Periods of solutions of periodic differential equations

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Abstract

Smooth non-autonomous T-periodic differential equations $x'(t) = f(t, x(t))$ defined in $\mathbb{R} \times \mathbb{K}^n$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and $n \geq 2$ can have periodic solutions with any arbitrary period S. We show that this is not the case when $n = 1$. We prove that in the real $\mathcal{C}^1$-setting the period of a non-constant periodic solution of the scalar differential equation is a divisor of the period of the equation, that is $T/S \in \mathbb{N}$. Moreover, we characterize the structure of the set of the periods of all the periodic solutions of a given equation. We also prove similar results in the one-dimensional holomorphic setting. In this situation the period of any non-constant periodic solution is commensurable with the period of the equation, that is $T/S \in \mathbb{Q}$.

Mathematics Subject Classification 2010: 34C25, 37C27, 37G15.
Keywords: Periodic orbits; periodic differential equations; periodic solutions; holomorphic differential equations.

1 Introduction and main results

Consider a non-autonomous differential equation

$$x'(t) = f(t, x(t)), \tag{1}$$

where f is of class $\mathcal{C}^1$ in $\mathbb{R} \times \mathbb{K}^n$ and $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. It is said that (1) is a *T-periodic differential equation* if it exists some $T > 0$ such that $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{K}^n$ and T is the minimum number with this property. Similarly, a function $\varphi(t), t \in \mathbb{R}$ is said to be *S-periodic* if there exits $S > 0$ such that $\varphi(t + S) = \varphi(t)$, for all $t \in \mathbb{R}$, and S is the minimum number with this property. A solution of (1) which is periodic will be named a *periodic solution.* By convenience we will say that the constant functions have period 0. For simplicity we will use the following notations: when $y \in \mathbb{R}^+$, $y/0 = \infty$ and $y/\infty = 0$. Moreover we denote by $\mathbb{N}$ the set of positive natural numbers.

Given a T-periodic differential equation, we study the relation between T and the periods of its periodic solutions. Our first result is:

Theorem 1. *Consider a T-periodic $\mathcal{C}^1$-differential equation $x' = f(t,x)$, defined on $\mathbb{R}\times\mathbb{K}^n$. Let S be the period of one of its periodic solutions. Then the following holds:*

(i) When $n = 1$ and $\mathbb{K} = \mathbb{R}$ then $T/S \in \mathbb{N} \cup \{\infty\}$. Moreover, for each $k \in \mathbb{N} \cup \{\infty\}$ there is an f and an S-periodic solution of the corresponding differential equation such that $T/S = k$.

(ii) When $n = 1$, $\mathbb{K} = \mathbb{C}$ and $z \to f(t,z)$ is holomorphic then $T/S \in \mathbb{Q}^+ \cup \{\infty\}$. Moreover, for each $q \in \mathbb{Q}^+ \cup \{\infty\}$ there is an f and an S-periodic solution of the corresponding differential equation such that $T/S = q$.

(iii) When $n \geq 2$, and $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$ and moreover $z \to f(t,z)$ is holomorphic, there is an f and an S-periodic solution of the corresponding differential equation such that T/S is any positive real number or infinity.

An example for $n = 2$ and $\mathbb{K} = \mathbb{R}$ for proving (iii) appears for instance in the classical book of Pliss ([12]) and is attributed there to Erugin, 1956. We recall it in the proof of the theorem. It can be easily adapted to give real analytic or holomorphic examples in $\mathbb{R}^n$ or $\mathbb{C}^n, n \geq 2$, respectively. As far as we know, items (i) and (ii) are new. We start studying them looking at Problems 1.531-2-3 of the useful book [2, p. 59]. In fact, the first part of item (ii) was already proved in [8, Prop. 2.1] when f is polynomial in x, that is for generalized Abel equations. To the best of our knowledge, it is the first time that complex 1-dimensional examples with $T/S \in \mathbb{Q}^+$, such that neither $T/S \in \mathbb{N}$ nor $S/T \in \mathbb{N}$, appear in the literature. We also remark that all the examples used to prove several parts of Theorem 1 are given by generalized Abel equations.

When an S-periodic solution of a T-periodic equation (1) is such that $S = mT$ for some $m \in \mathbb{N}$, it is usually said that it is an m-subharmonic, see [3, 7, 11]. Nevertheless, some authors simply call the m-harmonics, see [8, 9]. Notice that m-subharmonic solutions of T-periodic equation correspond to m-periodic points of the return map given by the flow at time T. This no more true when we look for S-periodic solutions with $S/T \notin \mathbb{N}$.

Real T-periodic equations $x' = f(t,x)$ having no uniqueness of solutions can also have m-subharmonic solutions, see [10]. In the papers [1, 11] the authors prove that the existence of just one subharmonic implies the existence of large sets of them.

Corollary 2. *Under the same notations of Theorem 1, if we consider T-periodic n-th order differential equations $x^{(n)} = g(t,x,x',x'',\ldots,x^{(n-1)})$ the same results hold.*

We denote by $\mathbb{P}(f)$ the set of the periods of all the periodic solutions of the T-periodic differential equation (1). For any $\emptyset \neq A \subset \mathbb{R} \cup \{\infty\}$ we denote by $\mathrm{Car}(A) \in \mathbb{N} \cup \{\infty\}$ its cardinal. Next results study $\mathbb{P}(f)$ when $n = 1$ and $K = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Theorem 3. *Consider a T-periodic differential equation $x' = f(t,x)$, defined on $\mathbb{R} \times \mathbb{R}$. Set $A \subset \mathbb{N} \cup \{\infty\}$. The following holds:*

(i) If $\mathrm{Car}(A) < \infty$ there exists a T-periodic real analytic function f such that $\mathbb{P}(f) = \{T/j : j \in A\}$.

(ii) For any set A there exists a real T-periodic $\mathcal{C}^\infty$-function f such that $\mathbb{P}(f) = \{T/j : j \in A\}$.

(iii) If the differential equation is real analytic and T-periodic and $\mathrm{Car}(\mathbb{P}(f)) = \infty$ then the set of all its periodic orbits is unbounded. Moreover, for any set A with $\mathrm{Car}(A) = \infty$, there exists a real analytic T-periodic function f such that $\mathbb{P}(f) \supset \{T/j : j \in A\}$.

The main difficulty for proving items (i) and (ii) of the above results is, once we have constructed an f with the given set of periods, to prove that the corresponding differential equation (1) has no other periods.

An easy way of proving the second part of item (iii) is to construct an f such that $\mathbb{P}(f) = \{T/j \,:\, j \in \mathbb{N} \cup \{\infty\}\}$. With the same approach, and for any given $A \subset \mathbb{N} \cup \{\infty\}$, we can also construct an f such that the equation (1) has periodic solutions with all periods T/j, $j \in A$. Unfortunately, in this construction we do not know how to ensure that other periodic solutions with different periods do not appear. Our construction is based on solving a problem of interpolation in two complex variables that we tackle using classical complex analysis techniques.

Next result proves, in the 1-dimensional holomorphic case, the coexistence of periodic orbits with different periods commensurable with T.

Theorem 4. *Consider a T-periodic holomorphic differential equation $x' = f(t,x)$, defined on $\mathbb{R} \times \mathbb{C}$. Set $A \subset \mathbb{Q}^+ \cup \{\infty\}$. If $\mathrm{Car}(A) < \infty$ then there exists a T-periodic holomorphic function f such that $\mathbb{P}(f) \supset \{T/j : j \in A\}$.*

We want to comment that the differential equations given to prove item (i) of Theorem 3 and Theorem 4 are again generalized Abel equations.

Our approach is also useful to prove results about the finiteness of $\mathbb{P}(f)$, see Theorem 13. In particular, we show that for trigonometrical generalized Abel equations $\mathrm{Car}(\mathbb{P}(f)) < \infty$. Recall that given a trigonometrical polynomial, its degree k is the highest harmonic, $\sin(kt)$ or $\cos(kt)$, of its corresponding Fourier series.

Proposition 5. *Let $x' = f(t,x) = a_m(t)x^m + a_{m-1}(t)x^{m-1} + \cdots + a_1(t)x + a_0(t)$ be a real 2π-periodic differential equation with all $a_j(t)$ trigonometrical polynomials of degree at most k. Then* $\mathrm{Car}(\mathbb{P}(f)) \leq 2k+1$.

In same of the examples presented in this work, for simplicity and without any explicit mention, instead of giving T-periodic differential equations, for any positive real number T, we simply will consider 2π-periodic or 1-periodic differential equations. The extensions of our examples to T-periodic ones is straightforward. In Section 2 we prove Theorem 1. Section 3 is devoted to give some preliminary results about the holomorphic and real analytic cases and to prove Proposition 5. Finally in Section 4, Theorems 3 and 4 are proved. We end the paper with several questions suggested by our work.

2 On the relation between T and S

We start with a well known preliminary result. We include its proof for the sake of completeness.

Proposition 6. *([12]) Let $\dot{x} = f(t,x)$ be a T-periodic $\mathcal{C}^1$-differential equation defined in some open set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$. Let $\varphi(t)$ be a periodic solution of period S, and assume that $T/S \notin \mathbb{Q} \cup \{\infty\}$. If $\gamma := \{\varphi(t)\,,\, t \in \mathbb{R}\}$, then $f(t,x)$ does not depend on t for all $x \in \gamma$.*

Proof. From the periodicity of $\varphi(t)$ we get that $f(t+kS, \varphi(t)) = f(t, \varphi(t))$ for all $t \in \mathbb{R}$ and for all $k \in \mathbb{Z}$. For each fixed $t^* \in \mathbb{R}$ and calling $x^* = \varphi(t^*)$ we consider $t_k \in [0,T]$, the representative of $t^* + kS$ modulus T. Since f is $T-$periodic we get that $f(t^*+kS, x^*) = f(t_k, x^*) = f(t^*, x^*)$ for all $k \in \mathbb{Z}$. On the other hand, the incommensurability of T and S implies that the points $\{t_k : k \in \mathbb{Z}\}$ fill densely the interval $[0,T]$. Then, by the continuity of f we get that $f(t,x^*) = f(t^*,x^*)$ for all $t \in \mathbb{R}$ and $x^* \in \gamma$. ∎

Proof of Theorem 1. (i) Consider an S-periodic solution $\phi(t)$ of equation (1) which is not constant. Let $\varphi(t) := \phi(t+T)$. Then

$$\varphi'(t) = \phi'(t+T) = f(t+T, \phi(t+T)) = f(t, \phi(t+T)) = f(t, \varphi(t)),$$

that is, $\varphi(t)$ is also a solution of (1). Let m (respectively M) be the maximum (resp. minimum) value of $\phi(t)$ on $[0,T]$, and let t_1 (resp. t_2) such that $\phi(t_1) = m$ and $\phi(t_2) = M$. Since $\varphi(t) = \phi(t+T)$ the inequalities

$$\phi(t_1) \leq \varphi(t_1) \text{ and } \phi(t_2) \geq \varphi(t_2)$$

hold. Hence, there exists some t^* between t_1 and t_2 such that $\phi(t^*) = \varphi(t^*)$. From the uniqueness of solutions we get that $\phi(t) = \varphi(t)$, that is $\phi(t+T) = \phi(t)$. Since $\phi(t)$ is an S-periodic function we deduce that $T = kS$ for some $k \in \mathbb{N}$.

The linear 2π-periodic differential equation

$$x' = \big(x - \sin(kt)\big)\sin(t) + k\cos(kt),$$

has the particular $2\pi/k$-periodic solution $x = \sin(kt)$. Hence $T/S = k$, as we wanted to prove. The case $S = 0$ is even easier, consider for instance $x' = x\sin(t)$ which has the solution $x = 0$.

(ii) Let γ be a S-periodic orbit of (1), with T and S incommensurable. Then given any $z^* \in \gamma$, by Proposition 6, $f(t, z^*)$ does not depend on t. Therefore

$$g(t, z^*) := \frac{\partial f(t, z^*)}{\partial t} = 0,$$

for all $t \in \mathbb{R}$ and $z^* \in \gamma$. Then, for each $t^* \in \mathbb{R}$, the holomorphic function $z \to g(t^*, z)$ has the continuum of zeros $z \in \gamma$ and as a consequence $g(t, z) \equiv 0$, fact which is in contradiction with the T-periodicity of f. Thus, S must be either 0 or commensurable with T, as we wanted to prove.

Given $q = m/k \in \mathbb{Q}$ with $(m, k) \in \mathbb{N}^2$ and $\gcd(m, k) = 1$, consider the following holomorphic 2π-periodic generalized Abel differential equation

$$z' = \frac{m}{k} iz + \big(z^k - e^{mit}\big)e^{it}. \tag{2}$$

Clearly, it has the particular solution $z = e^{mit/k}$, which has period $S = 2k\pi/m$. Therefore $T/S = m/k = q$, as we wanted to prove. The case $S = 0$ follows as in item (i).

(iii) We will give only 2-dimensional real or complex examples, because higher dimensional examples can be simply obtained by adding $n - 2$ more trivial equations $x_j' = 0, j = 3, \ldots, n$.

Next example is essentially the real 2-dimensional one appearing in [12, p. 10] due to Erugin. Given $z = x + iy$, consider

$$z' = \alpha iz + (z\overline{z} - 1)\sin(\beta t). \tag{3}$$

Notice that (3) is T-periodic with $T = 2\pi/\beta$ and has the solution $z = e^{i\,\alpha t}$ which is S-periodic with $S = 2\pi/\alpha$. It shows that T/S can be any real number. The construction of an example with $S = 0$ is similar that in items (i) and (ii).

From (3) we can construct an example in $(z, w) \in \mathbb{C}^2$, simply by taking the real variables (x, y) as the complex ones (z, w),

$$\begin{cases} z' = -\alpha w + (z^2 + w^2 - 1)\, \sin(\beta t), \\ w' = \alpha z. \end{cases} \tag{4}$$

A different extension to $\mathbb{C}^2$ of (3) is also

$$\begin{cases} z' = \alpha i z + (zw-1)\sin(\beta t), \\ w' = -\alpha i w. \end{cases}$$

Both cases are T-periodic, with $T = 2\pi/\beta$, and have some $2\pi/\alpha$-periodic solution. In the first case $z = \cos(\alpha t), w = \sin(\alpha t)$ and in the second one $z = e^{i\alpha t}$, $w = e^{-i\alpha t}$.

Notice that, as Proposition 6 shows, the above two periodic orbits with period incommensurable with T lay on the region where the differential equation becomes autonomous.

∎

Proof of Corollary 2. We start with the case $n = 2$. In fact, we only need to show that some of our real or complex examples can be written as a 2nd-order T-periodic differential equation. In the real case, notice that (3) writes as

$$y'' = h(t, y, y') := -\alpha^2 y + \alpha\left(\left(\frac{y'}{\alpha}\right)^2 + y^2 - 1\right)\sin(\beta t),$$

is $2\pi/\beta$-periodic and has the particular solution $y = \sin(\alpha t)$. In the complex case it suffices to consider $y \in \mathbb{C}$.

When $n > 2$ we can simply derive $n-2$ times the above equation, obtaining

$$y^{(n)} = H(t, y, y', \ldots, y^{(n-1)}) := \frac{d^{n-2}}{dt^{n-2}} h(t, y, y').$$

This nth-order differential equation is $2\pi/\beta$-periodic and has the $2\pi/\alpha$-periodic solution, $y = \sin(\alpha t)$. ∎

We end this section with a real $\mathcal{C}^\infty$-example with all their solutions periodic and such that $\mathbb{P}(f)$ contains a closed interval of real numbers.

Consider the $2\pi-$periodic system

$$\begin{cases} \dot{x} = -y\left(g(x^2+y^2) + h(x^2+y^2)\cos(t)\right), \\ \dot{y} = x\left(g(x^2+y^2) + h(x^2+y^2)\cos(t)\right). \end{cases} \tag{5}$$

In polar coordinates it reads as

$$\begin{cases} \dot{r} = 0, \\ \dot{\theta} = g(r^2) + h(r^2)\cos(t), \end{cases}$$

and hence its solutions are

$$r(t) = r_0, \quad \theta(t) = g(r_0^2)(t - t_0) + h(r_0^2)(\sin(t) - \sin(t_0)) + \theta_0.$$

Let h be a $\mathcal{C}^\infty$-function such that $h(r) = 0$ for all $r \in [0,1]$ and positive outside this interval, and let g be a positive $\mathcal{C}^\infty$-function such that $g(r) = 1$ for all $r \geq 1$.

- If $r_0 \le 1$ then the solutions of system (5) are

$$\varphi(t) = \big(r_0 \cos(g(r_0^2)(t-t_0)+\theta_0), r_0 \sin(g(r_0^2)(t-t_0)+\theta_0)\big)$$

  and hence $2\pi/g(r_0^2)$-periodic. Therefore the periods of these solutions take all values between $2\pi/g(0)$ and $2\pi/g(1)$, including, when $g(0) \neq g(1)$, commensurable and incommensurable values with 2π.

- If $r_0 > 1$, then $\theta(t) = t - t_0 + h(r_0^2)(\sin(t) - \sin(t_0)) + \theta_0$ and hence $\theta(t+2\pi) = \theta(t) + 2\pi$. This implies that then the solutions of (5), $\varphi(t) = (r_0 \cos(\theta(t)), r_0 \sin(\theta(t)))$ are all 2π-periodic.

3 Preliminary results for the real analytic and holomorphic cases

We begin proving a result that can be read as an extension to two dimensions of the following well-known result about interpolation of one variable holomorphic functions:

Theorem 7. *([6, Cor 7.27]) Let $\{a_j\} \subset \mathbb{C} \setminus \{0\}$ be such that $|a_j| < |a_{j+1}|$ for $j \in \mathbb{N}$ with $|a_j| \to +\infty$ and let $\{b_j\} \subset \mathbb{C}$ be arbitrary. Then there exists an entire function $f(z)$ such that $f(a_j) = b_j$, $j \in \mathbb{N}$.*

Theorem 8. *For each $j \in \mathbb{N}$, let $a_j, b_j : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ be holomorphic functions satisfying*

$$0 < m_j \le |a_j(w)|_{\mathcal{U}} \le M_j, \quad \textit{with} \quad \sum_j \frac{1}{m_j} < \infty, \quad M_j < m_{j+1}.$$

Then, there exists an holomorphic function $f(w,z)$, $f : \mathcal{U} \times \mathbb{C} \to \mathbb{C}$, such that

$$f(w, a_j(w)) = b_j(w), \quad \textit{for all} \quad w \in \mathcal{U}, j \in \mathbb{N}. \tag{6}$$

Proof. In this proof we will use several times the following well known Weirstrass theorem: let $\mathcal{V} \subset \mathbb{C}^n$, $n \in \mathbb{N}$, an open set and let $f_j : \mathcal{V} \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Then, if for any given compact set $K \subset \mathcal{V}$, f_j converges uniformly to some function f on K, then f is holomorphic on $\mathcal{V}$, see for instance [14, Thm 1.4.20].

First, consider the function $g : \mathcal{U} \times \mathbb{C} \to \mathbb{C}$,

$$g(w,z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j(w)}\right).$$

It is holomorphic on $\mathcal{U} \times \mathbb{C}$ because

$$\sum_{j=1}^{\infty} \left| \frac{z}{a_j(w)} \right| \le \sum_{j=1}^{\infty} \frac{|z|}{m_j},$$

and so the infinite product converges uniformly on compact sets, see again [6, Sec. 7.1&7.2]. Moreover g only vanishes on the sets $z = a_j(w), j \in \mathbb{N}$ and

$$c_j(w) := \frac{\partial g}{\partial z}(w, a_j(w)) \neq 0, \quad w \in \mathcal{U}.$$

Therefore we have that $g(w, a_j(w)) = 0$ on $\mathcal{U}$ for all $j \in \mathbb{N}$. Following the proof of Mittag-Leffer theorem ([6, Thm 7.24]) we will modify g to obtain a new function f satisfying (6).

Consider the holomorphic functions on $\mathcal{V}_j = \mathcal{U} \times \{z : |z| < m_j\}$,

$$S_j(w, z) = \frac{d_j(w)}{z - a_j(w)}, \quad \text{where} \quad d_j(w) = \frac{b_j(w)}{c_j(w)}.$$

Clearly, on each $\mathcal{V}_j$,

$$S_j(w, z) = -\frac{d_j(w)}{a_j(w)} \sum_{k=0}^{\infty} \left(\frac{z}{a_j(w)} \right)^k.$$

Moreover, on any compact set $C_j = K \times \{z : |z| \leq m_j/2\}$, where $K \subset \mathcal{U}$ is also a compact set, the above convergence is uniform. Define $P_\ell(x) = -\sum_{k=0}^{\ell} x^k$, and consider $\{\varepsilon_j\} \subset \mathbb{R}^+$ such that $\sum_{j=1}^{\infty} \varepsilon_j < \infty$. Then, for each $j \in \mathbb{N}$, there exists $\ell_j \in \mathbb{N}$ such that

$$\left| S_j(w, z) - \frac{d_j(w)}{a_j(w)} P_{\ell_j} \left(\frac{z}{a_j(w)} \right) \right| < \varepsilon_j. \tag{7}$$

Let us prove that the function

$$S(w, z) = \sum_{j=1}^{\infty} \left(S_j(w, z) - \frac{d_j(w)}{a_j(w)} P_{\ell_j} \left(\frac{z}{a_j(w)} \right) \right)$$

is meromorphic on $\mathcal{U} \times \mathbb{C}$. Fix any open set $\mathcal{W} \subset \mathcal{U} \times \mathbb{C}$, with $\overline{\mathcal{W}} = C$ compact. Therefore there exists $N_C \in \mathbb{N}$ such that all points $(w, a_j(w)/2)$ are outside C for $j > N_C$. Hence, write

$$S(w, z) = \sum_{j=1}^{N_C} \left(S_j(w, z) - \frac{d_j(w)}{a_j(w)} P_{\ell_j} \left(\frac{z}{a_j(w)} \right) \right) + \sum_{j=N_C+1}^{\infty} \left(S_j(w, z) - \frac{d_j(w)}{a_j(w)} P_{\ell_j} \left(\frac{z}{a_j(w)} \right) \right).$$

The first summand is a finite sum, while the second one converges uniformly on C because of (7). Hence $S(w, z) = S_1^{N_C}(w, z) + S_2^{N_C}(w, z)$, with $S_1^{N_c}$ meromorphic and $S_2^{N_C}$ holomorphic on $\mathcal{W}$. Since $\mathcal{W}$ is arbitrary we have proved that S is holomorphic on $\mathcal{U} \times \mathbb{C} \setminus \cup_{j=1}^{\infty} \{(w, z) : z = a_j(w)\}$.

Finally, consider $f(w, z) = g(w, z)\, S(w, z)$. This function is holomorphic on $\mathcal{U} \times \mathbb{C}$ and moreover

$$\begin{aligned} f(w, a_j(w)) &= \lim_{z \to a_j(w)} f(w, z) = \lim_{z \to a_j(w)} \frac{g(w, z) - g(w, a_j(w))}{z - a_j(w)} S(w, z)(z - a_j(w)) \\ &= c_j(w) \frac{b_j(w)}{c_j(w)} = b_j(w), \end{aligned}$$

as we wanted to prove. ∎

We need the two following technical lemmas.

Lemma 9. *Set* $C_\delta = \{w \in \mathbb{C} : |\operatorname{Im}(w)| \le \delta\}$ *and* $x_j(w) = q_j + \sin(jt)$, *for* $q_j \in \mathbb{R}^+$. *Then, for* $w \in \mathbb{C}_\delta$,

$$q_j - \cosh(j\delta) \le |x_j(w)| \le q_j + \cosh(j\delta).$$

Proof. It suffices to prove that $|\sin(jw)|_{C_\delta} \le \cosh(j\delta)$. It holds because on C_δ,

$$|\sin(jw)| = \left|\frac{e^{jwi} - e^{-jwi}}{2i}\right| \le \frac{e^{j\operatorname{Im}(w)} + e^{-j\operatorname{Im}(w)}}{2} = \cosh(j\operatorname{Im}(w)) \le \cosh(j\delta).$$

∎

Lemma 10. *Given* $\delta > 0$, *small enough, there exists a sequence of positive real numbers* $\{q_j\}$ *such that, for all* $w \in \mathbb{C}_\delta$ *and for all* $j \in \mathbb{N}$, *the functions*

$$a_j(w) = q_j + \sin(jw)$$

satisfy $0 < m_j < |a_j(w)| < M_j$, *for some sequences* $\{m_j\}$ *and* $\{M_j\}$ *such that* $M_j < m_{j+1}$ *and* $\sum_{j=1}^{\infty} 1/m_j < \infty$.

Proof. It suffices to take in Lemma 9, $q_j = 2 + j^2 + \sum_{k=1}^{j} \cosh(k\delta)$. ∎

Next result gives an example of autonomous differential equation with continua of periodic orbits with any given finite set of periods. It will be useful to construct the examples for proving Theorem 4.

Proposition 11. *Let* $S_1, S_2, \ldots, S_m$ *be* m *positive real numbers. Then, there exists a real polynomial* p *such that, for all* $1 \le k \le m$, *the holomorphic differential equation* $z' = ip(z)$ *has continua of periodic solutions of period* S_k.

Proof. Given a holomorphic differential equation $z' = f(z)$, it is well-known that if $z_0 \in \mathbb{C}$ is such that $f(z_0) = 0$ and $f'(z_0)$ is a non-zero purely imaginary number, then $z = z_0$ is an isochronous center for its associated planar system, see for instance [5]. In other words, $z = z_0$ is surrounded by periodic solutions of period $2\pi|i/f'(z_0)|$. Hence, to construct the polynomial p of the statement, consider the following Hermite interpolation problem:

$$g(x_j) = 0, \quad g'(x_j) = 2\pi/S_j, \quad j = 1, 2, \ldots, m,$$

where $x_1 < x_2 < \cdots < x_m$ are given real numbers. Then, taking p as the $2m-1$ degree Hermite interpolation polynomial of the above set of conditions, we get that the singularities $z = x_j \in \mathbb{R}$ of $z' = ip(z)$ are surrounded by periodic orbits with respective periods S_j, as we desired to prove. ∎

Finally we recall the local description of planar real analytic sets given by Sullivan in 1971, see [4, 13]. We use it to study the finiteness of $\mathbb{P}(f)$ in the 1-dimensional real analytic case. Recall that given any positive natural number $k \in \mathbb{N}$ it is said that a topological space is a k-star if it is homeomorphic to $\{z \in \mathbb{C} : z^k \in [0,1)\}$. The image of the origin under the corresponding homeomorphism is called a vertex of the star. Note that the vertex of a star is uniquely defined except in the case $k = 2$.

Theorem 12. *([4, Thm. 3.1]) Let $\mathcal{U} \subset \mathbb{R}^2$ be open and connected and $f : \mathcal{U} \to \mathbb{R}$ be a real analytic function. Let $C = \{z \in \mathcal{U} : f(z) = 0\}$ be the set of zeros of f. Then either $C = \mathcal{U}$ or given any non-isolated point of c, $z \in \mathbb{C}$, there exists a neighborhood $\mathcal{V}$ of z and a $k \in \mathbb{N}$ such that $\mathcal{V} \cap C$ is a $2k$-star with vertex z.*

Theorem 13. *Let f be a real analytic T-periodic map and consider the differential equation* (1)*. The following holds:*

(i) If $\sup_{\{x\in\mathbb{R}:\, f(t,x)\not\equiv 0\}} \mathrm{Car}(\{t \in [0,T) : f(t,x) = 0\}) \le K$, *then* $\mathrm{Car}(\mathbb{P}(f)) \le K + 1$.

(ii) If $\mathrm{Car}(\mathbb{P}(f)) = \infty$ *then the set of all periodic orbits is unbounded.*

Proof. We will use the following simple observation. Let $x = \varphi(t)$ a T/m-periodic solution of (1). Then, by periodicity it is clear the following property:
Property I : The function $x = \varphi(t)$ has at least m equidistantly distributed maxima on $[0,T]$, say $t_1, t_2, \ldots t_m$. Moreover, again by periodicity, if $\widehat{x} := \varphi(t_1) = \cdots = \varphi(t_m)$, it holds that

$$0 = \varphi'(t_j) = f(t_j, \widehat{x}), \quad j = 1, 2, \ldots, m.$$

Item (i) follows because by Theorem 3 the period of any solution are either 0 or T/j for some $j \in \mathbb{N}$. The first situation happens for all x^* such that $f(t, x^*) \equiv 0$ because $x = x^*$ is a constant periodic solution. Otherwise, by Property I, the minimum period is T/K. Hence at most $K + 1$ different periods can be achieved by the periodic solutions of differential equation.

To prove (ii), assume to arrive to a contradiction, that (1) has infinitely many bounded solutions, with infinitely many different periods. Call them $x = \varphi_k(t), k = 1, 2, \ldots$ and call their respective periods T/j_k, with all j_k different and tending to ∞. First notice that taking a subsequence, if necessary, we can assume that $\lim_{k\to\infty} \varphi_k(0) = x^*$. By continuity, if $x = \varphi^*(t)$ is the solution (1) such that $\varphi^*(0) = x^*$, then, a priori, either $x = \varphi^*(t)$ is a periodic solution or it is only defined on $[0, t_0) \subset [0, 1)$ and $\lim_{t\to t_0} |\varphi^*(t)| = \infty$. Let us see that, in fact, $\varphi^*(t) \equiv x^*$, that is, it is a constant periodic solution of (1). Assume that it is not. Then, in both situations, there exists $t^* \in [0, t_0)$ such that $\varphi'(t^*) = f(t^*, x^*) \ne 0$. Hence, in a small neighborhood of (t^*, x^*) the function f does not vanish. This is in contradiction with Property I, because $\{\varphi_k\}$ tends uniformly to φ^*, and on the graph of each of the functions there are at least j_k equidistantly distributed zeroes of $f(t,x)$. Therefore $\varphi^*(t) \equiv x^*$.

Finally, let us prove that it is also impossible that $\{\varphi_k\}$ converges uniformly to x^*. Assume that this holds, and take one point on $y = x^*$, (t^*, x^*). It is clear that $f(t^*, x^*) = 0$. Since f is a real analytic function, by Theorem 12 we know that in a neighborhood of this point the graph of f is a $2k$-star, with two of its edges contained in the line $y = x^*$. Choosing a new point in this neighborhood, $(\widehat{t}, x^*)$, $\widehat{t} \neq t^*$, on one of these edges, there is a smaller neighborhood of this point where the set $f(t,x) = 0$ coincides with a piece of the line $y = x^*$. Hence, arguing similarly that in the previous paragraph, we get that this fact is again in contradiction with Property I. Hence all the periodic orbits form an unbounded set, as we wanted to prove. ∎

Proof of Proposition 5. Fixed $x = x^*$, the equation

$$H_{x^*}(t) := f(t, x^*) = a_m(t)(x^*)^m + a_{m-1}(t)(x^*)^{m-1} + \cdots + a_1(t)x^* + a_0(t) = 0$$

is either identically zero or a trigonometrical polynomial of degree k. In the second case $H_{x^*}(t)$ is a trigonometrical polynomial of degree at most k. It is well known that non-trivial trigonometrical polynomial of degree d have at most $2d$ solutions in $[0, 2\pi)$. Therefore, the value K in item (i) of Theorem 13 is $K = 2k$ and as a consequence $\mathrm{Car}(\mathbb{P}(f)) \leq K + 1 = 2k + 1$. ∎

4 Proof of Theorems 3 and 4

Proof of item (i) of Theorem 3. Set $A = \{k_1, k_2, \ldots, k_m\}$ and suppose first that $A \subset \mathbb{N}$. Consider the m periodic functions $x_{i,\varepsilon}(t) = i + \varepsilon \sin(k_i t)$ of periods $2\pi/k_1, 2\pi/k_2, \ldots, 2\pi/k_m$. We take $|\varepsilon| \leq 1/3$ to assure that $x_{i,\varepsilon}(t) \neq x_{j,\varepsilon}(t)$ for all $t \in \mathbb{R}$ and all $i \neq j \in \{1, 2, \ldots, m\}$.

We want to construct a generalized Abel T-periodic equation having only the above functions as periodic solutions. More concretely, we look for a function

$$f(t, x, \varepsilon) = x^m + \sum_{j=0}^{m-1} b_j(t, \varepsilon) x^j,$$

such that for $i = 1, \ldots m$, each $x_{i,\varepsilon}$ is a solution of the equation $x' = f(t, x, \varepsilon)$. To this end we will use one adaption of the method of divided differences of Newton :

For $i = 1, \ldots, m$ set $\Delta_i(t, \varepsilon) = x'_{i,\varepsilon}(t) - x^m_{i,\varepsilon}(t)$ and for $j > 0$ such that $i + j \leq m$ define recursively

$$\Delta_{i,i+1,\ldots,i+j}(t, \varepsilon) = \frac{\Delta_{i+1,i+2,\ldots,i+)}(t, \varepsilon) - \Delta_{i,i+1,\ldots,i+j-1}(t, \varepsilon)}{x_{i+j,\varepsilon}(t) - x_{i,\varepsilon}(t)}.$$

In addition, for $j = 1, \ldots, m-1$, set

$$g_j(t,x,\varepsilon) = \prod_{i=1}^{j}(x - x_{i,\varepsilon}(t)) \quad \text{and} \quad g_0(t,x) \equiv 1.$$

Lastly consider

$$f(t,x,\varepsilon) = x^m + \sum_{j=0}^{m-1} a_j(t,\varepsilon)\, g_j(t,x,\varepsilon), \tag{8}$$

where $a_0(t,\varepsilon) = \Delta_1(t,\varepsilon)$, $a_1(t,\varepsilon) = \Delta_{1,2}(t,\varepsilon), \ldots, a_{m-1}(t,\varepsilon) = \Delta_{1,2\ldots,m}(t,\varepsilon)$. Then, equation (8) can we written as

$$\begin{aligned} f(t,x,\varepsilon) \quad &= x^n + \Delta_1(t,\varepsilon) + \Delta_{1,2}(t,\varepsilon)(x-x_1) + \Delta_{1,2,3}(t,\varepsilon)(x-x_1)(x-x_2) + \cdots + \\ &+ \Delta_{1,2,\cdots,n}(t,\varepsilon)(x-x_1)(x-x_2)\cdots(x-x_{m-1}), \end{aligned}$$

where $x_i = x_{i,\varepsilon}(t)$.

A calculation gives that for all $\varepsilon < 1/3$ the equation $x' = f(t,x,\varepsilon)$ has the solutions x_i for all $i \in \{1,2,\ldots,m\}$. We claim that for ε small enough there are no more periodic solutions. To control the periodic orbits we consider the Poincaré return map. Let $\varphi(t,x,\varepsilon)$ be the solution of the equation $x' = f(t,x,\varepsilon)$ which satisfies $\varphi(0,x,\varepsilon) = x$, and for each ε let I_ϵ be the open interval on which the return map

$$\phi_\varepsilon(x) = \varphi(2\pi, x, \varepsilon)$$

is well defined. Thus for each ε the periodic orbits of the equation $x' = f(t,x,\varepsilon)$ are determined by the fixed points of the map $\phi_\varepsilon : I_\epsilon \longrightarrow \mathbb{R}$.

Notice, that for $\varepsilon = 0$, $f(t,x,0) = (x-1)(x-2)\cdots(x-m)$, hence we deal with a differential equation that can be written as

$$x' = f(t,x,\varepsilon) = (x-1)(x-2)\cdots(x-m) + \varepsilon\, g(t,x,\varepsilon), \tag{9}$$

for a certain function $g(t,x,\varepsilon)$ that is polynomial on x of degree $m-1$ with periodic coefficients on t. This implies that there exists $M > 0$ such that if $|x| > M$ then $|f(t,x,\varepsilon)| > 0$ for all $t \in \mathbb{R}$ and for all $\varepsilon \in (-1/3, 1/3)$. Thus, for these values of ε, the fixed points of the return map are contained in the interval $[-M, M]$. To arrive a contradiction assume that for all $\epsilon \in (-1/3, 1/3)$ the map ϕ_ε has more than m fixed points. Since $\phi_\varepsilon(i) = i$ for $i = 1, \ldots, m$ and for all $\varepsilon \in (-1/3, 1/3)$ it follows that there exists $y_\epsilon \in I_\epsilon \setminus \{1, \ldots, m\}$ which is a fixed point of ϕ_ε. Since for all ε the fixed points of ϕ_ε belong to $[-M, M]$ it follows that the sequence y_ε must accumulate in some fixed point of ϕ_0. This gives a contradiction because the set of fixed points of ϕ_0 is exactly $\{1, \ldots, m\}$ and a simple computation shows that all of them are hyperbolic (simple). Thus the claim is proved.

We notice that equation $x' = f(t,x,\varepsilon)$ is a periodic equation of period T/k where k is the greatest common divisor between $k_1, k_2, \ldots, k_n$ and $T = 2\pi$. Since we want a differential equation of period T we consider

$$x' = F(t,x,\mu) := f(t,x,\bar{\varepsilon}) + \mu\, \cos(t), \tag{10}$$

where $\bar{\varepsilon}$ is such that equation $x' = f(t,x,\bar{\varepsilon})$ has only the n periodic solutions $x_{1,\bar{\varepsilon}}(t), x_{2,\bar{\varepsilon}}(t), \ldots, x_{m,\bar{\varepsilon}}(t)$, and the fixed points $1, \ldots, m$ of $\phi_{\bar{\varepsilon}}$ are all hyperbolic. Thus as before we can see that for μ small enough the equation (10) has exactly m periodic orbits. The fact that each of these orbits has the prescribed period follows using the same arguments, but considering the return maps given by $\pi_{\mu,i}(x) = \xi(2\pi/k_i, x, \mu)$, where $\xi(t,x,\mu)$ is the solution of (10) with $\xi(0,x,\mu) = x$, instead of ϕ_ε. This ends the proof of item (ii) in this case. When some $k_i = \infty$ the same procedure works by considering $x_{i,\varepsilon}(t) = i$. ∎

Proof of item (ii) of Theorem 3. We start by taking the function given by the Blaschke product,

$$f(x) = \prod_{m \geq 1} \left(1 - \frac{x}{m}\right) e^{x/m}. \tag{11}$$

It is known that f is an entire function and $f(x) = 0$ if and only if x is a positive natural number. Moreover all of these zeroes are simple. We will do the proof in three steps.

STEP 1. We prove that for the equation

$$x' = g(x,t,\varepsilon) = f(x) + \varepsilon\, s_m(x) \left(k\, \cos(kt) - \frac{f(x)}{x-m}\, \sin(kt)\right),\ \varepsilon \in [-1/4, 1/4]\ ,\ m \in \mathbb{N} \tag{12}$$

where $s_m(x)$ is a $\mathcal{C}^\infty$-function such that

$$s_m(x) = \begin{cases} 1 & if \quad |x-m| < 1/4, \\ 0 & if \quad |x-m| > 1/2, \end{cases} \tag{13}$$

it exists $0 \neq \varepsilon_m \in [-1/4, 1/4]$, such that (12) with $\varepsilon = \varepsilon_m$ has only the periodic solutions $x_m(t) = m + \epsilon_m\, \sin(kt)$ and $x(t) = k$ for $k \in \mathbb{N} \setminus \{m\}$.

A simple calculation proves that $x_{m,\varepsilon}(t) = m + \epsilon\, \sin(kt)$ is a $2\pi/k$-periodic solution of (12), for all $\epsilon \in [-1/4, 1/4]$. Moreover, since for $x \notin (m-1/2, m+1/2)$, $g(t,x,\varepsilon) \equiv f(x)$ it follows that $x(t) = k \in N \setminus \{m\}$ is a solution of (12). Moreover they are the only periodic solutions with initial condition $x \notin (m-1/2, m+1/2)$.

Let $\varphi(t,x,\varepsilon)$ be the solution of (12) with $\varphi(0,x,\varepsilon) = x$ and consider the Poincaré return map

$$\phi_\varepsilon(x) = \varphi(2\pi, x, \varepsilon) \quad \text{with} \quad (x,\varepsilon) \in [m-1/2, m+1/2] \times [-1/4, 1/4]\,.$$

Since for $\varepsilon = 0$, $x' = f(x)$ and $f(m) = 0$ we have that $x = m$ is the unique fixed point of ϕ_0. Now since it is also a hyperbolic fixed point it follows that for ε small enough ϕ_ε has also a unique fixed point. So we can choose ε_m small enough with the required property.

STEP 2. Consider $A = \{k_1, k_2, k_3, \ldots\}$ and suppose first that $A \subset \mathbb{N}$. Consider the non-autonomous differential equation

$$x' = f(x) + \sum_{m\geq 1} m\, s_m(x) \left(k_m \cos(k_m\, t) - \frac{f(x)}{x-m} \sin(k_m\, t) \right), \tag{14}$$

where $f(x)$ and $s_m(x)$ are defined in (11) and (13) respectively and ε_m is selected such that equation (12) with $\varepsilon = \varepsilon_m$ only has the periodic solution $x_m(t) = m + \varepsilon_m \sin k_m t$ in $[m-1/2, m+1/2]$. Then, equation (14) only has the periodic solutions $x_m(t) = m + \varepsilon_m \sin k_m t$, $m \in \mathbb{N}$.

It is so because from step 1 we know that in each strip $|x - m| < 1/2$ only the periodic solution $m + \varepsilon_m \sin k_m t$ exists and for $x < 1/2$, equation (14) reads as $x' = f(x)$. Hence, if $\varphi(t, x, \varepsilon) = 1/2$ for some $t > 0$, since $f(1/2) > 0$, $\varphi(t, x, \varepsilon) > 1/2$ for all $t > 0$, and the solution can not be periodic. If $\varphi(t, x, \varepsilon) < 1/2$ for all $t > 0$, then since autonomous equations in dimension 1 have no periodic solutions other than constants, $\varphi(t, x, \varepsilon)$ is not periodic.

In the case that $\infty \in A$ it suffices to consider a new summand in (14), corresponding to $x_0(t) = 0 + \varepsilon_0 \sin(0t) = 0$.

Notice that in both cases the differential equation (14) is $2\pi/\gcd(k_1, k_2, \ldots)$ periodic.

STEP 3. In order to get a differential equation of period 2π we consider in the case when $A \subset \mathbb{N}$,

$$x' = f(x) + \sum_{m\geq 1} \varepsilon_m\, s_m(x) \left(k_m \cos(k_m\, t) - \frac{f(x)}{x-m} \sin(k_m\, t) \right) + \mu\, s_0(x) \cos(t). \tag{15}$$

Then in each strip $|x - m| < 1/2$, equation (15) only has the prescribed periodic solutions $m + \varepsilon_m \sin k_m t$.

Since for $\mu = 0$ equation (15) has no periodic solutions in the strip $|x| < 1/2$, the same is true for μ small enough.

Finally, for $x < -1/2$ the solution $\varphi(t, x, \mu)$ of equation (15) is increasing in t for $t > 0$ small, because $x' = f(x)$ and $f(-1/2) > 0$. This implies that $\varphi(t, x, \mu) > -1/2$ and the solution can not be periodic.

In the case that $\infty \in A$ the proof follows similarly. ∎

Proof of item (iii) of Theorem 3. The first part is proved in item (ii) of Theorem 13. To simplify the proof of the second result we will consider first the case $T = 2\pi$ and $A = \mathbb{N}$.

Some comments about how to modify some steps to prove the result for other A and for including the period zero case will be given at the end of the proof.

Consider the f constructed in Theorem 8 associated to the following data:

- $\mathcal{U} = \mathrm{Int}(C_\delta) = \{w \in \mathbb{C} \,:\, |\operatorname{Im}(w)| < \delta\}$,
- The functions $a_j(w)$ appearing in Lemma 10,
- The functions $b_j(w) = a_j'(w)$.

Then, it holds that $a_j'(w) = f(w, a_j(w))$ on $\mathcal{U} \times \mathbb{C}$. Therefore, the same happens with the real analytic $2\pi/j$-periodic functions $a_j(t) = p_j + \sin(jt)$, that is $a_j'(t) = f(t, a_j(t))$. Notice that by construction, f is 2π-periodic and real restricted to $w = t + i\,t_2 \in \mathcal{U} \cap \mathbb{R}$ and $z = x + i\,y \in \mathbb{R}$.

To include the period zero case it suffices to take one more a_j, as $a_0(w) = 1$ and follow the same steps. When A is an arbitrary set it may happen that the greatest common divisor of all the elements of A, $\gcd(A) \neq 1$, then the constructed differential equation has period $2\pi/\gcd(A)$ instead 2π. To force f to be 2π-periodic we can add to the list of function a_j and b_j a new one, say $a_0(w) = 1 + \sin(w)$ but with $b_0(w) = 0$ instead of $a_0'(w)$. In this way $a_0(w)$ is not a solution of the differential equation, but f becomes 2π-periodic. ∎

Proof of Theorem 4. Consider first the case when A is a finite subset of $\mathbb{Q}^+$ and let $k_j/m_j, j = 1, 2, \ldots, \ell$, with $\gcd(m_j, k_j) = 1$, be its elements. The case when $\infty \in A$ will be treated at the end.

By using Proposition 11 we consider a real polynomial p such that $z' = ip(z)$ has ℓ periodic solutions $z_j(t)$ with respective periods $S_j = m_j/k_j$, $j = 1, 2, \ldots, \ell$.

Associated to each of these solutions, consider the functions

$$P_j(t,z) := \prod_{l=0}^{m_j-1} \Big(z - z_j\Big(t + \frac{l}{k_j}\Big)\Big).$$

Observe that

$$P_j\Big(t + \frac{1}{k_j}, z\Big) = P_j(t,z),$$

because $z_j(t + m_j/k_j) = z_j(t)$. Moreover, each function $z = z_j\Big(t + \frac{l}{k_j}\Big), l = 1, 2, \ldots, m_j - 1$, is also a S_j-periodic solution of $z' = ip(z)$. Finally consider the holomorphic differential equation

$$z' = ip(z) + e^{2\pi i t} \prod_{j=1}^{\ell} P_j(t,z). \tag{16}$$

This equation is 1-periodic, so $T = 1$ and by construction, for each j, has at least m_j periodic solutions $z = z_j\left(t + \frac{l}{k_j}\right), l = 0, 1, 2, \ldots, m_j - 1$, of period S_j. Hence, for each j, $T/S_j = k_j/m_j \in \mathbb{P}(f)$ as we wanted to prove.

If $\infty \in A$, write $A = \{k_j/m_j, j = 1, 2, \ldots, \ell\} \cup \{\infty\}$. Then we can modify (16) to be

$$z' = ip(z) + e^{2\pi it}(z - x_0) \prod_{j=1}^{\ell} P_j(t, z),$$

where $x_0 \in \mathbb{R}$ is one of the zeros of p. Clearly, $z = x_0$ is a constant periodic solution of this differential equation and $0 \in \mathbb{P}(f)$. Hence $\{T/j : j \in A\} \subset \mathbb{P}(f)$ and the theorem follows. ∎

Remark 14. Notice that in the proofs of item (iii) of Theorem 3 and the proof of Theorem 4 we can not ensure that the set of periods of all the periodic orbits is exactly $\{T/j : j \in A\}$. In the first case, an attempt for proving this fact consists in considering instead of the functions $a_j(w) = q_j + \sin(jw)$, the new ones $a_j(w, \varepsilon) = q_j + \varepsilon \sin(jw)$, with ε small enough, but we do not know how to control possible periodic orbits (and their corresponding periods) bifurcating from infinity when $\varepsilon \approx 0$.

Open questions.

We end this paper with several questions suggested by our results.

(I) For any set $A \subset \mathbb{N} \cup \{\infty\}$, with $\mathrm{Car}(A) = \infty$, improve item (iii) of Theorem 3 giving a real analytic map f such that $\mathbb{P}(f)$ is exactly $\{T/j : j \in A\}$.

(II) Improve Theorem 4 giving a holomorphic map f such that $\mathbb{P}(f)$ is exactly $\{T/j : j \in A\}$. Extend also this result to sets A with $\mathrm{Car}(A) = \infty$.

(III) In the $\mathcal{C}^\infty$-setting, the real 2-dimensional example (5) shows that $\mathbb{P}(f)$ can contain closed intervals. Is this possible in the real analytic or holomorphic settings? More in general: Which is the structure of $\mathbb{P}(f)$ when the differential equation in real analytic or holomorphic and the dimension $n \geq 2$? For instance, in the paper [3] the authors give real analytic planar differential equations ($n = 2$) for which there exists $j_0 \in \mathbb{N}$ such that $\{jT : j \geq j_0\} \subset \mathbb{P}(f)$. These examples are given by forced pendulum equations and the proof of the existence of all the j-subharmonics is based on the the Poincaré-Birkhoff fixed point theorem.

Acknowledgements.

The authors are supported by Ministry of Economy and Competitiveness of the Spanish Government through grants MTM2013-40998-P (first and second authors) and MTM2014-52209-C2-1-P (third author). They are also supported by the grant 2014-SGR-568 (first and second authors) and 2014-SGR-859 (third author) from AGAUR, Generalitat de Catalunya.

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