to vanish to know when a given a couple A and B forms a composition pair.

Observe the parallelism between the problem that we have solved and the detection of non-degenerated centers for planar polynomial vector fields of a given degree. Similarly that in our problem, the centers are characterized by the cancellation of the Lyapunov quantities, which are also polynomials in the coefficients of the system. Again the Hilbert's basis Theorem ensures that only finitely many of them are needed. Nevertheless, even for cubic vector fields this number is nowadays unknown.

We state and prove separately our results for the trigonometric and polynomial cases. From now one, we will write the time $s = t$ in the polynomial case and $s = \theta$ in the trigonometric one. Recall moreover that in this later situation $a = 0$ and $b = 2\pi$.

Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients and given $A \in \mathbb{R}[x]$ we denote by δA its degree. Similarly we introduce $\mathbb{R}[x, y]$ as the ring of polynomials in two variables, also with real coefficients. Our first result is:

Theorem 2. *Given $A, B \in \mathbb{R}[x]$ with $\max(\delta A, \delta B) = n$ the following statements are equivalent:*

(i) For all $i, j \in \mathbb{N}$ satisfying $i + j \leq 2n - 3$, $\int_a^b \widetilde{A}^i \widetilde{B}^j A = \int_a^b B = 0$.

(ii) For all $i, j \in \mathbb{N}$ satisfying $i + j \leq 2n - 3$, $\int_a^b \widetilde{A}^i \widetilde{B}^j B = \int_a^b A = 0$.

(iii) The polynomials $\widetilde{A}$ and $\widetilde{B}$ form a composition pair.

(iv) For all $i, j \in \mathbb{N}$, $\int_a^b \widetilde{A}^i \widetilde{B}^j A = \int_a^b \widetilde{A}^i \widetilde{B}^j B = 0$.

All the known centers for Abel equations (1) with A and B polynomials are such that $\widetilde{A}$ and $\widetilde{B}$ form a composition pair. If there were no other type of centers the above result would provide a finite and explicit number of conditions to solve the center-focus problem in this setting. This would be very interesting because, similarly that for planar vector fields, given the degrees of A and B and computing a kind of Lyapunov quantities, see [4], it can be proved that only finitely many polynomials relations, involving the coefficients of A and B, have to vanish to characterize the centers of (1). As in the case of planar systems there is no explicit upper bound for this number of conditions.

We will denote by $\mathbb{R}_t[\theta]$ the ring of trigonometric polynomials with real coefficients. Given $A \in \mathbb{R}_t[\theta]$ we write δA for the degree of the Fourier series corresponding to A, see also Lemma 10. When A is not a constant we will say that τ is the minimal period of A, if $\tau > 0$ is the smallest positive number such that $A(\theta + \tau) = A(\theta)$ for all $\theta \in \mathbb{R}$. It is easy to see that $\tau = 2\pi/m$, for some $m \in \mathbb{N}^+$. Notice that if $\tau = 2\pi/m$ then m is a divisor of δA. Given $A, B \in \mathbb{R}_t[\theta]$, with minimal periods $2\pi/m_1$ and $2\pi/m_2$, respectively, we will say that A and B have minimal common period $2\pi/\gcd(m_1, m_2)$. We prove:

Theorem 3. *Given $A, B \in \mathbb{R}_t[\theta]$ with $\max(\delta A, \delta B) = n$ and minimal common period $2\pi/k$, $k \in \mathbb{N}^+$, the following statements are equivalent:*

(i) For all $i, j \in \mathbb{N}$ satisfying $i + j \leq 4n/k - 3$, $\int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j A = \int_0^{2\pi} B = 0$.

(ii) For all $i, j \in \mathbb{N}$ satisfying $i + j \leq 4n/k - 3$, $\int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j B = \int_0^{2\pi} A = 0$.

(iii) $\widetilde{A}$ and $\widetilde{B}$ form a composition pair.

(iv) There exists $0 \neq S \in \mathbb{R}[x, y]$ with $\delta S \leq 2n/k - 1$ such that $S(\widetilde{A}, \widetilde{B}) = 0$.

(v) For all $i, j \in \mathbb{N}$, $\int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j A = \int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j B = 0$.

Contrary to what happens for the polynomial case it is well known that there are centers for the trigonometric Abel equation (1) with $\widetilde{A}$ and $\widetilde{B}$ not forming a composition pair; see for instance [1, 3, 13]. In any case, centers of this type are important because they are persistent under some particular perturbations and so they seem to be the biggest class of centers for trigonometric Abel equations.

This is the first time that an effective method involving finitely many computations is given for knowing when a couple of trigonometric polynomials or polynomials form a composition pair. This was not the case using Theorem 1 or the results of [12, 23], because infinitely many conditions have to be checked. Indeed, given a couple A and B either using one of the items (i)-(ii) of Theorems 2 or 3 or item (iv) of Theorem 3 it is easy to check if they form a composition pair. Moreover, notice that the approach given in item (iv) of Theorem 3 is also new.

As we will see, although the proofs for the polynomial and the trigonometrical polynomial cases share many points there is a main difference between the subfields of quotients associated to both families of functions, see Theorems 4 and 11. This difference makes the proofs different.

2 The polynomial case

We will write $\mathbb{K}$ to represent either $\mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}[x]$ denotes the set of polynomials with coefficients in $\mathbb{K}$ and $\mathbb{K}(x)$ its corresponding quotient field. Given $p, q \in \mathbb{K}(x)$, we denote by $\mathbb{K}(p)$ (resp. $\mathbb{K}(p, q)$) the smallest subfield of $\mathbb{K}(x)$ containing p (resp. p and q).

The next result, proved in [13], is a consequence of Lüroth's Theorem.

Theorem 4. *Let $\mathbb{L}$ be a subfield of $\mathbb{R}(x)$ containing a non-constant polynomial. Then $\mathbb{L} = \mathbb{R}(p)$ for some polynomial p. Moreover, if a polynomial $m \in \mathbb{L}$ then $m = f(p)$ for some polynomial f.*

We will say that $p, q \in \mathbb{K}[x]$ are *dependent* if there exist $u, r, s \in \mathbb{K}[x]$ with $\delta u > 1$ such that $p(x) = r(u(x))$ and $q(x) = s(u(x))$. We will say that $p, q \in \mathbb{K}[x]$ are *independent* if they are not dependent.

In view of the above theorem it is clear that $p, q \in \mathbb{K}[x]$ are dependent if and only if $\mathbb{K}(p, q) = \mathbb{K}(u)$ for some $u \in \mathbb{K}[x]$ with $\delta u > 1$. Reciprocally, $p, q \in \mathbb{K}[x]$ are independent if and only if $\mathbb{K}(p, q) = \mathbb{K}(x)$. Note that this last condition is equivalent to the existence of polynomials $R, S \in \mathbb{K}[x, y]$ such that $x = \frac{R(p(x),q(x))}{S(p(x),q(x))}$. Then if $p, q \in \mathbb{R}[x]$ are independent then they are also independent as elements of $\mathbb{C}[x]$.

Lemma 5. *Let $p, q \in \mathbb{R}[x]$ be dependent with $\max(\delta p, \delta q) = n$. Then there exists a polynomial $0 \neq S \in \mathbb{R}[x, y]$ with $\delta S < n - 1$ such that $S(p, q) = 0$.*

Proof. Consider the equation

$$xS(p(x), q(x)) = R(p(x), q(x)),$$

where the coefficients of the polynomials $R, S \in \mathbb{R}[x, y]$ are the unknowns and $\max(\delta R, \delta S) < n - 1$. From this equation we obtain a homogeneous linear system of $(n-2)n+2$ equations with $(n-1)n$ unknowns. So it has non-trivial solutions. Let R_1 and S_1 be a non-trivial one. If $S_1(p, q) \neq 0$ then we obtain $x = \frac{R_1(p(x),q(x))}{S_1(p(x),q(x))}$. Therefore $\mathbb{R}(p, q) = \mathbb{R}(x)$ which implies that p and q are independent, in contradiction with our hypothesis. So $S_1(p, q) = 0$ as we want to prove. □

Proposition 6. *Let $p, q \in \mathbb{R}[x]$ be independent with $\max(\delta p, \delta q) = n$ and let $0 \neq S \in R[x, y]$ be such that $S(p, q) = 0$. Then $\delta S > n - 1$.*

Proof. First we decompose $S(x, y) = \Pi_{i=1}^{k} S_i(x, y)$, in irreducible factors on $\mathbb{C}[x, y]$. Clearly for some $j \in \{1, \ldots, k\}$, $S_j(p, q) = 0$. If we show that $\delta S_j = n$ then the proposition will follow.

To prove this, let $V \subset \mathbb{C}^2$ be the affine variety associated to S_j, that is

$$V = \{(x, y) \in \mathbb{C}^2 \,:\, S_j(x, y) = 0\},$$

and consider the morphism

$$\phi : \mathbb{C} \longrightarrow V,$$

given by $\phi(t) = (p(t), q(t))$.

We claim that if an irreducible algebraic plane curve is parameterized through a pair of independent polynomials, then any regular point on the curve has associated a unique value of the parameter.

To prove the claim, notice first that the morphism ϕ extends to a morphism $\overline{\phi}$ between the projective complex line (that we denote by $\mathcal{P}^1$) and the closure of $\phi(\mathbb{C})$ on the projective complex plane. Since S_j is irreducible, this closure is the projective curve associated to S_j that we denote by $\overline{V}$. Therefore we have a morphism of varieties

$$\overline{\phi} : \mathcal{P}^1 \longrightarrow \overline{V}.$$

Since V has a polynomial parametrization, $\overline{V}$ is a rational curve having a unique point at infinity, $\overline{r}$, which is the image by $\overline{\phi}$ of r, the infinity point of $\mathcal{P}^1$. Let $\widehat{V}$ be the desingularization of $\overline{V}$. Since $\overline{V}$ is rational and non-singular it follows that $\widehat{V} = \mathcal{P}^1$; see for instance [18, 20]. By the universal property of the desingularization we know that there exists a morphism $\widehat{\phi} : \mathcal{P}^1 \longrightarrow \widehat{V}$ such that $\overline{\phi} = \pi \circ \widehat{\phi}$, i.e. the following diagram

$$\begin{tikzcd} & \widehat{V} = \mathcal{P}^1 \arrow[d, "\pi"] \\ \mathcal{P}^1 \arrow[ur, "\widehat{\phi}"] \arrow[r, "\overline{\phi}"'] & \overline{V} \end{tikzcd}$$

commutes, being π the projection between $\widehat{V}$ and $\overline{V}$; see again [18, 20]. Thus, if we denote by $\widehat{r} = \widehat{\phi}(r)$, we have that $\widehat{r}$ is the only point in $\mathcal{P}^1$ verifying that $\pi(\widehat{r}) = \overline{r}$. Hence $\widehat{\phi}$ can be viewed as a map from $\mathcal{P}^1$ to $\mathcal{P}^1$ that sends the infinity point of $\mathcal{P}^1$ to itself and no other points are sent to infinity. Thus, it follows that the restriction of $\widehat{\phi}$ to the afine local chart, $\widehat{\phi}_a$, is a polynomial.

Similarly, the restriction of π to $\mathbb{C}$, π_a, has polynomial components: $\pi_a(t) = (f(t), g(t))$ with $f, g \in \mathbb{C}[t]$.

Hence

$$\phi(t) = (p(t), q(t)) = \pi_a(\widehat{\phi}_a(t)) = \left(f(\widehat{\phi}_a(t)), g(\widehat{\phi}_a(t)) \right).$$

Since, by hypothesis, p and q are independent it follows that the degree of the polynomial $\widehat{\phi}_a$ is one.

On the other hand, since π is the projection of the normalized variety $\widehat{V}$ over $\overline{V}$, for almost all $x \in \overline{V}$ we have that $\pi^{-1}(x)$ is only one point. Thus the topological degree of π is one. So the topological degree of $\overline{\phi}$ coincides with the topological degree of $\widehat{\phi}$ that also coincides with its degree as polynomial. So we conclude that the topological degree of $\overline{\phi}$ is one and the claim follows.

Lastly note that δS_j is equal to the number of intersections of the affine variety V with a generic straight line $ax+by+c=0$. Since the parametrization $p(t), q(t)$ has topological degree one it passes only one time for almost all points of V. Hence this number is equal to the number of complex values of t satisfying $ap(t)+bq(t)+c=0$ which is also equal to the $\max(\delta p, \delta q)=n$. So $\delta S_j = n$ and the result follows. $\square$

Proposition 7. *Let $p, q \in \mathbb{R}[x]$ be such that $\max(\delta p, \delta q)=n$. Then p, q are dependent if and only if there exists $0 \neq S \in \mathbb{R}[x,y]$ with $\delta S < n-1$ such that $S(p,q)=0$. Moreover if they are independent then there exist polynomials $U, V \in \mathbb{R}[x,y]$ with $\max(\delta U, \delta V) < n-1$ such that $x = \frac{U(p(x),q(x))}{V(p(x),q(x))}$.*

Proof. The first statement follows from Lemma 5 and Proposition 6.

Now let $p, q \in \mathbb{R}[x]$ be independent. Arguing as in Lemma 5, consider the linear system determined by the equality $U(p(x),q(x)) = x\,V(p(x),q(x))$ with $\delta U = \delta V = n-2$, which has non-trivial solutions. Let U_1, V_1 be one of these solutions. From Proposition 6 we know that $V_1(x,y) \neq 0$. Hence

$$x = \frac{U_1(p(x),q(x))}{V_1(p(x),q(x))}$$

as we wanted to prove.

$\square$

Proposition 8. *Let A, B in $\mathbb{R}[x]$, with $\max(\delta A, \delta B)=n$ and satisfying*

$$\int_a^b \widetilde{A}^i \widetilde{B}^j A = \int_a^b B = 0 \text{ for all } i,j \geq 0 \text{ such that } i+j \leq 2n-3.$$

Then there exists $u \in \mathbb{R}[x]$ such that $\mathbb{R}(\widetilde{A}, \widetilde{B}) = \mathbb{R}(u)$ and $u(a)=u(b)$. In particular

$$\int_a^b \widetilde{A}^i \widetilde{B}^j A = \int_a^b B = 0 \text{ for all } i,j \geq 0.$$

Proof. First of all note that integrating by parts we obtain that

$$\int_a^b \widetilde{A}^i \widetilde{B}^j B = \int_a^b A = 0,$$

for all $i,j \geq 0$ satisfying $i+j \leq 2n-3$. By Theorem 4, since $\widetilde{A}, \widetilde{B}$ are polynomials, we have that $\mathbb{R}(\widetilde{A}, \widetilde{B}) = \mathbb{R}(u)$ with $u \in \mathbb{R}[x]$. To prove the implication it suffices to show that $u(a)=u(b)$. Clearly, u is a non-constant polynomial. We know that

$$u = \frac{P(\widetilde{A}, \widetilde{B})}{Q(\widetilde{A}, \widetilde{B})}, \tag{2}$$

for some $P, Q \in \mathbb{R}[x,y]$. Moreover $\widetilde{A} = r(u)$ and $\widetilde{B} = s(u)$ for some polynomials r, s with $\max(\delta r, \delta s) \leq n+1$. Hence

$$u = \frac{P(r(u), s(u))}{Q(r(u), s(u))},$$

or equivalently,

$$x = \frac{P(r(x), s(x))}{Q(r(x), s(x))}.$$

Therefore r and s are independent and by Proposition 7 we can choose P and Q such that $\max(\delta P, \delta Q) \leq n-1$.

Derivating (2) we obtain

$$u' = \frac{(Q\,\partial_1 P - P\,\partial_1 Q)(\widetilde{A}, \widetilde{B})A + (Q\,\partial_2 P - P\,\partial_2 Q)(\widetilde{A}, \widetilde{B})B}{Q^2(\widetilde{A}, \widetilde{B})}.$$

Since $\widetilde{A}$ and $\widetilde{B}$ are polynomial functions of u we have that

$$Q^2(\widetilde{A}, \widetilde{B}) = Q^2(r(u), s(u)) = N'(u),$$

for some polynomial N. Thus

$$N'(u)u' = Q^2(\widetilde{A}, \widetilde{B})u' = (Q\,\partial_1 P - P\,\partial_1 Q)(\widetilde{A}, \widetilde{B})A + (Q\,\partial_2 P - P\,\partial_2 Q)(\widetilde{A}, \widetilde{B})B.$$

Integrating both sides of this equality on $[a,b]$ and using that all the double moments of order at most $2n-3$ vanish we obtain that $N(u(b)) - N(u(a)) = 0$. If $u(a) = u(b)$ we are done. Assume, to arrive a contradiction, that $u(a) \neq u(b)$ and denote by I the interval with extremes these two values. Since $N'(u) = Q^2(\widetilde{A}, \widetilde{B}) \geq 0$ we have that $N'(x) \geq 0$ for all x in I. Therefore N is increasing on I and $N(u(b)) \neq N(u(a))$, given the desired contradiction. □

Proof of Theorem 2. $(i) \Rightarrow (ii) \Rightarrow (iii)$. These implications are given in Proposition 8.

$(iii) \Rightarrow (iv)$. It follows by direct computations.

$(iv) \Rightarrow (i)$. This implication is obvious. □

3 The trigonometric case.

We will denote by $\mathbb{R}_t(\theta)$ the quotient field of $\mathbb{R}_t[\theta]$. In fact $\mathbb{R}_t[\theta] = \mathbb{R}[\sin\theta, \cos\theta]$, $\mathbb{R}_t(\theta) = \mathbb{R}(\sin\theta, \cos\theta)$ and it is well known that $\mathbb{R}_t(\theta)$ is isomorphic to $\mathbb{R}(x)$ by means of the map $\Phi : \mathbb{R}_t(\theta) \longrightarrow \mathbb{R}(x)$ defined by

$$\Phi(\sin\theta) = \frac{2x}{1+x^2} \quad \text{and} \quad \Phi(\cos\theta) = \frac{1-x^2}{1+x^2}. \tag{3}$$

In particular, this morphism satisfies that

$$\Phi\Big(\tan\Big(\frac{\theta}{2}\Big)\Big) = \Phi\left(\frac{\sin\theta}{1+\cos\theta}\right) = x.$$

Next lemma characterizes the image by Φ of the set of trigonometric polynomials.

Lemma 9. *([13]) It holds that*

$$\Phi(\mathbb{R}_t[\theta]) = \bigcup_{m\geq 0}\left\{\frac{r(x)}{(1+x^2)^m} : r\in\mathbb{R}[x] \;\; and \;\; \delta r \leq 2m\right\}.$$

Recall that the degree of a trigonometric polynomial has been introduced as the degree of its Fourier series. Next result gives an equivalent interpretation of the degree.

Lemma 10. *Set $p\in\mathbb{R}_t[\theta]$ with $\delta p = n$. Then*

$$\Phi(p(\theta)) = \frac{r(x)}{(1+x^2)^n} \qquad with \qquad \gcd(r(x),(1+x^2)) = 1.$$

Proof. The Fourier series of p is

$$p(\theta) = \sum_{k=-n}^{n} a_k e^{k\theta i}, \quad a_{-k} = \overline{a}_k \in \mathbb{C} \quad \text{and} \quad a_n \neq 0.$$

Equivalently,

$$\begin{aligned}\Phi(p(\theta)) =& \sum_{k=1}^{n}\overline{a}_k\left(\frac{1-x^2}{1+x^2} - \frac{2x}{1+x^2}i\right)^k + a_0 + \sum_{k=1}^{n} a_k\left(\frac{1-x^2}{1+x^2} + \frac{2x}{1+x^2}i\right)^k\\ =& \frac{\sum_{k=1}^{n}\overline{a}_k(1-x^2-2xi)^k(1+x^2)^{n-k} + a_0(1+x^2)^n}{(1+x^2)^n} +\\ &+ \frac{\sum_{k=1}^{n} a_k(1-x^2+2xi)^k(1+x^2)^{n-k}}{(1+x^2)^n} =: \frac{r(x)}{(1+x^2)^n}.\end{aligned}$$

To end the proof we need to show that $\gcd(r(x),(1+x^2)) = 1$. This follows because $r(i) = 4^n\overline{a}_n \neq 0$. ◻

We also will use the following characterization of some subfields of $\mathbb{R}(\theta)$.

Theorem 11. *([14, 19]) Let $\mathbb{L}$ be a subfield of $\mathbb{R}_t(\theta)$ containing a non-constant trigonometric polynomial. Then either $\mathbb{L} = \mathbb{R}(\tan(k\theta/2))$ for some $k\in\mathbb{N}^+$ or $\mathbb{L} = \mathbb{R}(p(\theta))$ for some trigonometric polynomial p. Moreover, when $\mathbb{L} = \mathbb{R}(p(\theta))$, if $q\in\mathbb{L}$ is a trigonometric polynomial then $q(\theta) = f(p(\theta))$ for some polynomial $f\in\mathbb{R}[x]$.*

Any $p \in \mathbb{R}_t[\theta]$ can be thought as a real periodic function. Its minimal period is a real number $2\pi/k$, for some $k \in \mathbb{N}^+$, and then p can be written as a real polynomial in $\sin(k\theta), \cos(k\theta)$. Notice that k divides δp. From now on for $k \in \mathbb{N}^+$ we will denote by $\mathbb{R}_t[k\theta]$ the set of real trigonometric polynomials in $k\theta$ that is $\mathbb{R}[\cos(k\theta), \sin(k\theta)]$. Also we denote by $\mathbb{R}_t(k\theta)$ its quotient field. Clearly $\mathbb{R}_t(k\theta) = \mathbb{R}(\tan(k\theta/2))$.

Given two trigonometric polynomials p, q we will say that they form a *composition pair* if $\mathbb{R}(p, q) = \mathbb{R}(u)$ for some trigonometric polynomial u. In view of Theorem 11 then there exist $\widehat{p}, \widehat{q} \in \mathbb{R}[x]$ such that $p(\theta) = \widehat{p}(u(\theta))$ and $q(\theta) = \widehat{q}(u(\theta))$.

We will say that $p, q \in \mathbb{R}_t[\theta]$ are *k-independent* if $\mathbb{R}(p(\theta), q(\theta)) = \mathbb{R}(\tan(k\theta/2))$. When $p, q \in \mathbb{R}_t[\theta]$ are 1-independent we simply say that they are *independent*. Notice that in this case $\mathbb{R}(\Phi(p), \Phi(q)) = \mathbb{R}(x)$, where Φ is given in (3).

Observe that given a pair of polynomials then either they form a composition pair or they are k-independent for some $k \geq 1$.

From these definitions and the previous theorem we obtain next result.

Lemma 12. *The following assertions hold*

(i) $\mathbb{R}_t[\theta] \cap \mathbb{R}(\tan(k\theta/2)) = \mathbb{R}_t[k\theta]$.

(ii) If $p_1, p_2 \in \mathbb{R}_t[\theta]$ have minimal common period $2\pi/k$, there exist $\widehat{p}_1, \widehat{p}_2 \in \mathbb{R}_t[\theta]$ such that the minimal common period of $\widehat{p}_1, \widehat{p}_2$ is 2π, $\widehat{p}_i(k\theta) = p_i(\theta)$ and $\delta(\widehat{p}_i) = \delta p_i/k$ for $i = 1, 2$. Moreover, p_1 and p_2 form a composition pair if and only if the same holds for $\widehat{p}_1, \widehat{p}_2$. Equivalently, p_1 and p_2 are k-independent if and only if $\widehat{p}_1, \widehat{p}_2$ are independent.

Proof. Set $p \in \mathbb{R}_t[\theta] \cap \mathbb{R}(\tan(k\theta/2))$. Then p is a rational function in $\tan(k\theta/2)$ that implies that its minimal period is $2\pi/(sk)$, for some $s \in \mathbb{N}^+$. Therefore $p \in \mathbb{R}_t[sk\theta] \subset \mathbb{R}_t[k\theta]$. This proves the first assertion.

Set $p_1, p_2 \in \mathbb{R}_t[\theta]$ with minimal common period $2\pi/k$. Both polynomials can be written as Fourier polynomials in $k\theta$. For instance, $p_1(\theta) = \sum_{j=0}^{n} a_j \cos(jk\theta) + b_j \sin(jk\theta)$. Thus we can take $\widehat{p}_1(\theta) = \sum_{j=0}^{n} a_j \cos(j\theta) + b_j \sin(j\theta)$ and similarly for p_2.

If p_1 and p_2 are k-independent, there exist $R, S \in \mathbb{R}[x, y]$ such that

$$\tan\left(\frac{k\theta}{2}\right) = \frac{R(p_1(\theta), p_2(\theta))}{S(p_1(\theta), p_2(\theta))} = \frac{R(\widehat{p}_1(k\theta), \widehat{p}_2(k\theta))}{S(\widehat{p}_1(k\theta), \widehat{p}_2(k\theta))}.$$

Therefore $\widehat{p}_1, \widehat{p}_2$ are independent.

Conversely, if p_1 and p_2 form a composition pair, $\mathbb{R}(p_1, p_2) = \mathbb{R}(u)$ for some $u \in \mathbb{R}_t[k\theta]$. Let $\widehat{u} \in \mathbb{R}_t[\theta]$ such that $u(\theta) = \widehat{u}(k\theta)$. Thus $\mathbb{R}(\widehat{p}_1, \widehat{p}_2) = \mathbb{R}(\widehat{u})$ and the result follows. $\square$

Proposition 13. *Let $p, q \in \mathbb{R}_t[\theta]$ be independent with* $\max(\delta p, \delta q) = n$ *and let* $0 \neq S \in R[x, y]$ *be such that* $S(p, q) = 0$. *Then* $\delta S > 2n - 1$.

Proof. Arguing as in the proof of Proposition 6 it suffices to prove that $\delta S = 2n$ assuming that S is irreducible. Following also that proof we can suppose that the following diagram commutes

$$\begin{tikzcd} & \widehat{V} = \mathcal{P}^1 \arrow[d, "\pi"] \\ \mathcal{P}^1 \arrow[ur, "\widehat{\phi}"] \arrow[r, "\overline{\phi}"'] & \overline{V} \end{tikzcd}$$

where $V \subset \mathbb{C}^2$ is the affine variety associated to S, that is

$$V = \{(x, y) \in \mathbb{C}^2 \, : \, S(x, y) = 0\},$$

the morphism $\phi : \mathbb{C} \longrightarrow V$ is given by

$$\phi(t) = \Big(\frac{r(t)}{(1+t^2)^n}, \frac{s(t)}{(1+t^2)^n}\Big), \quad \text{where} \quad \frac{r(t)}{(1+t^2)^n} = \Phi(p(\theta)) \quad \text{and} \quad \frac{s(t)}{(1+t^2)^n} = \Phi(q(\theta)),$$

and $\overline{\phi}, \widehat{\phi}, \overline{V}$ and $\widehat{V}$ are defined as in that proof and Φ is given in (3). Notice that again $\widehat{V}$ admits a rational parametrization and is non-singular. Therefore $\widehat{V} = \mathcal{P}^1$. Since the maps $\widehat{\phi}$ and π are rational maps we obtain

$$\Big(\frac{r(t)}{(1+t^2)^n}, \frac{s(t)}{(1+t^2)^n}\Big) = \phi(t) = \pi_a(\widehat{\phi}_a(t)) = (f(\widehat{\phi}_a(t)), g(\widehat{\phi}_a(t))), \tag{4}$$

for some rational maps $f, g \in \mathbb{R}(t)$. Here $\widehat{\phi}_a$ and π_a are the expressions of $\widehat{\phi}$ and π in the corresponding affine charts.

Recall that by definition of the independence of p and q,

$$\mathbb{R}(t) = \mathbb{R}\Big(\frac{r(t)}{(1+t^2)^n}, \frac{s(t)}{(1+t^2)^n}\Big),$$

and by (4), $\mathbb{R}(t) \subset \mathbb{R}(\widehat{\phi}_a(t))$. As a consequence, $\widehat{\phi}_a(t)$ is a Möbius map, i.e. $\widehat{\phi}_a(t) = \frac{v(t)}{w(t)}$ with $v, w \in \mathbb{R}[t]$, $\gcd(v, w) = 1$ and $\max(\delta v, \delta w) = 1$. So the topological degree of $\widehat{\phi}_a(t)$ is one.

On the other hand since π is the projection of the normalized variety $\widehat{V}$ over $\overline{V}$ for almost all $x \in \overline{V}$ we have that $\pi^{-1}(x)$ is only one point. Thus the topological degree of π is one. So we conclude that the topological degree of $\overline{\phi}$ is one. The same argument used in the proof of Proposition 6 let us to say that the topological degree of $\phi(t)$ is also one.

Lastly note that δS is equal to the number of intersections of the affine variety V with a generic straight line $ax + by + c = 0$. Since the parametrization $\phi(t) = \big(\frac{r(t)}{(1+t^2)^n}, \frac{s(t)}{(1+t^2)^n}\big)$ has topological degree one it passes only one time for almost all points of V. Therefore this number is equal to the number of complex values of t satisfying $a\frac{r(t)}{(1+t^2)^n} + b\frac{s(t)}{(1+t^2)^n} + c = 0$ which is $2n$. So $\delta S = 2n$ as we wanted to prove. $\square$

Proposition 14. *Let $p, q \in \mathbb{R}_t[\theta]$ be such that $\max(\delta p, \delta q) = n$. If they are k-independent then there exist polynomials $R, S \in \mathbb{R}[x, y]$ with $\max(\delta R, \delta S) \le 2n/k - 1$ such that*

$$\tan\Big(\frac{k\theta}{2}\Big) = \frac{R(p(\theta), q(\theta))}{S(p(\theta), q(\theta))}.$$

Proof. We prove first the case $k = 1$, i.e. when p and q are independent. Set $\frac{r(t)}{(1+t^2)^n} = \Phi(p(\theta))$ and $\frac{s(t)}{(1+t^2)^n} = \Phi(q(\theta))$, where Φ is given in (3), and consider the equation

$$R\Big(\frac{r(t)}{(1+t^2)^n}, \frac{s(t)}{(1+t^2)^n}\Big) = tS\Big(\frac{r(t)}{(1+t^2)^n}, \frac{s(t)}{(1+t^2)^n}\Big)$$

where $R, S \in \mathbb{R}[x, y]$ and $\max(\delta R, \delta S) = 2n - 1$. Thus, we obtain an homogeneous linear system of equations with unknowns the coefficients of R and S. This system has $4n^2 - 2n + 1$ equations and $4n^2 + 2n$ unknowns so it has non-trivial solutions. If R and S is a non-trivial one the result follows from Proposition 13 because $\delta S < 2n$ implies $S(p, q) \neq 0$.

Consider now the case $k > 1$. First of all note that if $p, q \in R_t[\theta]$ are k-independent then $p, q \in R_t[k\theta]$, that is they are polynomials in $\cos(k\theta)$ and $\sin(k\theta)$. This follows from the fact that since $p, q \in R(\tan k\theta/2)$ they are $2\pi/k$-periodic trigonometric polynomials.

By Lemma 12 we can write $p(\theta) = \widehat{p}(k\theta)$ and $q(\theta) = \widehat{q}(k\theta)$, with $\delta\widehat{p} = \delta p/k$ and $\delta\widehat{q} = \delta q/k$. Moreover $\widehat{p}$ and $\widehat{q}$ are independent and $\max(\delta\widehat{p}, \delta\widehat{q}) = n/k$. Then the result follows by using that it holds the case $k = 1$. $\square$

Proposition 15. *Let $p_1, p_2 \in \mathbb{R}_t[\theta]$ with $\max(\delta p_1, \delta p_2) = n$ and minimal common period $2\pi/k$. Then p_1, p_2 form a composition pair if and only if there exists $0 \neq S \in \mathbb{R}[x, y]$ with $\delta S < 2n/k$ such that $S(p_1, p_2) = 0$.*

Proof. Let $\widehat{p}_1, \widehat{p}_2$ be the trigonometric polynomials given by Lemma 12 such that $\widehat{p}_i(k\theta) = p_i(\theta)$ and $\delta\widehat{p}_i = n/k, i = 1, 2$.

If p_1 and p_2 do not form a composition pair then they are k-independent and, by Lemma 12, $\widehat{p}_1$ and $\widehat{p}_2$ are independent. By Lemma 13 it follows that $S(\widehat{p}_1, \widehat{p}_2) \neq 0$ for all $S \in \mathbb{R}[x, y]$ with $\delta S < 2n/k$. Thus we get $S(p_1(\theta), p_2(\theta)) = S(\widehat{p}_1(k\theta), \widehat{p}_2(k\theta)) \not\equiv 0$ for all $S \in \mathbb{R}[x, y]$ with $\delta S < 2n/k$.

Conversely, if p_1 and p_2 form a composition pair, again by Lemma 12, $\widehat{p}_1$ and $\widehat{p}_2$ form also a composition pair. Arguing as in the proof of the Proposition 14 we consider the equation

$$R(\Phi(\widehat{p}_1(\theta)), \Phi(\widehat{p}_2(\theta))) = tS(\Phi(\widehat{p}_1(\theta)), \Phi(\widehat{p}_2(\theta))),$$

where $R, S \in \mathbb{R}[x, y]$, $\max(\delta R, \delta S) = 2n/k - 1$ and Φ is given in (3). Thus we obtain a linear system with non-trivial solutions. Let R, S be a non trivial solution. We claim that

$S(\widehat{p}_1, \widehat{p}_2) = 0$. If not, we will have

$$\frac{R(\Phi(\widehat{p}_1(\theta)), \Phi(\widehat{p}_2(\theta)))}{S(\Phi(\widehat{p}_1(\theta)), \Phi(\widehat{p}_2(\theta)))} = t,$$

that implies

$$\frac{R(\widehat{p}_1(\theta), \widehat{p}_2(\theta)))}{S(\widehat{p}_1(\theta), \widehat{p}_2(\theta))} = \tan\left(\frac{\theta}{2}\right).$$

This last equality contradicts the the fact that $\widehat{p}_1$ and $\widehat{p}_2$ form a composition pair. So $S(\widehat{p}_1, \widehat{p}_2) = 0$. Therefore $S(p_1(\theta), p_2(\theta)) = S(\widehat{p}_1(k\theta), \widehat{p}_2(k\theta)) = 0$ and the proof follows. □

Proposition 16. *Let A, B be in $\mathbb{R}_t[\theta]$ with $\max(\delta A, \delta B) = n$ and minimal common period $2\pi/k$, $k \in \mathbb{N}^+$. Assume that*

$$\int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j A = \int_0^{2\pi} B = 0 \textit{ for all } i, j \geq 0 \textit{ satisfying } i + j \leq 4n/k - 3.$$

Then $\widetilde{A}$ and $\widetilde{B}$ form a composition pair. In particular

$$\int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j A = \int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j B = 0 \textit{ for all } i, j \geq 0.$$

Proof. First of all note that integrating by parts we obtain

$$\int_0^{2\pi} \widetilde{A}^i \widetilde{B}^j B = \int_0^{2\pi} A = 0 \text{ for all } i, j \geq 0 \text{ satisfying } i + j \leq 4n/k - 3.$$

Consider the field $\mathbb{L} := \mathbb{R}(\widetilde{A}, \widetilde{B})$. Since $\int_0^{2\pi} A = \int_0^{2\pi} B = 0$, the functions $\widetilde{A}$ and $\widetilde{B}$ are trigonometric polynomials. Therefore we can apply Theorem 11 and $\mathbb{L} = \mathbb{R}(p)$, with p either a trigonometric polynomial or $p = \tan(k\theta/2)$ for some $k > 0$. Notice that if we prove that the second possibility does not occur then we are done.

Assume that the second possibility happens. Then

$$\frac{P(\widetilde{A}(\theta), \widetilde{B}(\theta))}{Q(\widetilde{A}(\theta), \widetilde{B}(\theta))} = \tan\left(\frac{k\theta}{2}\right),$$

for some $P, Q \in \mathbb{R}[x, y]$ and $k \in \mathbb{N}^+$. By Proposition 14 we can choose P, Q such that $\max(\delta P, \delta Q) \leq 2n/k - 1$. Derivating with respect to θ we get

$$\frac{(Q\,\partial_1 P - P\,\partial_1 Q)(\widetilde{A}(\theta), \widetilde{B}(\theta))A(\theta) + (Q\,\partial_2 P - P\,\partial_2 Q)(\widetilde{A}(\theta), \widetilde{B}(\theta))B(\theta)}{Q^2(\widetilde{A}(\theta), \widetilde{B}(\theta))} = \frac{k}{2}\left(1 + \tan^2\left(\frac{k\theta}{2}\right)\right).$$

So

$$(Q\,\partial_1 P - P\,\partial_1 Q)(\widetilde{A}(\theta),\widetilde{B}(\theta))A(\theta) + (Q\,\partial_2 P - P\,\partial_2 Q)(\widetilde{A}(\theta),\widetilde{B}(\theta))B(\theta) = \frac{k}{2}(P^2+Q^2)(\widetilde{A}(\theta),\widetilde{B}(\theta)).$$

Note that the integral in the interval $[0,2\pi]$ of the left side of this equality is zero by our hypotheses, because it is the sum of a finite number of integrals of monomials of the form $\widetilde{A}^i\widetilde{B}^jA$ or $\widetilde{A}^i\widetilde{B}^jB$ with $i+j\le 2(2n/k-1)-1$. On the other hand the right side of the equality is a positive continuous function. This gives the desired contradiction. □

Proof of Theorem 3. $(i)\Rightarrow(ii)\Rightarrow(iii)$. These two implications follow from Proposition 16.

$(iii)\Rightarrow(iv)$. It si proved in Proposition 15.

$(iv)\Rightarrow(v)$. Using again Proposition 15 we get that $(iv)\Rightarrow(iii)$ and the proof that $(iii)\Rightarrow(v)$ follows by simple computations.

$(iv)\Rightarrow(v)$. This last implication is obvious. □

Acknowledgements. We thank our colleague Joaquim Roe for his valuable comments.

References

[1] Alwash, M.A.M. *On a condition for a centre of cubic non-autonomous equations*, Proc. Roy. Soc. Edinburgh 113A (1989), 289–291.

[2] Alwash, M.A.M. *Periodic solutions of Abel differential equations*, J. Math. Anal. Appl. 329 (2007), 1161–1169.

[3] Alwash, M.A.M. *The composition conjecture for Abel equation*, Expo. Math. 27 (2009), 241–250.

[4] Alwash, M.A.M. and Lloyd, N.G. *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Roy. Soc. Edinburgh 105A (1987), 129–152.

[5] Blinov, M., Roytvarf, N. and Yomdin, Y. *Center and moment conditions for Abel equation with rational coefficients*, Functional differential equations and applications (Beer-Sheva, 2002). Funct. Differ. Equ. 10 (2003), 95–106.

[6] Briskin, M., Françoise, J.P. and Yomdin, Y. *Center conditions, compositions of polynomials and moments on algebraic curve*, Ergodic Theory Dyn. Syst. 19 (1999), 1201–1220.

[7] Briskin, M., Françoise J.P. and Yomdin, Y. *Center condition II: Parametric and model center problem*, Isr. J. Math. 118 (2000), 61–82.

[8] Briskin, M., Françoise, J.P. and Yomdin, Y. *Center condition III: Parametric and model center problem*, Isr. J. Math. 118 (2000), 83–108.

[9] Briskin, M, Roytvarf, N. and Yomdin, Y. *Center conditions at infinity for Abel differential equations*, Annals of Math. 172 (2010), 437–483.

[10] Brudnyi, A. *An algebraic model for the center problem*, Bull. Sci. Math. 128 (2004), 839–857.

[11] Brudnyi, A. *On center sets of ODEs determined by moments of their coefficients*, Bull. Sci. Math. 130 (2006), 33–48.

[12] Brudnyi, A. and Yomdin, Y. *Tree composition condition and moments vanishing*, Nonlinearity 23 (2010), 1651–1673.

[13] Cima, A., Gasull, A. and Mañosas, F. *Centers for trigonometric Abel equations*, Qual. Theory Dyn. Syst. 11 (2012), 19–37.

[14] Cima, A., Gasull, A. and Mañosas, F. *A simple solution of some composition conjectures for Abel equations*, J. Math. Anal. Appl. 398 (2013), 477–486.

[15] Cherkas, L.A. *Number of limit cycles of an autonomous second-order system*, Differ. Eq. 5 (1976), 666–668.

[16] Christopher, C. *Abel equations: composition conjectures and the model problem*, Bull. London Math. Soc. 32 (2000), 332–338.

[17] Devlin, J., Lloyd, N.G. and Pearson, J.M. *Cubic systems and Abel equations*, J. Differential Equations 147 (1998), 435–454.

[18] Fulton, W. *Algebraic curves: an introduction to algebraic geometry*, The Bejamin/Cummings, 1969.

[19] Giné, J., Grau, M. and Llibre, J. *Universal centers and composition conditions*, to appear in Proc. London Math. Soc. doi:10.1112/plms/pds050

[20] Hartshorne, R. *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[21] Lins Neto, A. *On the number of solutions of the equation* $dx/dt = \sum_{j=0}^{n} a_j(t)x^j$, $0 \le t \le 1$, *for which* $x(0) = x(1)$, Invent. Math. 59 (1980), 67–76.

[22] Pakovich, F. *A counterexample to the "Composition Conjecture"*, Proc. Amer. Math. Soc. 130 (2002), 3747–3749.

[23] Pakovich, F. *On the polynomial moment problem*, Math. Res. Lett. 10 (2003), 401–410.

[24] van der Waerden, B.L. *Modern Algebra, Vol. 1*, 2nd ed. New York: Frederick Ungar, 1966.

[25] Yang, L. and Tang, Y. *Some new results on Abel equations* J. Math. Anal. Appl. 261 (2001), 100–112.

[26] Yomdin, Y. *The center problem for the Abel equation, composition of functions, and moment conditions*, Moscow Math. J. 3 (2003), 1167–1195.