Bifurcation of 2-periodic orbits from non-hyperbolic fixed points *

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Abstract

We introduce the concept of 2-cyclicity for families of one-dimensional maps with a non-hyperbolic fixed point by analogy to the cyclicity for families of planar vector fields with a weak focus. This new concept is useful in order to study the number of 2-periodic orbits that can bifurcate from the fixed point. As an application we study the 2-cyclicity of some natural families of polynomial maps.

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1 Introduction

The cyclicity of a family of vector fields having a weak focus or a center is a well known concept in the theory of planar vector fields and the problems surrounding the second part of the Hilbert's 16th problem [11, 16]. A grosso modo the cyclicity expresses the maximum number of small amplitude limit cycles that can effectively bifurcate from the singular point by varying the parameters in the family of considered vector fields.

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This cyclicity is given by the number of fixed points near the critical point of a family of orientation preserving maps (the so called return maps) with a non-hyperbolic fixed point. As we will see, the cyclicity also can be seen as the number of 2-periodic orbits of a related family of orientation reversing maps (the half-return maps), see for instance [4] or Section 4. Recall that given a map $f : \mathbb{R} \to \mathbb{R}$, a 2-periodic orbit is a set $\{x, y\}$ such that $f(x) = y, f(y) = x$ and $x \neq y$.

Hence it is natural, in the discrete setting, to study the bifurcation of 2-periodic orbits from non-hyperbolic fixed points of orientation reversing one-dimensional analytic diffeomorphisms of the form

$$f(x) = f_a(x) = -x + \sum_{j\geq 2} a_j x^j. \tag{1}$$

This will be the main goal of this paper.

To fix the problem we start introducing the concept of 2-*cyclicity* of a family of maps of the form (1), by analogy with the concept of cyclicity for planar vector fields. Here, given $x \in \mathbb{R}^m$ and $\rho \in \mathbb{R}^+$, $D_\rho(x) := \{y \in \mathbb{R}^m \,:\, ||y - x|| < \rho\}$.

Definition 1. *Set $a = (a_1, \ldots, a_n)$ varying in an open set of $\mathcal{V} \subseteq \mathbb{R}^n$, and consider the family of analytic reversing orientation maps from $\mathbb{R}$ into itself,*

$$f_a(x) = -x + \sum_{i\geq 2} c_i(a) x^i. \tag{2}$$

We will say that the origin of a map f_{a^*}, with $a^* \in \mathcal{V}$, has 2-cyclicity $N \in \mathbb{N} \cup \{0\}$ *if:*

(i) it is possible to find $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that the maximum of isolated 2-periodic orbits within $D_{\delta_0}(0) \subset \mathbb{R}$ for every map (2) with $a \in D_{\varepsilon_0}(a^*) \subset \mathcal{V}$ is N.

(ii) for any $\varepsilon > 0$ and any $\delta > 0$ there exists $a \in D_\varepsilon(a^*) \subset \mathcal{V}$ such that f_a has N different isolated 2-periodic orbits within $D_\delta(0)$.

A family of maps f_a, with $a \in \mathcal{V} \subseteq \mathbb{R}^n$, has 2-cyclicity N at the origin if N is the maximum 2-cyclicity achieved by a map in the family.

We remark that it has no sense to study the 2-cyclicity for locally orientation preserving diffeomorphisms because it is always 0, see Remark 9.

In the following, for the sake of simplicity, we will simply say *cyclicity* to refer to 2-*cyclicity* of a map or a family of maps at the origin.

We also remark that the cyclicity of a family (2) does not depend only on the number n of parameters but on their role, that is, on how the parameters $a \in \mathbb{R}^n$ appear in the expressions of the coefficients $c_i(a)$ of the map (2). As an example, we will show in Section 4 that there exist one-parametric families ($n = 1$) of maps with arbitrary large cyclicity.

In the recent paper [3], we have introduced what we call *stability constants* to study the stability of the origin of one-dimensional maps of the form (2) and also of periodic discrete dynamical systems with a common fixed point. A summary of results on this issue can also be found in [6]. The analysis of these constants plays also an important role in the study of the cyclicity, as the proof of our main result of this paper evidences. Let us recall them.

To know the local stability of the origin of an analytic map of the form (1), we consider

$$f \circ f(x) := f(f(x)) = x + \sum_{j \geq 3} W_j(a_2, \ldots, a_j) x^j. \tag{3}$$

If f is not an involution (i.e. $f \circ f \neq \mathrm{Id}$), we define a *stability constant of order* k (with $k \geq 3$) as

$$\begin{aligned} &V_3 = V_3(a) = W_3(a_2, a_3) \ \text{ if } \ W_3 \neq 0, \ \text{ or} \\ &V_k = V_k(a) = W_k(a_2, \ldots, a_k) \ \text{ if } \ W_j = 0, \ j = 3, \ldots, k-1. \end{aligned} \tag{4}$$

Notice that the stability constant V_k only has sense when all the previous $W_j, j < k$ vanish. Hence, any expression of the form $W_k + U$, where U belongs to the ideal generated by $W_3, W_4, \ldots, W_{k-1}$, $\mathcal{I}_{k-1} := \langle W_3, W_4, \ldots, W_{k-1} \rangle$, is a valid expression for V_k. In this work we will refer the expressions of the polynomials W_k as stability constants, but also we will consider the expressions V_k as the normal forms of W_k in the Gröbner basis of $\mathcal{I}_k$ when the graded reverse lexicographic order (called *grevlex* or *degrevlex* in the literature and $\mathrm{tdeg}(a_2, a_3, \ldots, a_k)$ in Maple) is used, see [5, p. 58]. In this order, the monomials are compared first by their total degree and ties are broken by reverse lexicographic order, that is, by smallest degree in $a_k, a_{k-1}, \ldots, a_2$. In order to avoid ambiguity, the expressions of V_k will be called *reduced* stability constants.

It is known that the first non-zero stability constant is of odd order (see [3]). For the sake of completeness, in Section 2 we include a proof of this fact, as well as their algebraic properties that are reminiscent of similar properties satisfied by the Lyapunov and period constants, see [1, 2, 8, 14, 18].

If for a value of a it holds that $V_3(a) = V_5(a) = \cdots = V_{2k-1}(a) = 0$ and $V_{2k+1}(a) \neq 0$ we will say that the origin is a *weak fixed point of order* $k-1$, by similitude with the concept of order of a weak focus for non-degenerated critical point of planar polynomial vector fields. As we will see in Proposition 6 the maximum number of 2-periodic orbits that bifurcate from a weak fixed point of order m is m.

Our main result, which is proved in section 3, deals with the simplest case: the maps f_a are polynomial of fixed degree d, and the parameters are the coefficients of the system. Notice that the only involution in these families corresponds to the trivial case $f_{\mathbf{0}}(x) = -x$. As we will see, even in this simple setting some questions are not easy to answer.

Theorem 2. *Consider the family of polynomial maps*

$$f_a(x) = -x + \sum_{j=2}^{d} a_j x^j, \quad a = (a_2, a_3, \ldots, a_d) \in \mathbb{R}^{d-1}. \tag{5}$$

It has only the trivial involution corresponding to $a = \mathbf{0}$ and its cyclicity is at most $\left[(d^2-1)/2\right]$, where $[\,\cdot\,]$ stands for the integer part. Furthermore:

(a) For d even, its cyclicity is at least $d-2$. Moreover,

- *(i) For $d = 2, 4$ it is $d-2$.*
- *(ii) For $d = 6, 8, 10$ it is at most $d-2$ for any f_a, $a \neq \mathbf{0}$.*
- *(iii) For $d = 6, 8, 10$ it is at most $5, 9, 13$, respectively, for $f_{\mathbf{0}}$.*

(b) For d odd, its cyclicity is at least $d-3$. Moreover,

- *(i) For $d = 3$ it is $d - 2 = 1$.*
- *(ii) For $d = 5, 7, 9$ it is at most $d-2$ for any f_a, $a \neq \mathbf{0}$, and there is some a such that it is $d-2$.*
- *(iii) For $d = 5, 7, 9$ it is at most $4, 7, 10$, respectively, for $f_{\mathbf{0}}$.*
- *(iv) For $d = 4m+3$, $m \geq 0$, there are some values of a such that the origin is a weak fixed point of order $d-2$ for the corresponding f_a.*

Observe that the above result only accounts for the number of local (near $x = 0$) isolated 2-periodic orbits. For instance, with respect to statement (a) with $d = 4$, and although the cyclicity of the family is 2, it is easy to find examples with 3 global 2-periodic orbits. This is the case, for instance, for the map $f(x) = -x - 7x^2 + 10x^4$, which has also four fixed points. Notice that, the first statements of the above result are straightforward. If for some a, f_a has degree k then $f_a \circ f_a$ has degree k^2. Hence the only involution is $f_{\mathbf{0}}(x) = -x$. Moreover, a priori, the maximum number of isolated fixed points of $f_a \circ f_a$ for any polynomial map of degree d is d^2. Hence excluding the fixed point $x = 0$, we have that the maxim number of global 2-periodic orbits is $\left[(d^2-1)/2\right]$. It is not difficult to construct examples of polynomial maps of degree d (for instance using Chebyshev polynomials) with $\left[(d^2-d)/2\right]$ global 2-periodic orbits.

It seems natural to think that for any d the cyclicity is $d-2$. For d even, we have been able to prove that this value is a lower bound of this cyclicity by using the algebraic properties of the stability constants. When d is odd the problem is more difficult. In particular, for $d = 4m+5$, it is not easy at all to prove the existence of weak fixed points of order $d-2$, see our proofs for cases $d = 5, 9$ in item (ii) of part (b) of the theorem.

To prove that $d-2$ is an upper bound for values of a for which the origin is a weak fixed point is sometimes possible because we can use again some algebraic computations together with the Weierstrass preparation theorem, see Proposition 6 and Lemma 8. Nevertheless, when $a = \mathbf{0}$, our approach needs to show that the ideal generated by the first $d-1$ stability constants, say $\mathcal{I}$, is *radical* and contains all the functions $W_j(a)$ given in (4). This is only true for $d = 2, 3, 4$.

For $d \geq 4$, the proof of statements (iii) of parts (a) and (b) of Theorem 2 are based on large symbolic computations.

In Section 4 we study the relation between the cyclicity of weak foci or centers of planar vector fields and our results. In particular we show that any map of type (1) is a model for the half-return map associated to a weak focus, see Proposition 10.

2 Stability constants and preliminary results

In this section, first we prove some properties of the stability constants, including also their algebraic properties. Secondly, we include some standard tools to prove upper or lower bounds for the cyclicity of families of maps.

A related result to next theorem is also given in [6], first in terms of the derivatives of the map $f \circ f$ (Theorem 5.1), and also using some explicit expressions that are closely related with the stability constants (Theorem 5.4), which are obtained using the Faà di Bruno Formula, [12].

Theorem 3. *Let f_a be an analytic map of the form* (1)*. If f is not an involution, then there exists $m \geq 1$ such that $V_3 = V_5 = \cdots = V_{2m-1} = 0$, $V_{2m+1} \neq 0$. Moreover, if $V_{2m+1} < 0$ (resp. > 0), the origin is locally asymptotically stable (resp. a repeller). In particular, all $V_{2k+1} = V_{2k+1}(a), k \geq 1$, are polynomials in the variables $a_2, a_3, \ldots, a_{2k+1}$ and the first reduced stability constants are:*

$$
\begin{aligned}
V_3 =& -2{a_2}^2 - 2a_3,\\
V_5 =& -6\,a_4a_2 + 4\,{a_3}^2 - 2\,a_5,\\
V_7 =& 3\,a_2a_3a_4 - 8\,a_6a_2 + 13\,a_3a_5 - 4\,{a_4}^2 - 2\,a_7,\\
V_9 =& \frac{242}{17}\,a_2a_3a_6 - \frac{121}{17}\,a_2a_4a_5 - 10\,a_8a_2 + \frac{358}{17}\,a_3a_7 - 10\,a_4a_6 + \frac{69}{17}\,{a_5}^2 - 2\,a_9,\\
V_{11} =& \frac{4563}{121}\,a_2a_3a_8 - \frac{11765}{242}\,a_2a_4a_7 + \frac{13}{2}\,a_2a_5a_6 + \frac{4407}{242}\,a_3a_4a_6 - \frac{936}{121}\,a_3{a_5}^2\\
& -12\,a_{10}a_2 + \frac{3865}{121}\,a_3a_9 - 12\,a_4a_8 + \frac{515}{242}\,a_5a_7 - 6\,{a_6}^2 - 2\,a_{11},
\end{aligned}
$$

$$
\begin{aligned}
V_{13} =& \frac{94587200}{1428271}\,a_2a_3a_{10} - \frac{304305945}{2856542}\,a_2a_4a_9 + \frac{2992379}{219734}\,a_2a_5a_8 + \frac{1939207}{329601}\,a_2a_6a_7 \\
&+ \frac{145516929}{2856542}\,a_3a_4a_8 - \frac{138885638}{4284813}\,a_3a_5a_7 + \frac{4183988}{1428271}\,a_3{a_6}^2 - \frac{273943}{329601}\,{a_4}^2a_7 \\
&+ \frac{383791}{109867}\,a_4a_5a_6 - 14\,a_{12}a_2 + \frac{62421386}{1428271}\,a_3a_{11} - 14\,a_4a_{10} - \frac{29912981}{2856542}\,a_5a_9 \\
&- 14\,a_6a_8 + \frac{3323839}{329601}\,{a_7}^2 - 2\,a_{13}, \\
V_{15} =& - \frac{6188200}{465637}\,a_2a_3a_5a_8 + \frac{964610838}{8847103}\,a_2a_3a_{12} - \frac{1932055066}{8847103}\,a_2a_4a_{11} \\
&+ \frac{2073461406}{115012339}\,a_2a_5a_{10} + \frac{102777002}{1396911}\,a_2a_6a_9 + \frac{10885500630}{1070499463}\,a_2a_7a_8 \\
&+ \frac{1324158696}{8847103}\,a_3a_4a_{10} - \frac{70657783876}{345037017}\,a_3a_5a_9 + \frac{178495020}{8847103}\,a_3a_6a_8 \\
&- \frac{10948144126}{1070499463}\,a_3{a_7}^2 + \frac{888498472}{26541309}\,{a_4}^2a_9 + \frac{2562962080}{115012339}\,a_4a_5a_8 \\
&- \frac{4032962292}{1070499463}\,a_4a_6a_7 - \frac{150876019048}{13916493019}\,{a_5}^2a_7 + \frac{546329272}{115012339}\,a_5{a_6}^2 \\
&- 16\,a_{14}a_2 + \frac{511907618}{8847103}\,a_3a_{13} - 16\,a_4a_{12} - \frac{4393292988}{115012339}\,a_5a_{11} \\
&- 16\,a_6a_{10} + \frac{6893660012}{169026231}\,a_7a_9 - 8\,{a_8}^2 - 2\,a_{15}.
\end{aligned}
$$

Proof. First, observe that by the definition of normal form of W_k using a Gröbner basis $\mathcal{G}$ of the ideal $\langle W_3, W_4, \ldots, W_{k-1}\rangle$ it holds that $V_k = W_k + \sum_{g\in\mathcal{G}} p_g\, g$ where p_g are polynomials in a, see [5, p. 82]. Hence $\operatorname{sign}(V_k) = \operatorname{sign}(W_k)$.

Next we prove that that the order of the first non-zero stability constant is odd. Suppose, to arrive to a contradiction, that $f(f(x)) - x = W_{2m}x^{2m} + O(x^{2m+1}) = V_{2m}x^{2m} + O(x^{2m+1})$ with $V_{2m} \neq 0$. For instance assume that $V_{2m} > 0$. Then we can consider a neighborhood of the origin $\mathcal{U}$ such that for all $x \in U \setminus \{0\}$, f is strictly monotonically decreasing and $f(f(x)) - x > 0$. Let $x_0 \in U \setminus \{0\}$ and consider its orbit $x_n = f(x_{n-1}), n \geq 1$. We also take $|x_0|$ small enough, such that $x_1, x_2, x_3 \in U$. We know that $x_2 - x_0 = f(f(x_0)) - x_0 > 0$. Since f is decreasing, it implies that $f(x_2) < f(x_0)$, that is, $f(f(x_1)) < x_1$, a contradiction with $f(f(x)) - x > 0$.

A simple argument gives that the stability of the origin for $f \circ f$ is determined by the sign of $x\big(f(f(x)) - x\big)$ in a neighborhood of the origin. Observe that when $V_{2m+1} \neq 0$, it holds that for $x \in U \setminus \{0\}$ the function $x\big(f(f(x)) - x\big) = V_{2m+1}x^{2m+2} + O(x^{2m+3})$ has the same sign that V_{2m+1}. As a consequence, the stability of the origin for both maps $f \circ f$ and f is characterized by the sign of the stability constants. ∎

We continue this section by proving an algebraic property of the stability constants W_k. This property is analogous to the one possessed by the Lyapunov constants of weak foci and the period constants of centers for planar vector fields, see [2]. In fact, these constants play a similar role to the *Lyapunov constants* in the study of small amplitude limit cycles of planar analytic differential systems with weak focus or a center, or the the *Period constants* in the study of the critical periods arising in planar centers, [1, 2, 8, 14, 18]. Ending with this list of similarities, we can say that the case where $f \circ f = \mathrm{Id}$ is the one corresponding with either the center or the isochronous cases, in each of the above two analogous situations.

Proposition 4. *The stability constants W_j, introduced in* (3)*, associated to an orientation reversing diffeomorphism of the form* (1) *are quasi-homogeneous polynomials of quasi-degree $j-1$ and weights $(1, 2, \ldots, j-1)$ in the coefficients $(a_2, a_3, \ldots, a_j)$, that is*

$$W_j(\lambda a_2, \lambda^2 a_3, \ldots, \lambda^{j-1} a_j) = \lambda^{j-1} W_j(a_1, \ldots, a_j).$$

Proof. It can be seen straightforwardly that each coefficient W_j is a polynomial function of the coefficients of a_i for $i = 2, \ldots, j$.

Observe that the change of variables $x = \lambda u$ conjugates the map $f(x) = -x + \sum_{i\geq 2} a_i x^i$ with the map $g(u) = -u + \sum_{i\geq 2} b_i u^i$ where $b_i = \lambda^{i-1} a_i$, and so conjugates the map $f(f(x)) = x + \sum_{j\geq 3} W_j(a_2, \ldots, a_j) x^j$, with

$$g(g(u)) = u + \sum_{j\geq 3} W_j(b_2, \ldots, b_j) u^j.$$

Since $g(g(u)) = \frac{1}{\lambda} f(f(\lambda u))$ we have

$$\begin{aligned} u + \sum_{j\geq 3} W_j(b_2, \ldots, b_j) u^j &= \frac{1}{\lambda} \left(\lambda u + \sum_{j\geq 3} W_j(a_2, \ldots, a_j)(\lambda u)^j \right) \\ &= u + \sum_{j\geq 3} \lambda^{j-1} W_j(a_2, \ldots, a_j) u^j. \end{aligned}$$

Hence $W_j(b_2, \ldots, b_j) = \lambda^{j-1} W_j(a_2, \ldots, a_j)$. ∎

As we will see, the above result is a key tool to prove part of item (a) of Theorem 2. It is also useful to find algebraic relations among the polynomials W_j because a priori they give some restrictions on them.

As we have already explained in their definition, the explicit expressions of the reduced stability constants have been obtained first by computing coefficients of the Taylor expansion of $f \circ f$ and afterwards, by taking the normal form of W_k in the Gröbner basis of $\langle W_3, W_k, \ldots, W_{k-1} \rangle$ when the graded reverse lexicographic order is used. The above results states that the stability constants W_k are quasi-homogeneous polynomials in the coefficients of the maps. Notice that the reduced stability constants V_k, given in Theorem 3, are also quasi-homogeneous polynomials.

Next results collect and adapt some tools for studying the number of zeroes of families of smooth maps that are borrowed from the techniques used to study the number of small amplitude limit cycles bifurcating from weak foci or centers.

Proposition 5. *Let $W_j = W_j(a)$ and $V_j = V_j(a)$ be the polynomials associated to the family of maps (5) given in (4). Assume that there exists $m = m(d)$ such that for all $k = 1, 2, \ldots, m-1$,*

$$\langle W_3, W_4, \ldots, W_{2k+1}\rangle = \langle W_3, W_4, \ldots, W_{2k+2}\rangle = \langle V_3, V_5, \ldots, V_{2k+1}\rangle$$

and $\langle V_3, V_5, \ldots, V_{2m+1}\rangle = \langle W_3, W_4, \ldots W_{d^2}\rangle$. Then the cyclicity of the family is at most $m-1$.

Proof. We need to study the number of isolated positive zeroes in a neighborhood of the origin of the maps

$$h_a(x) = \frac{f_a(f_a(x)) - x}{x^3} = \sum_{j=3}^{d^2} W_j(a)x^{j-3} = \sum_{j=1}^{m} V_{2j+1}(a)\big(1 + x\psi_{2j+1}(x,a)\big)x^{2j-2}, \tag{6}$$

where, to write the last equality, we have used the hypotheses on the polynomials W_j and V_{2j+1} and ψ_{2j+1} are polynomial functions. Notice that these zeroes always correspond to 2-periodic orbits of f_a and are not fixed points because, locally, f_a sends positive values of x to negative ones, and viceversa.

The procedure that we follow is rather standard and it is usually called *division-derivation algorithm*. Other examples of its application can be seen in [1, 9, 16, 18].

We will prove by induction that any map of the form

$$h_a(x) = \sum_{j=1}^{k} g_j(a)\big(1 + x\psi_j(x,a)\big)x^{2j-2}, \tag{7}$$

where ψ_j are smooth functions in x, has at most $k-1$ positive isolated zeroes is any small enough neighborhood of the origin.

When $k = 1$, then obviously the function (7) has not zeroes. Assume that the result holds for $k = m-1$. Set $k = m$, then

$$\begin{aligned}\frac{h_a(x)}{1 + x\psi_1(x,a)} =& \sum_{j=1}^{m} g_j(a)\frac{1 + x\psi_j(x,a)}{1 + x\psi_1(x,a)}x^{2j-2}\\ =& g_1(a) + \sum_{j=2}^{m} g_j(a)\big(1 + x\phi_j(x,a)\big)x^{2j-2},\end{aligned}$$

where ϕ_j are smooth functions in x. Then, for some new smooth functions φ_j and ζ_j:

$$\begin{aligned}\frac{d}{dx}\Big(\frac{h_a(x)}{1+x\psi_1(x,a)}\Big) &= \sum_{j=2}^{m} g_j(a)\big(2j-2+x\varphi_j(x,a)\big)x^{2j-3} \\ &= \sum_{j=2}^{m}(2j-2)\, g_j(a)\big(1+x\zeta_j(x,a)\big)x^{2j-3}.\end{aligned}$$

Observe that the map

$$\begin{aligned}\frac{1}{x}\frac{d}{dx}\Big(\frac{h_a(x)}{1+x\psi_1(x,a)}\Big) &= \sum_{j=2}^{m}(2j-2)\, g_j(a)\big(1+x\zeta_j(x,a)\big)x^{2j-4} \\ &= \sum_{i=1}^{m-1} \tilde{g}_i(a)\big(1+x\xi_i(x,a)\big)x^{2i-2},\end{aligned}$$

where $\tilde{g}_i(a) = 2i\, g_{i+1}(a)$ and $\xi_i(x,a) = \zeta_{i+1}(x,a)$, is of the form (7) with $k = m-1$. Hence, by the induction hypothesis it has at most $m-2$ zeroes in any positive neighborhood of the origin. Hence, by the Rolle's Theorem the map h_a has at most $m-1$ zeroes.

Of course, since the map (6) is in the form (7), the result follows. Observe that if for some values of a, one of the V_{2j+1} vanishes, the division derivation procedure for this value of a can be accelerated and gives rise to less number of positive zeroes. ∎

Proposition 6. *Let $V_j = V_j(a)$ be the reduced stability constants associated to the family of maps* (5) *given in* (4). *Assume that for $a = a^*$ the map has a weak fixed point of order $m-1$, that is, $V_3(a^*) = V_5(a^*) = \cdots = V_{2m-1}(a^*) = 0$ and $V_{2m+1}(a^*) \neq 0$. Then, the maximum cyclicity of f_{a^*} is $m-1$.*

Moreover, if the $m-1$ vectors

$$\nabla V_3(a^*), \nabla V_5(a^*), \ldots, \nabla V_{2m-1}(a^*),$$

where $\nabla = (\partial/\partial a_2, \partial/\partial a_3, \ldots, \partial/\partial a_m)$, are linearly independent, the cyclicity of the map f_{a^*} is $m-1$.

Proof. To prove that the maximum cyclicity of the origin of f_{a^*} is $m-1$, as usual, we will apply the Weierstrass preparation theorem ([10]) to the function $h_a(x)$ introduced in (6). More precisely, write $H(x,a) = H(x, a_2, a_3, \ldots, a_d) = h_a(x)$ as a holomorphic function with d-variables. Notice that

$$H(x, a_2^*, a_3^*, \ldots, a_d^*) = V_{2m+1}(a^*)x^{2m-2} + O(x^{2m-1})$$

and hence, we are under the hypotheses of that theorem. Therefore, in a neighborhood in $\mathbb{C}^d$ of $(0, a^*)$, it holds that

$$H(x, a_2, a_3, \dots, a_d)$$
$$= \left[x^{2m-2} + A_{2m-3}(a)x^{2m-3} + A_{2m-3}(a)x^{2m-3} + \cdots + A_1(a)x + A_0(a)\right] g(x, a), \quad (8)$$

where A_j and g are holomorphic functions, $g(0, a^*) = V_{2m+1}(a^*) \neq 0$ and $A_j(a^*) = 0$. As a consequence, for parameters in a neighborhood of $a = a^*$ the function h_a has at most $2m - 2$ non-zero roots in a neighborhood of the origin. Since the non-zero roots of this function appear in couples (for each positive zero corresponding to a 2-periodic orbit, there is a negative one corresponding to the other point of this orbit), we have proved that the number of positive zeroes in a neighborhood of the origin is at most $m-1$, giving the desired bound for the cyclicity.

The proof of the second part is also based on a well-known approach, see for instance [4]. It simply uses Bolzano's theorem and consists on producing successive changes of stability of the origin. We give the details when $m = 3$. The general case follows by using the same type of arguments. Recall that $h_a(x) = (f_a(f_a(x)) - x)/x^3$ and its positive zeroes give rise to the 2-periodic orbits.

We have that for $a = a^*$ it holds that $V_3(a^*) = V_5(a^*) = 0$ and $V_7(a^*) \neq 0$. Assume without loss of generality that $V_7(a^*) < 0$. If δ_2 is small enough then for all $0 < \delta < \delta_2$ there exists $x_0 > 0$ such that $|x_0| < \delta$ such that $h_{a^*}(x_0) < 0$. Consider the mapping Φ from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $\Phi(a_2, a_3) = (V_3(a_2, a_3, a_4^*), V_5(a_2, a_3, a_4^*))$. Then $\Phi(a_2^*, a_3^*) = (0, 0)$ and since by hypothesis $\nabla V_3(a^*), \nabla V_5(a^*)$ are linearly independent, $\det\left(D\Phi(a^*)\right) \neq 0$. This fact implies that Φ is locally exhaustive. Hence, we can find values $a = (a_2, a_3, a_4^*)$ as near as we want of a^*, say $|a - a^*| < \epsilon_1$, with $V_5(a) > 0$ and $V_3(a) = 0$. This fact implies that there exists $0 < x_1 < x_0 < \delta$ such that $h_a(x_1) > 0$ but still $h_a(x_0) < 0$. Hence, there exists at least a positive root of h_a in (x_1, x_0). Now let a with $|a - a^*| < \epsilon_2 < \epsilon_1$ such that $V_3(a) < 0$ and, yet $h_a(x_2)h_a(x_1) < 0$. Finally, we get that there exists $0 < x_2 < x_1$ satisfying $h_a(x_2) < 0$ with $h_a(x_1) > 0$ and $h_a(x_0) < 0$. Hence, for $(x, a) \in (0, \delta) \times D_{\epsilon_2}(a^*)$, $h_a(x)$ has two positive zeros corresponding with the two announced 2-periodic orbits. ∎

Next proposition extends the second part of the previous one, when instead of dealing with the reduced stability constants we consider the stability constants. Its proof is similar and we omit it.

Proposition 7. *Let $W_j = W_j(a)$ be the stability constants associated to the family of maps* (5) *given in* (4)*. Assume that for $a = a^*$ the map has a weak fixed point of order $m - 1$, that is, $W_3(a^*) = W_4(a^*) = \cdots = W_{2m-1}(a^*) = W_{2m}(a^*) = 0$ and $W_{2m+1}(a^*) \neq 0$.*

Then, if the $m-1$ vectors

$$\nabla W_3(a^*), \nabla W_5(a^*), \ldots, \nabla W_{2m-1}(a^*),$$

where $\nabla = (\partial/\partial a_2, \partial/\partial a_3, \ldots, \partial/\partial a_m)$, are linearly independent, the cyclicity of the map f_{a^*} is at least $m-1$.

Lemma 8. *Let $W_j = W_j(a)$ and $V_j = V_j(a)$ be the stability constants associated to the family of maps* (5) *given in* (4)*. Assume that there exist $k \geq 3$ and $0 < n_j \in \mathbb{N}$ such that for all $j = 3, 4, \ldots, d^2$,*

$$W_j^{n_j} \in \langle V_3, V_5, \ldots, V_{2k+1} \rangle. \tag{9}$$

Let ℓ denote the minimum k such that (9) *holds. Then, the highest order of the origin as weak fixed point is $\ell - 1$. Moreover, the maximum cyclicity of any map f_a, with $a \neq \mathbf{0}$, is also $\ell - 1$.*

Proof. Assume, to arrive to a contradiction, that the family has some weak fixed point with order bigger than $\ell - 1$ for some $a = a^* \neq \mathbf{0}$. In particular, for this a it holds that $V_3(a^*) = V_5(a^*) = \cdots = V_{2\ell+1}(a^*) = 0$. By hypotheses, for any $j \geq 3$,

$$W_j^{n_j}(a) = \sum_{i=1}^{\ell} p_{2i+1,j}(a) V_{2i+1}(a),$$

for some polynomials $p_{2i+1,j}(a)$. Hence, $W_j^{n_j}(a^*) = 0$ for all $j \geq 3$, giving that $W_j(a^*) = 0$. As a consequence, $f_{a^*}(x) = -x$, a contradiction with our initial assumption.

Finally, the maximum cyclicity for any map f_a, with $a \neq \mathbf{0}$, is $\ell - 1$ because of the first part of Proposition 6. ∎

We end this list of preliminary results with a remark about the cyclicity of families of orientation preserving diffeomorphisms.

Remark 9. For any family of maps $f_a(x) = x + \sum_{i \geq 2} c_i(a) x^i$ with a in an open set $\mathcal{V} \subseteq \mathbb{R}^n$, depending continuously with respect to a, the origin has 2-*cyclicity* 0. This holds because, given any $a = a^*$ there is a neighborhood of the origin and a^* for which f_a is monotonous increasing.

3 Proof of Theorem 2

For any $d \geq 2$ the family of maps (5) is a $(d-1)$-parametric family, with parameters $a = (a_2, \ldots, a_d) \in \mathbb{R}^{d-1}$. As we have already argued in the introduction, if for some a, f_a has degree k then $f_a \circ f_a$ has degree k^2. Hence the only involution in the family is the trivial one $f_{\mathbf{0}}(x) = -x$. To prove the second assertion of the statement, notice that $x = 0$ is a fixed point of f_a and $f_a \circ f_a$. Hence, the maximum number of global 2-periodic orbits of a polynomial map in the family (5) is $\left[(d^2-1)/2\right]$.

(a) Consider first the case $d = 2n$, even. We start proving that its cyclicity is at least $d-2$. In this situation it is very easy to prove that taking $a^* = (0, 0, \ldots, 0, 1)$ the origin is a weak fixed point of order $d - 2 = 2(n-1)$, with $W_{4n-1}(a^*) = V_{4n-1}(a^*) = -2n \neq 0$, because when $f_{a^*}(x) = -x + x^{2n}$,

$$\begin{aligned} f_{a^*}(f_{a^*}(x)) =& x - x^{2n} + \big(-x + x^{2n}\big)^{2n} = x - x^{2n} + x^{2n}\big(1 - x^{2n-1}\big)^{2n} \\ =& x - 2nx^{4n-1} + O(x^{4n}). \end{aligned}$$

To show that the cyclicity of the map f_{a^*} is $2(n-1)$ we will apply Proposition 7. Therefore we must prove that the vectors in $\mathcal{W} := \{\nabla W_{2k+1}(a^*),\ k = 1, 2, \ldots, 2n-2\}$, are linearly independent, where we recall that $\nabla = (\partial/\partial a_2, \partial/\partial a_3, \ldots, \partial/\partial a_{2n-1})$.

Using the quasi-degree properties of the stability constants $W_{2k+1}(a)$ proved in Proposition 4, it is clear that for a general family of maps (5) with $d = 2n$, for any $W_{2k+1}(a)$, $k = 1, 2, \ldots, n-1$, the only degree 1 monomial of each of them is $\alpha_{2k+1}a_{2k+1}$ for some real constants α_{2k+1}. To determine these constants notice that when $f(x) = -x + x^{2k+1}$ then

$$f(f(x)) = x - x^{2k+1} + \big(-x + x^{2k+1}\big)^{2k+1} = x - x^{2k+1} - x^{2k+1}\big(1 + O(x)\big) = x - 2x^{2k+1} + O(x^{2k+2}).$$

Hence, for these values of k, $\alpha_{2k+1} = -2$. In consequence

$$\nabla W_{2k+1}(a^*) = (0, 0, \ldots, -2, 0, \ldots, 0), \quad k = 1, 2, \ldots, n-1, \tag{10}$$

where the -2 is placed at the $2k$ position of the $(2n-2)$-dimensional vector, because for these values of k all the other monomials of $W_{2k+1}(a)$ have degree at least 2, and their derivatives, evaluated at a^* vanish.

For k from n until $2n-2$, and due again to the algebraic property given in Proposition 4, the corresponding stability constant $W_{2k+1}(a)$ (again, for a general family of maps (5) with $d = 2n$) has no monomials of degree 1. Similarly it can have several monomials of degree 2, all of them of the form $\beta_{s,t}a_{2s}a_{2t}$, for some real values $\beta_{s,t}$, to be determined, and with $(s, t) \in \mathbb{N}^2$, both bigger than 1 and such that $s + t = k + 1$. Because we are only interested on computing $\nabla W_{2k+1}(a^*)$, the only relevant monomial of degree 2 in $W_{2k+1}(a)$ will be $\beta_{k+1-n,n}a_{2(k+1-n)}a_{2n}$. To obtain these values of $\beta_{k+1-n,n}$, consider $f(x) = -x + x^{2(k+1-n)} + x^{2n}$. Similar computations than the ones done above give that this coefficient in $W_{2k+1}(a)$ is $-2(k+1)$. Therefore $W_{2k+1}(a)$ has the monomial $-2(k+1)a_{2(k+1-n)}a_{2n}$ and it holds that

$$\nabla W_{2k+1}(a^*) = (0, 0, \ldots, -2(k+1), 0, \ldots, 0), \quad k = n, n+1, \ldots, 2n-2, \tag{11}$$

where the value $-2(k+1)$ is placed at the $2(k-n)+1$ position of this $(2n-2)$-dimensional vector.

Joining (10) and (11), we obtain that the vectors in $\mathcal{W}$ are linearly independent. Hence we have proved that when $d=2n$ the cyclicity of the whole family is at least $d-2$, because for this specific value of $a=a^*$ it is so.

Now we are going to consider the maps (5) for small values of d.

Case $\mathbf{d}=\mathbf{2}$. In this simple case $f_a(x)=-x+a_2\,x^2$ and $f_a(f_a(x))=x-2\,a_2^2\,x^3+a_2^3\,x^4$. The equation $f_a(f_a(x))=x$ only gives the solutions $x=0$ and $x=\frac{2}{a_2}$ which in fact are fixed points of f_a. Hence f_a has not 2-periodic orbits.

Case $\mathbf{d}=\mathbf{4}$. In this case $f_a(f_a(x))=x+V_3\,x^3+\sum_{j=4}^{16}W_j\,x^j$. It is straightforward, either by hand, or using the Gröbner basis package in Maple that we are under the hypotheses of Proposition 5 with $m=3$. Hence the cyclicity of the family is at most 2 and, therefore, since we have proved that it is at least $d-2=2$, it is exactly 2. As an example of the computations that we have done, next we give some details of the first algebraic relations.

In this case

$$V_3=-\,2{a_2}^2-2a_3,\quad V_5=-6\,a_4a_2+4\,{a_3}^2,\quad V_7=3\,a_2a_3a_4-4\,{a_4}^2,$$

and it holds that $V_3=W_3$,

$$\begin{aligned}W_4=&-\frac{1}{2}\,a_2\,V_3,\quad W_5=V_5+\frac{1}{2}\,a_3\,V_3,\\W_6=&-\frac{3}{2}\,a_2\,V_5+\frac{1}{2}(a_4-a_2\,a_3)V_3,\\W_7=&V_7+\frac{3}{4}(a_2^2-a_3)V_5-\frac{1}{4}a_2a_4V_3,\end{aligned}$$

and $W_j\in\langle V_3,V_5,V_7\rangle$, for $j=8,9,\ldots,16$.

Case $\mathbf{d}=\mathbf{6}$. As when $d=4$, we want to apply Proposition 5. In this case we prove that we are under the hypotheses of this result with $m=6$, and hence the cyclicity of the family will be at most 5. Indeed, using the Maple's Gröbner basis package again we find that,

$$W_j\in\langle V_3,V_5,V_7,V_9,V_{11},V_{13}\rangle\text{ for }3\le j\le 36.$$

Moreover, it also holds that

$$W_j^2\in\langle V_3,V_5,V_7,V_9,V_{11}\rangle:=\mathcal{I}\text{ for }3\le j\le 36,\ \text{ and also }W_{13}\notin\mathcal{I},$$

showing that we are under the hypotheses of Lemma 8 with $\ell=5$, proving that the cyclicity of any map f_a, with $a\neq\mathbf{0}$, is at most $\ell-1=4=d-2$.

Notice that the above two relations imply in particular that the ideal $\mathcal{I}$ is not radical.

Cases $\mathbf{d = 8, 10}$. Doing similar computations that when $d = 6$ we can apply the same results.

For $d = 8$ we get that $m = 10$ and $\ell = 7$, because,

$$W_j \in \langle V_3, V_5, \ldots, V_{19}, V_{21}\rangle \text{ for } 3 \leq j \leq 64$$

and no similar relation appears before. Moreover,

$$W_j^2 \in \langle V_3, V_5, \ldots, V_{13}, V_{15}\rangle \text{ for } 3 \leq j \leq 64.$$

Hence, by Proposition 5 the cyclicity of $f_{\mathbf{0}}$ is at most 9 and, by Lemma 8 and the fact that the cyclicity is at least $d - 2 = 6$, we get the desired result. We remark that for some W_j there is no need to take W_j^2 to be in the ideal, but it is essential for instance for W_{17}.

For $d = 10$, $m = 14$ and $\ell = 9$, we prove that the cyclicity of $f_{\mathbf{0}}$ is at most 13 and that the cyclicity of any f_a, for $a \neq \mathbf{0}$, is once again $d - 2 = 8$. We remark that in this case it happens that

$$W_j \in \langle V_3, V_5, \ldots, V_{27}, V_{29}\rangle \text{ for } 3 \leq j \leq 100,$$

without similar relations appearing before, and

$$W_j^3 \in \langle V_3, V_5, \ldots, V_{17}, V_{19}\rangle := \mathcal{I} \text{ for } 3 \leq j \leq 100.$$

We remark that not all W_j need the exponent 3 to be in $\mathcal{I}$. Nevertheless, for instance, neither W_{21} nor W_{21}^2 are in $\mathcal{I}$.

(b) When $d = 2n + 1$ is odd it is clear that the cyclicity of the family is at least the one to the case of degree $2n$, that we have proved that it is at least $2n - 2$. Hence it is at least $d - 3$.

Now we are going to consider the cases $d = 3, 5, 7, 9$.

Case $\mathbf{d = 3}$. Doing similar computations that the ones of case $d = 4$ we get that we are under the hypotheses of Proposition 5 with $m = 2$. Hence an upper bound of the cyclicity of this family is 1. To prove that this bound is attained we take $a^* = (a_2^*, a_3^*) = (1, -1)$. Then $V_3(a^*) = 0$ and $V_5(a^*) = 4 > 0$. Since $V_3(a) = -2{a_2}^2 - 2a_3$, it holds that

$$\nabla V_3(a^*) = \frac{\partial}{\partial a_2} V_3(a)\bigg|_{a^*=(1,-1)} = -4 \neq 0.$$

Therefore the cyclicity of the map f_{a^*} is 1, and so it is the cyclicity of the family.

Case $\mathbf{d = 5}$. Proceeding as in case $d = 6$, first we will get some upper bounds of the cyclicity. In fact we can apply Lemma 8 with $\ell = 4$, proving that the cyclicity of any map f_a, with $a \neq \mathbf{0}$, is at most $\ell - 1 = 3 = d - 2$, and Proposition 5 with $m = 5$, showing that the cyclicity of the family is at most 4. Next we present one example with cyclicity 3.

By solving the system $\{V_3(a) = V_5(a) = V_7(a) = 0\}$ with respect $a_2, \dots, a_5$, and by direct inspection of its solutions, we obtain that taking $a^* = (1, -1, (9+\sqrt{55})/2, -(23+3\sqrt{55})/2)$ it holds that

$$V_3(a^*) = V_5(a^*) = V_7(a^*) = 0,\ V_9(a^*) = 1701 + 229\sqrt{55} > 0.$$

A computation shows that

$$\det\left(\nabla V_3(a^*), \nabla V_5(a^*)\nabla V_7(a^*)\right) = \det\begin{pmatrix} -4 & -27-3\sqrt{55} & -\frac{1}{2}\left(27-3\sqrt{55}\right) \\ -2 & -8 & -136-18\sqrt{55} \\ 0 & -6 & -39-4\sqrt{55} \end{pmatrix}$$

$$= 5280 + 736\sqrt{55},$$

where $\nabla = (\partial/\partial a_2, \partial/\partial a_3, \partial/\partial a_4)$. So, the three vectors $\nabla V_3(a^*), \nabla V_5(a^*)$ and $\nabla V_7(a^*)$, are linearly independent and therefore, by Proposition 6, the cyclicity of f_{a^*} is exactly 3.

Case $\mathbf{d = 7}$. We start proving that the cyclicity is at least five by finding an example with this cyclicity. Proceeding as in the above case, or looking at the proof of item (iv), we find that taking $a^* = (0, 0, 1, 0, 0, -2)$, we obtain that $V_3(a^*) = V_5(a^*) = V_7(a^*) = V_9(a^*) = V_{11}(a^*) = 0$ and $V_{13} = 42$. A computation gives

$$\det\left(\nabla V_3(a^*), \nabla V_5(a^*), \nabla V_7(a^*), \nabla V_9(a^*), \nabla V_{11}(a^*)\right) = \det\begin{pmatrix} 0 & -6 & 0 & 0 & \frac{11765}{121} \\ -2 & 0 & 0 & -\frac{716}{17} & 0 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & -2 & 0 & 0 & -\frac{515}{121} \\ 0 & 0 & 0 & -10 & 0 \end{pmatrix}$$

$$= -35200.$$

Hence the cyclicity of f_{a^*} is 5, as desired. Finally, using Maple again we get that

$$W_j \in \langle V_3, V_5, \dots, V_{17} \rangle \text{ for } 3 \le j \le 48$$

and

$$W_j^2 \in \langle V_3, V_5, \dots, V_{13} \rangle \text{ for } 3 \le j \le 48.$$

Hence the cyclicity of $f_{\mathbf{0}}$ is at most 7 and the cyclicity of f_a, for any $a \neq \mathbf{0}$ is at most $d - 2 = 5$.

Case $\mathbf{d = 9}$. Once again, some computations using the Maple's Gröbner basis package give

$$W_j \in \langle V_3, V_5, \dots, V_{23} \rangle \text{ for } 3 \le j \le 81$$

and

$$W_j^3 \in \langle V_3, V_5, \ldots, V_{17}\rangle \text{ for } 3 \le j \le 81.$$

Hence the cyclicity of $f_{\mathbf{0}}$ is at most 10 and the cyclicity of f_a, for any $a \neq \mathbf{0}$ is less or equal to $d-2=7$. To end the proof we prove that there is a value of $a \neq \mathbf{0}$ such that the cyclicity at this value is 7.

In this case, the study of the solutions of the system of equations described by the first five reduced stability constants is complicated. So we will propose an alternative method for obtaining weak fixed points of high order. This method is based on the knowledge of the structure of 1-dimensional involutions.

It is know that any analytic 1-dimensional non-trivial involution h can be written as

$$h(x) = g(-g^{-1}(x)),$$

where g is an analytic diffeomorphism such that $g(0) = 0$, see [13]. Notice that it is straightforward to check that $g(-g^{-1})$ is an involution. Take any map of the form

$$g(x) = x + \sum_{j\ge 2} b_j x^j$$

and compute the Taylor series of its inverse,

$$g^{-1}(x) = x - b_2x^2 + (2b_2^2 - b_3)x^3 + (-5b_2^3 + 5b_2b_3 - b_4)x^4 + \sum_{j\ge 5} D_j(b)x^j,$$

where $b = (b_2, b_3, \ldots)$ and we do not detail the polynomial functions D_j, that are given by the so called Bell polynomials. It holds that $h \circ h = \mathrm{Id}$.

Now, to find a map with a weak fixed point of high order, we can fix some degree d, and consider the Taylor approximation of h of degree d, at the origin, $h_d = T_d(h)$. Then

$$h_d(x) = -x + \sum_{j=2}^{d} B_j(b)x^j,$$

where

$$\begin{aligned} &B_2(b) = 2b_2,\ B_3(b) = -4b_2^2,\ B_4(b) = 10b_2^3 - 4b_2b_3 + 2b_4,\\ &B_5(b) = -28b_2^4 + 24b_2^2b_3 - 12b_2b_4, \end{aligned} \tag{12}$$

and $B_j(b)$, for $j = 6, \ldots, d$ are some polynomials that we do not detail. This map has a high order weak fixed point at the origin. For instance when $d = 9$, it holds that

$$h_9(h_9(x)) = x + W_{11}(b)x^{11} + \sum_{j=12}^{81} W_j(b)x^j.$$

Now, to increase the level of weakness of the fixed point, that is the order of h_9, we have to select the values of b such that the associated stability constants up to order 15 vanish, i.e.

$$\Big\{W_{11}(b) = W_{13}(b) = W_{15}(b) = 0, \tag{13}$$

where we omit the expression of these stability constants. Since $W_{11}(b)$ linear with respect b_7 we can isolate and fix this parameter, obtaining

$$\begin{aligned} b_7 := \quad & \frac{1}{4b_2\left(107\,b_2^3 + 6\,b_2b_3 - 3\,b_4\right)}\left(20774b_2^{10} - 64272b_2^8b_3 + 32136b_2^7b_4 + 52962b_2^6b_3^2\right. \\ & -7644b_2^6b_5 - 41496b_2^5b_3b_4 - 9464b_2^4b_3^3 + 3822b_2^5b_6 + 4836b_2^4b_3b_5 + 6552{b_2}^4b_4^2 \\ & +6942b_2^3b_3^2b_4 - 507{b_2}^2b_3^4 - 1776b_2^3b_3b_6 - 1348b_2^3b_4b_5 + 300b_2^2b_3^2b_5 - 684b_2^2b_3b_4^2 \\ & +564b_2b_3^3b_4 + 214b_2^3b_8 + 246b_2^2b_4b_6 - 12b_2^2b_5^2 - 114b_2b_3^2b_6 - 204b_2b_3b_4b_5 - 50b_2b_4^3 \\ & \left.-156b_3^2b_4^2 + 12b_2b_3b_8 + 12b_2b_5b_6 + 60b_3b_4b_6 + 24b_4^2b_5 - 6b_4b_8 - 3b_6^2\right). \end{aligned}$$

To reduce the number of parameters we impose $b_2 = 1$ and $b_3 = 0$, and solve the system $\{W_{13}(b) = W_{15}(b) = 0\}$ using the Maple algebra software, obtaining the following solution, among others: $b_4 = \xi$, b_5 as a free parameter and $b_6 = n(b_5,\xi)/d(b_5,\xi)$ where

$$\begin{aligned} n(b_5,\xi) = \quad & -4830249480\xi^9 + 78255450\xi^8b_5 - 309996323910\xi^8 + 121885399860\xi^7b_5 \\ & +499588480916\xi^7 - 3569620983180\xi^6b_5 + 433844538538740\xi^6 \\ & -3036308656220\xi^5b_5 + 10120115599755700\xi^5 - 1400107036991768\xi^4b_5 \\ & +78554454691772584\xi^4 - 16364417170088484\xi^3b_5 + 278979787186921660\xi^3 \\ & -60913553653703380\xi^2b_5 + 434487144164761772\xi^2 - 150424031357777588\xi b_5 \\ & -147634409601271244\xi + 253882004776386078b_5 - 155128034441262745 8, \\ d(b_5,\xi) = \quad & 39127725\xi^8 + 60942699930\xi^7 - 1784810491590\xi^6 - 1518154328110\xi^5 \\ & -700053518495884\xi^4 - 8182208585044242\xi^3 - 30456776826851690\xi^2 \\ & -75212015678888794\xi + 126941002388193039, \end{aligned}$$

and where ξ is any real root of the polynomial

$$\begin{aligned} P(x) = \quad & 160228033875x^{16} + 221432009870400x^{15} + 13936473199884004x^{14} \\ & -683923454204391464x^{13} + 26429954882084 03832x^{12} \\ & -385227003687957189136x^{11} - 3012116857431809290604x^{10} \\ & +4502608443142798941 3608x^9 + 75208088751808820472 9142x^8 \\ & +503289652282701719851 6064x^7 + 177791087322145265163 15308x^6 \\ & +298171711915238799261 81416x^5 - 142127933256060524840 90592x^4 \\ & -123365732211297823524 968592x^3 - 274115367296634168846 158244x^2 \\ & -325682563327763246441 199080x - 133940574254498343421 555617. \end{aligned}$$

Notice that, using the Sturm algorithm, one can check that $P(x)$ has 8 different simple real roots.

Finally we set $b_5 = 0$. With this choice of the parameters each constant $W_j(b)$ writes as $W_j(\xi)$. A computation shows that for $j = 11, \ldots, 16$:

$$\text{Resultant}\left(P(\xi), \text{Numer}\left(W_j(\xi)\right); \xi\right) = 0,$$
$$\text{Resultant}\left(P(\xi), \text{Denom}\left(W_j(\xi)\right); \xi\right) \neq 0,$$

and

$$\text{Resultant}\left(P(\xi), \text{Numer}\left(W_{17}(\xi)\right); \xi\right) \neq 0,$$
$$\text{Resultant}\left(P(\xi), \text{Denom}\left(W_{17}(\xi)\right); \xi\right) \neq 0.$$

Hence, when $x = \xi^*$ is any of the real roots of $P(x)$ the map h_9 has a weak fixed point of order 7. Now we prove that it has cyclicity 7. Using the expressions of the functions B_j (see (12)) we set $a_j^* = B_j(\xi^*)$ for $j = 2, \ldots 9$ and take $a^* = (a_2^*, \ldots, a_9^*)$. A computation gives that

$$\det\left(\nabla V_3(a^*), \ldots, \nabla V_{15}(a^*)\right) = \frac{R(\xi)}{Q(\xi)},$$

where R and Q are co-prime polynomials with degree 77 and 68 respectively in ξ. Again, one can check that Resultant $(P(\xi), R(\xi); \xi) \neq 0$, and Resultant $(P(\xi), Q(\xi); \xi) \neq 0$, hence the vectors $\nabla V_3(a^*), \ldots, \nabla V_{15}(a^*)$ are linearly independent and, by Proposition 6, the cyclicity of f_{a^*} is at least 7. This ends the proof of statements (b) (i)–(iii).

To prove statement (iv), we consider for $d = 4m + 3$:

$$f_{a^*}(x) = -x + x^{2m+2} - (m+1)\, x^{4m+3}.$$

A routine computation shows that

$$f_{a^*}(f_{a^*}(x)) = x + \frac{(m+1)(5m+4)(4m+3)}{3}\, x^{8m+5} + O(x^{8m+6}).$$

Hence f_{a^*} has a weak fixed point of order $d - 2 = 4m + 1$ as we wanted to show.

4 Poincaré maps and 2-cyclicity

Locally orientation reversing diffeomorfisms appear naturally when studying *the Poincaré maps* associated to the origin of planar differential systems of the form

$$\begin{cases} \dot{x} = -y + P(x, y), \\ \dot{y} = x + Q(x, y), \end{cases} \tag{14}$$

where P and Q are analytic functions starting with at least second order terms. It is well known that the origin of the above system is *monodromic*, i.e. there is a well defined associated Poincaré map. In this situation, using polar coordinates $r^2 = x^2 + y^2$ and

$\theta = \arctan(y/x)$ the solution of (14) that passes through the point $(x, 0)$ with $x > 0$ small enough can be expressed by $r(\theta; x) = x + \sum_{i\geq 2} a_i(\theta)x^i$, and the Poincaré map is given by $\Pi(x) = r(2\pi; x)$.

Let $\Pi_+(x)$ be the map defined over an interval $(0, \epsilon) \subset \mathbb{R}^+$ given by $\Pi_+ : (0, \epsilon) \to \mathbb{R}^-$ where $(0, \epsilon)$ is on the semi-axis OX^+, such that it gives the first intersection, in positive time, of the orbit that at time $t = 0$ passes through the point $(x, 0)$. We call this map the *half-return map.* In [4] it is proved that $\Pi_+(x) = -r(\pi; x)$, hence it is of the form (1). As can be seen in this reference, $\Pi_+(x)$ has an analytic extension to $\mathbb{R}$. Using this analytic extension, the authors prove that $\Pi = \Pi_+ \circ \Pi_+$.

It is clear, then, that given a parametric family of vector fields of the form

$$X_a(x, y) = (-y + P_a(x, y)) \frac{\partial}{\partial x} + (x + Q_a(x, y)) \frac{\partial}{\partial y},$$

with $a \in \mathbb{R}^n$ and $P_a(x, y)$ and $Q_a(x, y)$ starting with second order terms, the cyclicity of X_a (that is, the number of small amplitude limit cycles of the differential equation associated to X_a) is exactly the cyclicity of the associated family of maps $\Pi_{+,a}(x)$ (the number of 2-periodic orbits). Conversely, observe that the following result proves that any given map of the form (1) is conjugate with the corresponding half-return map of a polynomial vector field.

Proposition 10. *Given an analytic map with $f(0) = 0$ and $f'(0) = -1$, there exists a polynomial vector field of the form* (14) *such that $f(x)$ is locally $\mathcal{C}^\infty$-conjugate to the half-return map $\Pi_+(x)$ of the vector field.*

Proof. Suppose that f is an involution. By the Bochner linearization Theorem [15], the local diffeomorphism $\psi(x) = x - f(x)$ conjugates f with the linear map $L(x) = -x$ (it is straightforward to check that $\psi \circ f = L \circ \psi$). Hence, f is analytically conjugate with the half-return map of a the linear center

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x. \end{cases}$$

Suppose now, that f is not an involution. Following [17], there exists a local $\mathcal{C}^\infty$-diffeomorphism φ_1, that conjugates f with its $\mathcal{C}^\infty$-normal form

$$f_N(x) = -x + \sigma x^{2\ell+1} + c x^{4\ell+1},$$

where $\sigma = \pm 1$.

Consider the polynomial vector field given by

$$\begin{cases} \dot{x} = -y + x\left(\delta(x^2 + y^2)^{2\ell} + \gamma(x^2 + y^2)^{4\ell}\right), \\ \dot{y} = x + y\left(\delta(x^2 + y^2)^{2\ell} + \gamma(x^2 + y^2)^{4\ell}\right), \end{cases} \tag{15}$$

with $\delta=-\sigma/\pi$ and $\gamma=-(c+(2\ell+1)\sigma^2/2)/\pi$. We claim that, using the notation introduced above,

$$\Pi_+(x)=-r(\pi;x)=-x+\sigma x^{2\ell+1}+cx^{4\ell+1}+O(x^{4\ell+2}), \tag{16}$$

and therefore, there exists a $\mathcal{C}^\infty$-diffeomorphism φ_2, that conjugates Π_+ with f_N. In consequence,

$$f_N=\varphi_1^{-1}\circ f\circ\varphi_1 \text{ and } f_N=\varphi_2^{-1}\circ\Pi_+\circ\varphi_2,$$

so

$$f=(\varphi_1\circ\varphi_2^{-1})\circ\Pi_+\circ(\varphi_2\circ\varphi_1^{-1})$$

and f is conjugate with Π_+. To finish, we prove (16). We apply similar arguments than the ones used in the proof of Lemma 2.7 in [17].

Observe that the system (15) has the associated polar equation

$$\dot r=\delta r^{2\ell+1}+\gamma r^{4\ell+1}, \tag{17}$$

with analytic solution $r(\theta;x)=\sum_{i\geq 1}a_i(\theta)x^i$. By substituting this expression in (17), taking into account that $r(0;x)=x$, and comparing powers we obtain that $a_i'(\theta)=0$ for all $i=1,\ldots,2\ell$, so $a_1(\theta)\equiv 1$ and $a_i(\theta)\equiv 0$ for all $i=2,\ldots,2\ell$. Applying the same argument we have

$$\begin{aligned}\sum_{i\geq 2\ell+1}a_i'(\theta)x^i &= \delta\left(x+\sum_{i\geq 2\ell+1}a_i(\theta)x^i\right)^{2\ell+1}+\gamma\left(x+\sum_{i\geq 2\ell+1}a_i(\theta)x^i\right)^{4\ell+1}\\ &= \delta x^{2\ell+1}+\left(\delta(2\ell+1)a_{2\ell+1}(\theta)+\gamma\right)x^{4\ell+1}+O(x^{4\ell+2}).\end{aligned}$$

Hence, comparing powers, integrating term by term, and using again that $r(0;x)=x$ we have that $a_{2\ell+1}(\theta)=\delta\,\theta$, $a_i(\theta)\equiv 0$ for all $i=2\ell+2,\ldots,4\ell$, and $a_{4\ell+1}(\theta)=\gamma\,\theta+\delta^2(2\ell+1)\theta^2/2$. The result follows using that $\delta=-\sigma/\pi$ and $\gamma=-(c+(2\ell+1)\sigma^2/2)/\pi$. ∎

The result above establishes that each map (1) is conjugate to the half-return map of a polynomial vector field, but we remark that given a map (1) it is not easy to prove that *it is* the half-return map of a polynomial vector field.

We end the paper showing that there exist families of type (2) with a single parameter having cyclicity k for any $k\in\mathbb{N}$. This is a consequence of the results in [7]. Indeed, in this reference it is shown that for any $k\in\mathbb{N}$ there exists a suitable choice of fixed values of $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$, such that the one-parametric family of vector fields with associated differential system

$$\begin{cases}\dot x=-y+x(x^2+y^2)\left(a^k+\alpha_1a^{k-1}r^2+\cdots+\alpha_{k-1}ar^{2(k-1)}+\alpha_kr^{2k}\right),\\ \dot y=x+y(x^2+y^2)\left(a^k+\alpha_1a^{k-1}r^2+\cdots+\alpha_{k-1}ar^{2(k-1)}+\alpha_kr^{2k}\right),\end{cases}$$

with $r^2 = x^2 + y^2$, has cyclicity k and, in consequence the one-parametric family of locally orientation reversing analytic diffeomeorphisms $\Pi_{+,a}$ also has cyclicity k.

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