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PERIODS OF SOLUTIONS OF PERIODIC DIFFERENTIAL EQUATIONS

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Abstract. Smooth non-autonomous T-periodic differential equations $x'(t) = f(t, x(t))$ defined in $\mathbb{R} \times \mathbb{K}^n$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and $n \geq 2$ can have periodic solutions with any arbitrary period S. We show that this is not the case when $n = 1$. We prove that in the real $\mathcal{C}^1$-setting the period of a non-constant periodic solution of the scalar differential equation is a divisor of the period of the equation, that is $T/S \in \mathbb{N}$. Moreover, we characterize the structure of the set of the periods of all the periodic solutions of a given equation. We also prove similar results in the one-dimensional holomorphic setting. In this situation the period of any non-constant periodic solution is commensurable with the period of the equation, that is $T/S \in \mathbb{Q}$.

1. Introduction and main results

Consider a non-autonomous differential equation

$$x'(t) = f(t, x(t)), \tag{1.1}$$

where f is of class $\mathcal{C}^1$ in $\mathbb{R} \times \mathbb{K}^n$ and $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. It is said that (1.1) is a T-*periodic differential equation* if it exists some $T > 0$ such that $f(t+T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{K}^n$ and T is the minimum number with this property. Similarly, a function $\varphi(t), t \in \mathbb{R}$ is said to be S-*periodic* if there exits $S > 0$ such that $\varphi(t + S) = \varphi(t)$, for all $t \in \mathbb{R}$, and S is the minimum number with this property. A solution of (1.1), which is periodic will be named a *periodic solution*. By convenience we will say that the constant functions have period 0. For simplicity, we will use the following notations: when $y \in \mathbb{R}^+$, $y/0 = \infty$ and $y/\infty = 0$. Moreover, we denote by $\mathbb{N}$ the set of positive natural numbers.

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