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A simple solution of some composition conjectures for Abel equations

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ABSTRACT

Trigonometric Abel differential equations appear in the study of the number of limit cycles and the center-focus problem for certain families of planar polynomial systems. The composition centers are a class of centers for trigonometric Abel equations which have been widely studied during last years. We characterize this type of centers as the ones given by couples of trigonometric polynomials for which all the generalized moments vanish. They also coincide with the strongly and the highly persistent centers. Our result gives a simple and self-contained proof of the so called *Composition Conjecture for trigonometric* Abel differential equations. We also prove a similar version of this result for Abel equations with polynomial coefficients.

1. Introduction and main results

The study of Abel differential equations of the form

$$\dot{r} = \frac{dr}{d\theta} = A(\theta)\, r^3 + B(\theta)\, r^2 + C(\theta) r,$$

provides a useful tool for knowing either the number of limit cycles of certain planar polynomial differential equations or for studying the center-focus problem for them; see for instance [1–5]. These equations are also interesting in applications; see [6,7].

In this paper we consider Abel differential equations of the form

$$\dot{r} = A(\theta)\, r^3 + B(\theta)\, r^2, \tag{1}$$

defined on the cylinder $(r, \theta) \in \mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$, with A and B being trigonometric polynomials. We focus on the *center-focus* problem, and in particular, on obtaining conditions for A and B to ensure that all the solutions $r = r(\theta, r_0)$, with initial condition $r(0, r_0) = r_0$ and $|r_0|$ small enough, are 2π-periodic. Shortly, if this property holds, we will say that the Abel equation has a *center*. This question is relevant in the context of planar polynomial equations with homogeneous non-linearities, because the center-focus problem for them can be reduced to it; see [8,4].

In this work we give simple proofs of some *composition conjectures*. These are conjectures about the relation between a special type of centers, the ones satisfying the *composition condition*, the cancellation of some *moments* computed from A and B (see [9,10]) and the persistence under certain perturbations of the centers. To be more precise we introduce some definitions.

When there exist $\mathcal{C}^1$-functions A_1, B_1 and u, with u being 2π-periodic, such that

$$\widetilde{A}(\theta) := \int_0^\theta A(\psi)\, d\psi = A_1(u(\theta)) \quad \text{and} \quad \widetilde{B}(\theta) := \int_0^\theta B(\psi)\, d\psi = B_1(u(\theta)), \tag{2}$$

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