On the number of polynomial solutions of Bernoulli and Abel polynomial differential equations

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Abstract. In this paper we determine the maximum number of polynomial solutions of Bernoulli differential equations and of some integrable polynomial Abel differential equations. As far as we know, the tools used to prove our results have not been utilized before for studying this type of questions. We show that the addressed problems can be reduced to know the number of polynomial solutions of a related polynomial equation of arbitrary degree. Then we approach to these equations either applying several tools developed to study extended Fermat problems for polynomial equations, or reducing the question to the computation of the genus of some associated planar algebraic curves.

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1 Introduction

In this work we investigate the number of polynomial solutions of some differential equations of type

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_{n-1}(t)\,x^{n-1} + \cdots + p_1(t)\,x + p_0(t) \tag{1}$$

with q and p_i polynomials in real or complex coefficients for $i = 0, 1, 2, \ldots, n$, and $p_n(t) \not\equiv 0$. More specifically, we consider the real or complex Bernoulli equation ($p_{n-1} = p_{n-2} = \cdots = p_1 = 0$) and some special real Abel equations ($n = 3$) that will be fixed below.

There are several previous works asking for polynomial solutions of equation (1) for some values of n.

When $n = 2$, equation (1) is the well-known polynomial Riccati equation. In 1936, Rainville proved the existence of one or two polynomial solutions when $q(t) = 1$, see [20]. After, in the papers [7, 8] the authors presented some criteria determining the degree of polynomial solutions of $q(t)\,\dot{x} = p_2(t)\,x^2 + p_1(t)\,x + p_0(t)$ and show examples of these equations with 4 or 5 polynomial solutions. For them, in [11] the authors gave a complete answer: polynomial Riccati equations have at most $N + 1$ (resp. 2) polynomial solutions when $N \geq 1$ (resp. $N = 0$), where N is the maximum degree of $q(t), p_0(t), p_1(t), p_2(t)$; moreover, there are equations of this type having any number of polynomial solutions smaller than or equal to these upper bounds.

Also in [2, 3, 4] the degrees of the polynomial solutions of (1) are studied. In this setting in [14] it is shown that the degree of the polynomial solutions of (1) has to belong to a particular set of integers depending on the degrees of the coefficients. Finally, in [12] it is proved that equation (1) with $q = 1$ has at most n polynomial solutions and that this bound is sharp.

Notice that the question we are interested in is also reminiscent of a similar one proposed by Poincaré about the number and degree of the algebraic solutions of planar autonomous polynomial differential systems in terms of their degrees.

Our first result solves completely the problem for Bernoulli equations. It is not difficult to prove that linear equations have 0, 1 or all its solutions being polynomials. For instance the equation (2) with $n = 0$, $\dot{x} = t$, has the solutions $x = t^2/2 + c, c \in \mathbb{C}$. As we have already explained, the case $n = 2$, is solved in [11]. We include it in next theorem for the sake of completeness.

Theorem A. *Consider Bernoulli equations*

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_1(t)\,x, \tag{2}$$

with $q, p_n, p_1 \in \mathbb{C}[t]$ *and* $p_n(t) \not\equiv 0$. *Then:*

(i) For $n = 2$, *equation* (2) *has at most* $N+1$ *(resp. 2) polynomial solutions, where* $N \geq 1$ *(resp.* $N = 0$*) is the maximum degree of* q, p_2, p_1, *and these upper bounds are sharp. Moreover, when* $q, p_2, p_1 \in \mathbb{R}[t]$ *these upper bounds are reached with real polynomial solutions.*

(ii) For $n = 3$, *equation* (2) *has at most seven polynomial solutions and this upper bound is sharp. Moreover, when* $q, p_3, p_1 \in \mathbb{R}[t]$ *this upper bound is reached with seven polynomial solutions belonging to* $\mathbb{R}[t]$.

(iii) For $n \geq 4$, *equation* (2) *has at most* $2n - 1$ *polynomial solutions and this upper bound is sharp. Moreover, when* $q, p_n, p_1 \in \mathbb{R}[t]$ *it has at most three real polynomial solutions when* n *is even while it has at most five real polynomial solutions when* n *is odd, and both upper bounds are sharp.*

Notice also, that in general, given $n+1$ arbitrary polynomials $x_0, x_2, \ldots, x_n$ there exists always an equation of the form (1) having these solutions as particular solutions. To get this differential equation it suffices to plug them in the equation (1) with $q = 1$ and solve the linear system with $n + 1$ unknowns $p_n, p_{n-1}, \ldots, p_0$. Solving it we obtain a rational differential equation. Multiplying this equation by the least common multiple of all the denominators of the p_j, we obtain the desired polynomial differential equation. So, for general n, the problem is to know if there are equations (1) with more that $n + 1$ polynomial solutions. In particular, for Abel differential equations, we are interested in differential equations with at least five polynomials solutions.

Because of the difficulties that we have found to deal with the general Abel equation, in next result we fix our attention on real Abel differential equations, having at least three real polynomial solutions, that also have a very specific relative position in the space of polynomials: *they are collinear*. As we will see along the proof of next theorem, this geometric hypothesis implies that the Abel equation is *integrable*.

Theorem B. *If equation*

$$q(t)\,\dot{x} = p_3(t)\,x^3 + p_2(t)\,x^2 + p_1(t)\,x + p_0(t), \tag{3}$$

with coefficients in $\mathbb{R}[t]$ and $p_3(t) \not\equiv 0$, has three real polynomial solutions which are collinear then it has at most seven polynomial solutions and in this case one of the collinear solutions is the arithmetic average of the other two and the equation reduces to a Bernoulli equation with polynomial coefficients, as the one studied in item (ii) of Theorem A. If this relation between the three collinear solutions does not hold then equation (3) has at most six polynomial solutions and this upper bound is sharp.

Remark 1.1 Let x_1, x_2 and x_3 be the collinear solutions of (3) given in Theorem B. The case when one of the solutions is the arithmetic average of the other two can be described by the equation $x_3 - x_2 = x_2 - x_1$ where x_2 is the solution between the solutions x_1 and x_3. As we will see in the proof of the above Theorem when the three polynomial collinear solutions are not symmetric, that is none of them is the arithmetic average of the other two, and the equation has six polynomial solutions, then necessarily $x_3 - x_2 = 2(x_2 - x_1)$. In all other cases the equation has at most five polynomial solutions and this upper bound is again sharp. $\square$

In all the paper, given a polynomial $p \in \mathbb{C}[t]$ we will denote by $\delta(p)$ its degree and by $Z(p)$ the number of different complex zeroes of p, without counting their multiplicities. Moreover, given m polynomials $p_1, p_2, \ldots, p_m \in \mathbb{C}[t]$, $(p_1, p_2, \ldots, p_m)$ denotes their greatest common divisor.

As we will see, although the starting point for proving Theorems A and B is different, in both cases we will reduce the study of the polynomials solutions of the differential equation to the study of the polynomial solutions of a related polynomial equation with few monomials but arbitrarily high degree. More concretely, the equations that we will have to study will be

$$p^k + q^k = r^k, \quad \text{(Fermat equation)} \tag{4}$$

$$p^k - Mq^k = r^k - Ls^k, \tag{5}$$

$$(p^k - q^k)(r - s) = (r^k - s^k)(p - q), \tag{6}$$

where $p, q, r, s \in \mathbb{C}[t]$, $L, M \in \mathbb{C}$ and $0 < k \in \mathbb{N}$. The first equation will appear in the proof of Theorem A and the other two in the proof of Theorem B.

All these equations will be treated by using some results developed in [6] that we introduce in next section and some new related results (see for instance Theorem 2.1). These results extend the beautiful Fermat Theorem for polynomials, that deals with the first equation. We state here it for the sake of completeness.

Clearly, for all k, the Fermat equation (4) has the solutions $q = \alpha\, p$, $r = \beta\, p$ with $\alpha, \beta \in \mathbb{C}$ and $1 + \alpha^k = \beta^k$. It is said that two polynomials p_1 and p_2 are *similar* if p_1 and p_2 are linearly dependent.

Theorem 1.2. (Fermat Theorem for polynomials) *Equation* (4) *has polynomial solutions, not pairwise similar, if and only if $k = 1$ or $k = 2$.*

Notice that the above theorem implies that (4) has no non-trivial solutions when $k > 2$. Using the quoted results we will prove similar results for the other two equations. Equation (5) has no non-trivial solutions when $k > 7$ (see the proof of Proposition 5.1), and (6) has no non-trivial solutions when $k > 83$ (see Proposition 5.3).

The proof of Theorem 1.2 that we have found in the literature relies on a result, interesting by itself, called the "abc Theorem" for polynomials. It states that if a, b, c are pairwise coprime non-constant polynomials for which $a + b = c$, then the degree of each of these three polynomials cannot exceed $Z(a\,b\,c) - 1$. The "abc Theorem" for polynomials (also known as Mason's Theorem), was proved in 1981 by Stothers [23] and also later by Mason [16] and Silverman [22].

In the next section we give another proof of Theorem 1.2, based on the computation of the genus of a planar algebraic curve associated to (4). The key point will be that only curves of genus 0 can be rationally parameterized, see [13, 15]. The reason for introducing this proof of a known result is that the same idea will be used in several parts of the paper for studying the remainder low degree cases of equations (5) and (6).

In fact, the study of polynomial solutions of each of these equations, combined with other polynomial relations appearing in our approach, will derive the problem to know when the genus vanishes for the irreducible components of two families of algebraic curves, with polynomial unknowns $(u, v) \in \mathbb{C}^2$:

$$F(u, v) = v^n - u^n - (Lv^{n+m} - Mu^{n+m}) + u^n v^n (Lv^m - Mu^m),$$

with $n, m > 0$, $(n, m) = 1$, $L, M \in \mathbb{R}$, $L \neq 0 \neq M$, $M^n \neq 1 \neq L^n$ and $2 < k = n + m \leq 7$; and

$$G(u, v) = v^n - u^n - (v^{n+m} - u^{n+m}) + u^n v^n (v^m - u^m) = (u - v)(u - 1)(v - 1)P(u, v),$$

with $n, m > 0$, $n < m$, $(n, m) = 1$ and $2 < k = n + m \leq 83$.

The results for $F = 0$ correspond to Propositions 4.1 and 4.2. Because, for each n and m fixed, F is a 2-parameter family of algebraic curves with parameters $(L, M) \in \mathbb{R}^2$, but of low degree, we can prove its irreducibility by using a two steps procedure. First we find all its singular points (finite and infinite), studying the system

$$F(u, v) = 0, \quad \frac{\partial F}{\partial u}(u, v) = 0, \quad \frac{\partial F}{\partial v}(u, v) = 0,$$

and the corresponding ones in the other charts of the complex projective space, by using the resultants approach. Afterwards we develop an ad hoc method to prove the irreducibility of F. This method uses Bezout's Theorem and also computes some bounds of the intersection numbers at the found singularities.

The results for $G = 0$ (indeed for its component $P = 0$) are obtained in a different way. The main reason is that although the only parameters in G are the degrees, they arrive until 83 and the computational difficulty increases with the degree. In our computers, it is not possible to effectively obtain for $n + m$ big, neither the corresponding resultants needed to solve

$$P(u, v) = 0, \quad \frac{\partial P}{\partial u}(u, v) = 0, \quad \frac{\partial P}{\partial v}(u, v) = 0,$$

nor to solve directly the systems, by using the computer algebra system `Maple`. Instead we have to apply to our computations some Grôbner basis tools detailed in the proof of Proposition 4.3. Similarly, to prove that $P(u, v) = 0$ are irreducible curves we apply the algorithm proposed in [19], and also implemented in `Maple`, based on reducing the problem modulo some prime numbers, see again the proof of Proposition 4.3 for more details.

In short, the paper is organized as follows: In the next Section we introduce and prove some Fermat type results for polynomial equations. Section 3 is devoted to prove Theorem A using Fermat Theorem for polynomials. In Section 4 we compute the genus of several families of planar algebraic curves and, finally, Theorem B is proved in Section 5.

2 Generalized Fermat type theorems for polynomials

We start this section with our proof of Fermat Theorem for polynomials and with an extension when $k=2$, that will be the key step for studying Bernoulli equation with $n=3$.

A proof of Theorem 1.2. Assume without loss of generality that p is not a constant. Then since

$$1+\left(\frac{q}{p}\right)^k=\left(\frac{r}{p}\right)^k$$

we get that the algebraic curve $P(u,v):=1+u^k-v^k=0$ has a rational parametrization. On the other hand, it is easily seen that $P(u,v)=0$ has no singularities because the system

$$P(u,v)=0,\quad \frac{\partial}{\partial u}P(u,v)=0,\quad \frac{\partial}{\partial v}P(u,v)=0,$$

has no solutions. This implies that it is irreducible. Then, in this case its genus $g(P)$, only depends on its degree([10]) and is

$$g(P)=\frac{(k-1)(k-2)}{2}.$$

Since it is well-known that the only algebraic curves with rational parameterizations have genus 0, see [13], the fact that $k=1$ or $k=2$ follows.

The existence of many polynomial solutions when $k=1$ is trivial. For $k=2$ it is well known that (4) has non-trivial polynomial solutions and they can be expressed similarly to the ones of Fermat equation on the integers, that is,

$$p=2\,A\,B\,C,\quad q=C(A^2-B^2),\quad r=C(A^2+B^2),$$

for arbitrary polynomials A,B and C. $\square$

When $k=2$ and we prove a new result, related with the above one.

Theorem 2.1. *Let $p,q\in\mathbb{C}[t]$ be not similar polynomials such that*

$$p^2+q^2=r^2,\quad p^2+\varepsilon^2\,q^2=s^2,\quad \textit{with}\quad r,s\in\mathbb{C}[t],\quad \varepsilon\in\mathbb{C}. \tag{7}$$

Then $\varepsilon=0$ or $\varepsilon^2=1$.

Proof. First of all note that if p,q,r,s and ε satisfy equations (7) then $p^2=r^2-q^2=(r+q)\,(r-q)$ and $p^2=s^2-\varepsilon^2\,q^2=(s+\varepsilon\,q)(s-\varepsilon\,q)$. If $(p,q)=1$ denoting $A:=r+q$, $B:=r-q$, $C:=s+\varepsilon\,q$, $D:=s-\varepsilon\,q$ we get that $(A,B)=(C,D)=1$ and therefore A,B,C,D are perfect squares. Also

$$\varepsilon(A-B)\,=\,C-D. \tag{8}$$

The proof will be given by an induction process on the degree of p. We assume that $\varepsilon\neq 0$ and we are going to prove that $\varepsilon^2=1$. First of all note that the degree of p can not be 0 because this fact will imply that $p^2 = r^2 - q^2 = (r+q)\,(r-q)$ has degree 0 and hence the same holds for r and q contradicting the hypotheses.

Assume that the degree of p, $\delta(p) = 1$. If $(p,q) \neq 1$ we get $(p,q) = p$ and dividing the first equation of (7) by p^2 we obtain a new set of polynomials $\widetilde{p}, \widetilde{q}, \widetilde{r}$ satisfying the first equation of (7) with degree of $\widetilde{p}$ equals to zero; a contradiction. Then $(p,q) = 1$ and the decomposition $p^2 = A\,B$ must be $p^2 = \lambda\,\frac{p^2}{\lambda}$ with $0 \neq \lambda \in \mathbb{C}$ and $p^2 = C\,D$ is $p^2 = \mu\,\frac{p^2}{\mu}$ with $0 \neq \mu \in \mathbb{C}$. Then, from (8) we get that $\varepsilon\left(\lambda - \frac{p^2}{\lambda}\right) = \pm\left(\mu - \frac{p^2}{\mu}\right)$ which implies that $\varepsilon\,\lambda \mp \mu = \left(\frac{\varepsilon}{\lambda} \mp \frac{1}{\mu}\right) p^2$. Hence, $\varepsilon\,\lambda \mp \mu = 0$ and $\frac{\varepsilon}{\lambda} \mp \frac{1}{\mu} = 0$, which implies $\mu = \pm\varepsilon\lambda$ and $\varepsilon^2 = 1$. The proof is done when $\delta(p) = 1$.

Now assume that the result is true for all p with $\delta(p) \leq n-1$ and that $\delta(p) = n$.

If $M := (p,q)$ has degree greater than zero, then calling:

$$\bar{p} = \frac{p}{M}\,,\ \bar{q} = \frac{q}{M}\,,\ \bar{r} = \frac{r}{M}\,,\ \bar{s} = \frac{s}{M},$$

we have that

$$\bar{p}^2 + \bar{q}^2 = \bar{r}^2 \text{ and } \bar{p}^2 + \varepsilon^2\,\bar{q}^2 = \bar{s}^2,$$

and since $\delta(\bar{p}) < n$, by the induction hypothesis we get that $\varepsilon^2 = 1$.

So, from now on we are going to assume that $(p,q) = 1$. Notice that $\delta(A) + \delta(B) = 2\,\delta(p)$ and also $\delta(C) + \delta(D) = 2\,\delta(p)$. We are going to consider four different cases:

- Case 1. Assume that at least one over A, B, C, D has degree strictly between 0 and n. For instance, assume that $0 < \delta(A) < n$. Then (A,C) and (A,D) have degree less than n and at least one of them has positive degree (it is so because since $A\,B = C\,D$, the irreducible components of A have to be in C or D). Then, denote (A,C) by A^* and assume that assume that $0 < \delta(A^*) < n$. Also denote (B,D) by B^*. We observe that $(A^*, B^*) = 1$. From $A\,B = C\,D$ again we get that there exist two polynomials a, b and $0 \neq \lambda \in \mathbb{C}$ such that $A = A^*\,a$, $B = B^*\,b$, $C = \lambda\,b\,A^*$ and $D = \frac{1}{\lambda}\,a\,B^*$. Note also that $(a,b) = 1$. Now, from (8) we have that

  $$(\lambda\,b - \varepsilon\,a)\ A^* = \left(\frac{a}{\lambda} - \varepsilon\,b\right)\ B^*. \tag{9}$$

  Since $(a,b) = 1$ also $\left(\lambda\,b - \varepsilon\,a, \frac{a}{\lambda} - \varepsilon\,b\right) = 1$ and from (9) it must exist a complex number β such that $\lambda\,b - \varepsilon\,a = \beta\,B^*$ and $\frac{a}{\lambda} - \varepsilon\,b = \beta\,A^*$. Assume now $1 - \varepsilon^2 \neq 0$, we get

  $$a = \frac{(\lambda\,A^* + \varepsilon\,B^*)\,\beta}{1-\varepsilon^2} \quad \text{and} \quad b = \frac{(\varepsilon\,A^* + \frac{1}{\lambda}\,B^*)\,\beta}{1-\varepsilon^2}.$$

  We observe that since $p^2 = A^*\,B^*\,a\,b$ and any pair among A^*, B^*, a, b have no common divisors we have that A^*, B^*, a, b all are perfect squares. Hence, $A^* + \frac{\varepsilon}{\lambda}\,B^*$, $A^* + \frac{1}{\lambda\varepsilon}\,B^*$ are also perfect squares. But since $0 < \delta(A^*) < n$ and $(A^*, B^*) = 1$ by the induction hypothesis we get $\varepsilon^2 = 1$; a contradiction.

- Case 2. Assume that the degrees of A and B are $2n$ and 0 respectively and the degrees of C and D are $2n$ and 0 respectively. This case works as the case in which p has degree one: the decompositions $p^2 = A\,B = \frac{p^2}{\lambda}\,\lambda$ and $p^2 = C\,D = \frac{p^2}{\mu}\,\mu$ for certain non-zero $\lambda, \mu \in \mathbb{C}$, using (8) give $\mu = \lambda\,\varepsilon$ and $\varepsilon^2 = 1$.

- Case 3. Assume that the degrees of A and B are $2n$ and 0 respectively and $\delta(C) = \delta(D) = n$. It would imply that $\delta(A - B) = 2\,n$ and $\delta(C - D) \leq n$. But it is not consistent with (8).

- Case 4. Assume that the degrees of A, B, C, D are all equal n. Calling $A^* = (A, C)$ and $B^* = (B, D)$ as before, if $\delta(A^*) = n = \delta(B^*)$ then $A = \lambda\, C$, $B = \frac{1}{\lambda}\, D$, with $0 \neq \lambda \in \mathbb{C}$, and from this, one can easily obtain $\varepsilon^2 = 1$. So we can assume that $\delta(A^*) < n$. If $\delta(A^*) > 0$ then the proof follows as in Case 1. If $\delta(A^*) = 0$, then $(A, D) = A$ and $(B, C) = B$ and the proof follows as in the previous subcase $\delta(A^*) = n = \delta(B^*)$

$\square$

Several generalizations of Mason's Theorem and extensions of Fermat Theorem for polynomials have appeared in [1, 5, 6, 17, 18, 21]. Next we state the one dimensional version of two of these extensions, proved in [6] for polynomials in several variables, that we will use in the proof of Theorem B.

Theorem 2.2. *Let $g_1, \ldots, g_n \in \mathbb{C}[t]$ be not all zero, satisfying*

$$g_1^d + g_2^d + \cdots + g_n^d = 0, \quad \textit{with} \quad d \in \mathbb{N},$$

and suppose that $d \geq n\,(n-2)$. Then the vanishing sum $g_1^d + g_2^d + \cdots + g_n^d = 0$ decomposes into vanishing subsums $g_{i_1}^d + g_{i_2}^d + \cdots + g_{i_s}^d = 0$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, for which all g_{i_j} are pairwise similar.

Theorem 2.3. *Set $n \geq 3$ and let $f_1, \ldots, f_n \in \mathbb{C}[t]$ be not all constant, such that*

$$f_1 + f_2 + \ldots + f_n = 0.$$

Assume furthermore that for all $1 \leq i_1 < i_2 < \ldots < i_s \leq n$, $f_{i_1} + f_{i_2} + \ldots + f_{i_s} = 0$ implies $(f_{i_1}, f_{i_2}, \ldots, f_{i_s}) = 1$. Then for all $i \in \{1, \ldots, n\}$ we get

$$\delta(f_i) \leq \frac{(n-1)(n-2)}{2}\big(Z(f_1 f_2 \ldots . f_n) - 1\big).$$

3 Proof of Theorem A

(i) This result is proved in [11]. For instance, an equation with $N + 1$ polynomials solutions is

$$q(t)\dot{x} = x^2 + q'(t)x, \quad q(t) = \prod_{j=1}^{N}(t - j),$$

because all its solutions are $x = 0$ and $x_c(t) = -\frac{q(t)}{t-c}$, and its $N+1$ polynomial solutions are $x = 0$ and $x_j(t), j = 1, 2, \ldots, N$.

(ii) Consider equation (2) with $n = 3$,

$$q(t)\,\dot{x} = p_3(t)\,x^3 + p_1(t)\,x. \tag{10}$$

Performing the change $u(t) = x^2(t)$, it is transformed into

$$q(t)\,\dot u = 2\,p_3(t)\,u^2 + 2\,p_1(t)\,u, \tag{11}$$

a Ricatti equation. We are interested in polynomial solutions of (11) which are perfect squares. Suposse that $v(t), w(t) \in \mathbb{C}[t]$ are such that $v^2(t)$ and $w^2(t)$ are solutions of equation (11). Either solving directly the differential equation or by using the fact that the cross ratio of four solutions of the Ricatti equation is constant in time it follows that, any solution $u(t)$ of (11), different of $u = 0$, is of the form

$$u(t) \;=\; \frac{v^2(t)\,w^2(t)}{c\,v^2(t) + (1-c)\,w^2(t)}\;,\; c \in \mathbb{C}.$$

Hence any other polynomial solution of (11) being a perfect square is determined for a value of $c \in \mathbb{C}$ such that $c\,v^2(t) + (1-c)\,w^2(t)$ also is a perfect square, say $z^2(t)$. If the two solutions $v^2(t)$ and $w^2(t)$ are similar polynomials then $p_3 = 0$. Hence, by the hypotheses this possibility is excluded. Then, we can apply Proposition 2.1 to the equation $c\,v^2(t) + (1-c)\,w^2(t) = z^2(t)$, which assures that this can only happen for a unique value of $c \notin \{0,1\}$. This fact implies that equation (10) has at most seven polynomial solutions.

In order to find an equation of type (10) with seven polynomial solutions we look for an equation $\dot u = a(t)u^2 + b(t)u$ with $a(t)$ and $b(t)$ rational functions with three perfect squares polynomial solutions. If x_1^2 and x_2^2 are two of them, then the third one must be

$$x_3^2 := \frac{x_1^2 \cdot x_2^2}{c\,x_1^2 + (1-c)\,x_2^2}$$

for some value of c. Since we are interested in polynomial solutions we consider:

$$x_1 = \frac{r(t)\,s(t)\,(r(t)^2 + s(t)^2)}{2\sqrt{c}} \quad \text{and} \quad x_2 = \frac{r(t)\,s(t)\,(r(t)^2 - s(t)^2)}{2\sqrt{c-1}},$$

because in this way

$$x_3^2 = \frac{(r(t)^2 + s(t)^2)^2\,(r(t)^2 - s(t)^2)^2}{16\,c(c-1)}$$

and it also is a perfect square polynomial.

As we have explained in the introduction, given two arbitrary polynomial functions $r(t)$ and $s(t)$ we can find two rational functions $a(t)$ and $b(t)$, such that the differential equation $\dot u = a(t)u^2 + b(t)u$ has x_1^2, x_2^2 as solutions. Then taking $r(t) = t$, $s(t) = 1$ and $c = 25/16$ we find that the differential equation

$$4\,t\,(t^2+1)\,(t^2-1)\,(t^2-4)\,(4\,t^2-1)\,\dot x = 225\,x^3 + 16\,(3\,t^8 - 17\,t^6 + 6\,t^4 - 1)\,x$$

has the seven polynomial solutions, $x = 0$ and

$$x_1^{\pm}(t) = \pm\frac{2}{5}\,t\,(t^2+1), \quad x_2^{\pm}(t) = \pm\frac{2}{3}\,t\,(t^2-1), \quad x_3^{\pm}(t) = \pm\frac{4}{15}\,(t^4-1).$$

(*iii*) First observe that if $x(t)$ is a solution of (2) then $\alpha x(t)$ also is a solution for all $\alpha \in \mathbb{C}$ such that $\alpha^{n-1} = 1$. We perform the change of variable $u = x^{n-1}$ in (2). This equation is transformed into the Riccati equation

$$q(t)\,\dot u = (n-1)\,p_n(t)\,u^2 + (n-1)\,p_1(t)\,u. \tag{12}$$

If v^{n-1} and ω^{n-1} are different solutions of (12) and it exists another solution of type x^{n-1}, then

$$x^{n-1} = \frac{v^{n-1} \cdot \omega^{n-1}}{cv^{n-1} + (1-c)\omega^{n-1}}$$

for some number $c \in \mathbb{C}$. This fact implies that $\left(\sqrt[n-1]{c}\, v\right)^{n-1} + \left(\sqrt[n-1]{1-c}\, \omega\right)^{n-1} = y^{n-1}$ for some polynomial y. From Theorem 1.2 this last equation has no non-similar polynomial solutions for $n \geq 4$ and as a consequence such an x does not exist unless the solutions are similar. This situation is also impossible because $p_n \neq 0$.

Hence, apart of the solution $x = 0$, there can be only two sets of solutions for equation (2):

$$x_1, \alpha x_1, \alpha^2 x_1, \ldots, \alpha^{n-2} x_1 \quad \text{and} \quad x_2, \alpha x_2, \alpha^2 x_2, \ldots, \alpha^{n-2} x_2,$$

where α is a $(n-1)$-primitive root of unity. Therefore, for $n \geq 4$, equation (2) has most $2n-1$ polynomials solutions.

It is easy to find examples with this number of polynomial solutions. Imposing that t and t^2 are solutions we find the equation:

$$(t^{2n-1} - t^n)\, x' = x^n + (t^{2n-2} - 2\, t^{n-1})\, x$$

which has the solutions 0, $\alpha\, t$ and $\alpha\, t^2$ for each α satisfying $\alpha^{n-1} = 1$.

The real case follows from the fact that when n is even $\alpha^{n-1} = 1$ has only the real solution $\alpha = 1$ while it has the solutions $\alpha = \pm 1$ when n is odd.

4 The genus of some algebraic curves

To prove Theorem B we will use similar arguments as in our proof of Theorem 1.2. As we will see we will need to know the genus of some algebraic curves.

Proposition 4.1. *Consider the polynomial*

$$F(u,v) = v^n - u^n - (Lv^{n+m} - Mu^{n+m}) + Lu^n v^{n+m} - Mv^n u^{n+m}$$

with $L \neq 0 \neq M$, $(n,m) = 1$ and $2 < n+m \leq 7$. If $M^n \neq 1 \neq L^n$ and $L^n \neq M^n$ then $F(u,v)$ is irreducible. Furthermore, the genus of F is

$$g(F) = \frac{(2n+m-1)(2n+m-2)}{2} - \frac{3n(n-1)}{2}.$$

Proof. We start finding the singular points of F, and their multiplicities in the projective complex plane. We consider homogeneous coordinates (u,v,w). Then the curve $F=0$ always pass trough the points $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and these points are singular if $n > 1$. In fact, in the charts $U_1 = \{u \neq 0\}$, $U_2 = \{v \neq 0\}$, $U_3 = \{w \neq 0\}$ the local coordinates are (v,w), (u,w) and (u,v) respectively and the expressions of F near the singular points are:

$$F_1(v,w) := M(w^n - v^n) + Lv^{n+m} - w^{n+m} + v^n w^{n+m} - Lw^n v^{n+m}, \tag{13}$$

$$F_2(u,w) := L(u^n - w^n) + w^{n+m} - Mu^{n+m} + Mw^n u^{n+m} - u^n w^{n+m}, \tag{14}$$

$$F(u,v) = v^n - u^n + Mu^{n+m} - Lv^{n+m} + Lu^n v^{n+m} - Mv^n u^{n+m}. \tag{15}$$

From these expressions we see that each one of the points e_1, e_2 and e_3 are points of F with multiplicity n, and hence they are singular if $n > 1$.

In fact, we will prove that when the hypothesis on L, M are satisfied and $n > 1$, then e_1, e_2, e_3 are the only singularities of F.

Singular points on the affine plane have to satisfy $F(u,v) = 0$ and $\frac{\partial F}{\partial u}(u,v) = 0\,,\ \frac{\partial F}{\partial v}(u,v) = 0$. It is easy to see that $(u,v) = (0,0)$ is the unique singular point with $u = 0$. Then we are interested in to know which relations have to satisfy L and M in order to get more singularities. To this end we consider each one of the cases n, m with $n + m \leq 7$ and n, m relatively primes. We begin by calculating the resultant between F and $\frac{\partial F}{\partial u}$ in respect to v, and also compute the resultant between F and $\frac{\partial F}{\partial v}$ in respect to v, getting two polynomials R_1, R_2 depending on u, L, M. These two polynomials vanish when $u = 0$, hence we divide them for the common power of u getting $\widetilde{R_1}$ and $\widetilde{R_2}$. Finally we consider the resultant between $\widetilde{R_1}$ and $\widetilde{R_2}$ in respect to u which is a polynomial R_3 depending on L, M. For instance for $n = 3, m = 1$ this last polynomial is a constant multiplied by $L^{37}\, M^{24}\, (M^3 - 1)^{28}\, (L^3 - M^3)^{16}\, (L^3 - 1)^{21}$. From the properties of the resultant we have that when $R_3(L,M) \neq 0$ the system of equations $F = 0, \frac{\partial F}{\partial u} = 0, \frac{\partial F}{\partial v} = 0$ has no solutions with $u \neq 0$. Analogue calculations on the other local charts let us to say that when the hypothesis on L, M are satisfied, the only singular points of F are e_1, e_2 and e_3. The same result holds for all the other cases.

Now we prove the irreducibility of F. When $n = 1$, since the points e_1, e_2, e_3 are no multiple points of F, the result follows.

For $n \geq 2$, to see that F is irreducible, in order to arrive to a contradiction, we suppose that $F = fg$ for some polynomials of degrees r and $s = 2n + m - r$, for some $1 \leq r \leq 2n + m - 1$. Then from Bezout's Theorem, we know that $\Sigma_{p \in \{f \cap g\}} I(f \cap g, p) = rs$, where $I(f \cap g, p)$ denotes the intersection number of $f \cap g$ at p. Since the points in $f \cap g$ are singular points of F, and we know that e_1, e_2 and e_3 are the unique singularities of F, we only need to compute the numbers $I_i := I(f \cap g, e_i)$ for each $i = 1, 2, 3$. For instance, near the point e_3 which corresponds to the point $(0,0)$ in the usual affine chart, we write $f = f_k + hot$ and $g = g_{n-k} + hot$, for some $k \in \{0, 1, \ldots, n-1, n\}$. Since $f_k g_{n-k} = v^n - u^n = \Pi_{i=0}^{n-1}(v - \alpha^i u)$ with $\alpha^n = 1$ we get that f_k has k factors of $v^n - u^n$ and g_{n-k} has the other ones. Since all the factors of f_k are different from the ones of g_{n-k} we have that $I_3 = k(n-k)$, which implies that $I_3 \leq \left(\frac{n}{2}\right)^2$. The analysis near e_1, e_2 is exactly the same.

Hence, if F is reducible, for some $r \in \{1, 2, \ldots, 2n + m - 1\}$,

$$r(2n + m - r) = I_1 + I_2 + I_3 \leq 3\left(\frac{n}{2}\right)^2. \tag{16}$$

If $r \in \{1, 2, 2n+m-2, 2n+m-1\}$ the above inequality holds and we will arrive to a contradiction by using another approach afterwards. Otherwise, let us prove that (16) does not hold, giving us the desired contradiction.

To prove that when $3 \leq r \leq 2n + m - 3$ the inequality (16) does not hold, first, notice that for $r = 3$, $r(2n + 1 - r) - 3\left(\frac{n}{2}\right)^2 = 3(2n - 2) - 3\left(\frac{n}{2}\right)^2 = \frac{3}{4}\left(-n^2 + 8n - 8\right) > 0$ for $2 \leq n \leq 6$. Hence

$$r(2n + m - r) \geq r(2n + 1 - r) \geq r(2n + 1 - r)\big|_{r=3} > 3\left(\frac{n}{2}\right)^2$$

Therefore, if F is reducible, at least one of its irreducible components has degree one or two. Therefore, without loss of generality we haver that either $r = 1$ or $r = 2$.

Consider first the case $r = 1$. Then $F = fg$ and $\delta(f) = 1$. Then f pass at most for one critical point e_i (otherwise f would be one of the coordinates u, v or w, which is not possible for F). And at this point e_i we have that $I_i = n - 1$. Hence Bezout's Theorem says that $2n + m - 1 = n - 1$, that is $n + m = 0$.

Finally, consider the case $r = 2$. Then $F = fg$, $\delta(f) = 2$ and f is irreducible. Since irreducible conics have not multiple points, if f pass for some e_i then the multiplicity of f at e_i must be one, and hence $I_i = n-1$. This implies that $\Sigma_{j=1}^{3} I_j \in \{n-1, 2\,(n-1), 3\,(n-1)\}$. Since $2\,(2n+m-2) = \Sigma_{j=1}^{3} I_j$ we get the three possibilities $2\,(2n+m-2) = n-1$, $2\,(2n+m-2) = 2n-2$ or $2\,(2n+m-2) = 3n-3$ which are not compatible with $n \geq 2, m \geq 1$.

Hence, in all the cases that we are interested in, F is irreducible. To see that the formula for the genus is the announced in the statement we apply the well-known formula ([10]) that says that the genus of a curve G of degree k is

$$g(G) = \frac{(k-1)(k-2)}{2} - \Sigma_p \frac{m_p(G)(m_p(G)-1)}{2} \tag{17}$$

where $m_p(G)$ is the multiplicity of G at p, provided that near each multiple point p, G has $m_p(G)$ different tangents.

From equations (13), (14) and (15), since F has degree $2n + m$ and the multiplicity of F at each e_i is n, the result follows.

□

Proposition 4.2. *Consider the polynomial*

$$F(u, v) = v^n - u^n - (Lv^{n+m} - Mu^{n+m}) + Lu^n v^{n+m} - Mv^n u^{n+m}$$

with $L \neq 0 \neq M$, $L^n = M^n \neq 1$, $(n, m) = 1$ *and* $2 < n + m \leq 7$. *Then there exists* $\alpha \in \mathbb{C}$ *with* $\alpha^n = 1$ *such that the polynomial* $F(u, v)$ *can be written as* $F(u, v) = (v - \alpha\, u)\, P(u, v)$. *Furthermore* $P(u, v)$ *is irreducible and*

$$g(P) = \frac{(2n+m-2)(2n+m-3)}{2} - \frac{2n(n-1)}{2} - \frac{(n-1)(n-2)}{2}.$$

Proof. A simple calculation proves that $v - \alpha\, u$ is a factor of $F(u, v)$ if and only if $\alpha^n = 1$ and $L = \alpha^m\, M$ which clearly implies that $M^n = L^n$. If $(n, m) = 1$, using the Bezout identity it can be seen that the equality $M^n = L^n$ implies that it exists $\alpha \in \mathbb{C}$ with $\alpha^n = 1$ and $L = \alpha^m M$. This proves that under our hypothesis, if $M^n = L^n$ then $F(u, v)$ has a factor $v - \alpha u$. Considering the change $v = \alpha\, \bar{v}$ we have that the obtained polynomial in $(u, \bar{v})$ has the factor $\bar{v} - u$ and is $F(u, \bar{v})$ with $L = M$. Hence we consider:

$$F(u, v) = v^n - u^n + M(u^{n+m} - v^{n+m}) + Mu^n v^n (v^m - u^m).$$

The proof is similar to the proof of Proposition 4.1. Considering all the cases (n, m) with $n + m \leq 7$ and $(n, m) = 1$, our calculations imply that when $L = M \neq 1$, the singular points of $P(u, v)$ only can be e_1, e_2, e_3. And their multiplicities are $n, n, n - 1$ respectively.

When $n = 1$, $P(u,v)$ has no singular points and hence $P(u,v)$ is irreducible. Using again formula (17) we get that

$$g(P) = \frac{(2n+m-2)(2n+m-3)}{2} = \frac{m(m-1)}{2},$$

as we wanted to prove.

For $n \geq 2$, to see that $P(u,v)$ is irreducible, we suppose, as in the proof of previous proposition, that $P = f\,g$ for some polynomials of degrees r, s with $s = 2n+m-1-r$ and we apply Bezout's Theorem to $f \cap g$, considering $2 \leq n \leq 6$. Now the corresponding equality (16) is

$$r\,(2n+m-1-r) \leq 2\left(\frac{n}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2.$$

Arguing similarly as before we get that if F is reducible, then at least one of its irreducible components has degree one or two. Also these two possibilities can again be discarded for all the values of n, m. Applying once more (17) we obtain the desired result. □

Proposition 4.3. *Consider the algebraic curve* $G_{n,m}(u,v) = (1-u^{n+m})(1-v^n) - (1-v^{n+m})(1-u^n) = 0$ *with* $n, m > 0$. *This curve reduces in the following way*

$$G_{n,m}(u,v) = (u-v)(u-1)(v-1)P_{n,m}(u,v)$$

and when $2 < n+m \leq 83$, $n < m$ *and* $(n,m) = 1$, $P_{n,m}(u,v) = 0$ *is irreducible and has genus*

$$\frac{(2n+m-4)(2n+m-5)}{2} - 3\frac{(n-1)(n-2)}{2}.$$

Proof. The scheme of the proof is similar to the one of previous results but as we have already explained in the introduction, neither the computation of resultants approach, nor the use of the solve tools in `Maple` work for big $n+m$. Instead, for instance, for proving that in the finite plane for $n \geq 3$ the only solution of

$$P(u,v) = 0, \quad \frac{\partial P}{\partial u}(u,v) = 0, \quad \frac{\partial P}{\partial v}(u,v) = 0, \tag{18}$$

where $P = P_{n,m}$, is $(0,0)$, we proceed as follows. First we compute the Grobner basis of the three polynomials given in (18) with the order `plex(u, v)`. Then we solve the new system, usually given by many equations, obtaining that $(0,0)$ is its unique solution.

This method works for all n and m under the hypotheses of the Proposition, but the case $n = 29$, $m = 38$. For this case we use the specific package `algcurves` with the tool `singularities(P, u, v)`, obtaining the same result.

It is not difficult to prove that when $n = 1, 2$ the algebraic curve has no singularities. Moreover, when $n \geq 3$, by using the above result we obtain easily that the only singularities (in the complex projective space) are as in Proposition 4.1, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ (this last one corresponds to the solution of (18)).

Therefore, to apply formula (17) to get the genus of P given in the statement, we only need to prove that this algebraic curve is irreducible. To prove this, there is a very useful result implemented in `Maple`, based on [19]. This result uses the package `algcurves` and is the function `AIrreduc(P)`. This function uses test of irreducibility of P modulo some prime numbers p to know whether a polynomial P is irreducible over $\mathbb{C}[u,v]$. More specifically, this function looks for sufficient conditions of absolute reducibility and returns `true` if the polynomial P is detected absolutely irreducible, `false` if it is detected absolutely reducible, `FAIL` otherwise. When P has rational coefficients (our case), the prime p runs through a given set of prime integers: the first ten odd primes and the first five primes greater than the degree of P are automatically chosen.

Running the above algorithm we prove the irreducibility of P for all n and m, except for the pairs given in Table 1. For these pairs the irreducibility is proved by using the same test but with the prime numbers indicated in that table.

(n,m)	$(11,47)$	$(11,51)$	$(19,63)$	$(27,53)$	$(31,51)$
p	43	67	47	53	59

Table 1: Suitable prime number to prove the irreducibility of $P(u,v)=0$ for the given values of n and m.

□

5 Proof of Theorem B

A crucial point to prove Theorem B will be the analysis of the number of rational real solutions of the equation

$$(19)\qquad q(t)\,\dot z = p(t)z(z-1)(z-k)\,,\ \ k\in(-1,0)\,,\ p(t),q(t)\in\mathbb{R}[t].$$

Next results goes in this direction.

Proposition 5.1. *If $z_1(t)=\frac{y_1(t)}{x_1(t)}$ and $z_2(t)=\frac{y_2(t)}{x_2(t)}$ with $(y_1,x_1)=1=(y_2,x_2)$ are two non-constant rational solutions of equation (19) then there exists $0\neq c\in\mathbb{R}$ such that $y_2(t)=c\,y_1(t)$.*

Proof. Solving equation (19) we get that for $i=1,2$

$$\frac{(z_i-k)z_i^{k-1}}{(z_i-1)^k} = C_i\,\exp\left(k(k-1)H(t)\right)\quad\text{where}\quad H'(t)=\frac{p(t)}{q(t)}.$$

Then

$$\frac{(z_1-k)z_1^{k-1}}{(z_1-1)^k} = K\frac{(z_2-k)z_2^{k-1}}{(z_2-1)^k}\quad\text{where}\quad K=\frac{C_1}{C_2},$$

and writing $z_1(t)=\frac{y_1(t)}{x_1(t)}$ and $z_2(t)=\frac{y_2(t)}{x_2(t)}$ we get that

$$(20)\qquad (y_1-kx_1)(y_1-x_1)^{-k}y_2^{1-k} \;=\; K\,(y_2-kx_2)(y_2-x_2)^{-k}y_1^{1-k}.$$

Since $k<0$, and $(x_1,y_1)=1=(x_2,y_2)$ we have that $y_1=0$ if and only if $y_2=0$. Moreover, their zeroes have the same multiplicities. □

As we have seen in the previous Proposition, if $z(t)$ is a solution of equation (19), $z \neq 1$, and $H(t)$ is a prescribed primitive of $\frac{p(t)}{q(t)}$ then there exists $L \in \mathbb{R}$ such that

$$\frac{(z-k)z^{k-1}}{(z-1)^k} = L \exp\big(k(k-1)H(t)\big).$$

From now on we will say that the *energy level* of the solution $z(t)$ is L and we will denote it by $\pi(z)$.

Proposition 5.2. *Assume that $z_1(t) = \frac{y(t)}{x_1(t)}$ and $z_2(t) = \frac{y(t)}{x_2(t)}$, with $(y, x_1) = 1 = (y, x_2)$ are two non-constant rational solutions of equation (19) and set $M = \frac{\pi(z_2)}{\pi(z_1)}$. Then the following holds.*

(a) $k \in \mathbb{Q} \cap (-1, 0)$, that is, there exist $n, m \in \mathbb{N}$ such that $(n, m) = 1, n < m$ and $k = -\frac{n}{m}$.

(b) If $M^m \neq 1$ then there exist two polynomials $P, Q \in \mathbb{R}[t]$ with $(P, Q) = 1$, not simultaneously constant, such that

$$\begin{aligned} y =& \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right), \\ x_1 =& \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right) - \left(P^n - MQ^n\right)P^m, \\ x_2 =& \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right) - \left(P^n - MQ^n\right)Q^m. \end{aligned}$$

(c) If $M^m = 1$ then there exist two polynomials $P, Q \in \mathbb{R}[t]$ with $(P, Q) = 1$, not simultaneously constant such that

$$\begin{aligned} y =& \frac{n}{n+m}\,\frac{\left(P^{n+m} - Q^{n+m}\right)}{P-Q}, \\ x_1 =& \frac{n}{n+m}\,\frac{\left(P^{n+m} - Q^{n+m}\right)}{P-Q} - \frac{\left(P^n - Q^n\right)P^m}{P-Q}, \\ x_2 =& \frac{n}{n+m}\,\frac{\left(P^{n+m} - Q^{n+m}\right)}{P-Q} - \frac{\left(P^n - Q^n\right)Q^m}{P-Q}. \end{aligned}$$

Proof. (a) From (20) we know that

$$\left(\frac{y-kx_1}{y-kx_2}\right)\left(\frac{y-x_1}{y-x_2}\right)^{-k} = M, \tag{21}$$

for some $M \in \mathbb{R}$. If $k \in \mathbb{R}\backslash\mathbb{Q}$, then (21) is not consistent with the fact that y, x_1, x_2 are polynomials. Hence we get that $k = -\frac{n}{m} \in \mathbb{Q} \cap (-1, 0)$, with $(n, m) = 1$.

(b) From (21) we get

$$\left(\frac{y-kx_1}{y-kx_2}\right)^m\left(\frac{y-x_1}{y-x_2}\right)^n = M^m. \tag{22}$$

From this last equality we deduce that there exist polynomials $P, Q, a \in \mathbb{R}[t]$ with $(P, Q) = 1$ such that

$$y - x_1 = a\,P^m \quad \text{and} \quad y - x_2 = a\,Q^m \tag{23}$$

and also there exists $b \in \mathbb{R}[t]$ such that

$$y + \frac{n}{m}\, x_1 = M\, b\, Q^n \quad \text{and} \quad y + \frac{n}{m}\, x_2 = b\, P^n.$$

From the equalities $y - x_1 = a\, P^m$ and $y + \frac{n}{m}\, x_1 = M\, b\, Q^n$ (resp. $y - x_2 = a\, Q^m$ and $y + \frac{n}{m}\, x_2 = b\, P^n$) we get:

$$\frac{n+m}{m} y = \frac{n}{m} a P^m + M b Q^n \quad \text{and} \quad \frac{n+m}{m} y = \frac{n}{m} a Q^m + b P^n.$$

Hence,

$$\frac{n}{m} a(P^m - Q^m) \;=\; b(P^n - M Q^n). \tag{24}$$

Since $M^m \neq 1$ it follows that $(P^m - Q^m, P^n - M Q^n) = 1$ and hence it must exist a polynomial $T \in \mathbb{R}[t]$ such that

$$a = (P^n - M\, Q^n) T \quad \text{and} \quad b = \frac{n}{m}(P^m - Q^m) T.$$

Substituting the expressions of a and b in $\frac{n+m}{m} y = \frac{n}{m} a P^m + M b Q^n$ we get

$$y = \frac{nT}{n+m}(P^{n+m} - M\, Q^{n+m})$$

and consequently

$$\begin{aligned} x_1 =& \frac{nT}{n+m}(P^{n+m} - M\, Q^{n+m}) - (P^n - M Q^n)\, T\, P^m, \\ x_2 =& \frac{nT}{n+m}(P^{n+m} - M\, Q^{n+m}) - (P^n - M Q^n)\, T\, Q^m. \end{aligned}$$

Since $(y, x_1) = 1 = (y, x_2)$ it follows that T is constant and we get the desired result.

(c) In this case from (22) we get

$$\left(\frac{y - k x_1}{y - k x_2}\right)^m \left(\frac{y - x_1}{y - x_2}\right)^n = 1.$$

From this equality we again deduce that there exist polynomials $P, Q, a \in \mathbb{R}[t]$ with $(P, Q) = 1$ such that (23) holds. Arguing as in case (b) we arrive again to equation (24), but with $M = 1$.

Since $(m, n) = 1$ and $(P, Q) = 1$ in this situation it holds that $(P^m - Q^m, P^n - Q^n) = P - Q$ and hence it must exist a polynomial $T \in \mathbb{R}[t]$ such that

$$a = \frac{P^n - Q^n}{P - Q} T \quad \text{and} \quad b = \frac{n}{m} \frac{P^m - Q^m}{P - Q} T.$$

Now the result follows as in the previous case. □

Next proposition studies the solutions of the equation

$$(P^{n+m} - Q^{n+m})(R - S) = (R^{n+m} - S^{n+m})(P - Q)$$

where $P, Q, R, S, \in \mathbb{R}[t]$, $(P, Q) = (R, S) = 1$. This equation has the solutions $P = Q$ and $R = S$; $P = S$ and $Q = R$, and also when $n + m$ is odd $P = -R$ and Q=-S; $P = -S$ and $Q = -R$. We call these solutions *trivial solutions*.

Proposition 5.3. *When $k > 83$ the equation*

$$(P^k - Q^k)(R - S) - (R^k - S^k)(P - Q) = 0, \tag{25}$$

where $P, Q, R, S, \in \mathbb{R}[t]$, $(P, Q) = (R, S) = 1$ and $\delta(PQ) > 0, \delta(RS) > 0$, only has trivial solutions.

Proof. Assume first that no proper subsum of (25) is equal to zero. Then since $(P, Q) = (R, S) = 1$ we get

$$(P^k R, P^k S, Q^k R, Q^k S) = 1$$

and we are under the hypothesis of Theorem 2.3. Let $l = \max\{\delta(P), \delta(Q), \delta(R), \delta(S)\}$ and assume without loss of generality that $\delta(P) = l$. Thus, since equation (25) has 8 monomials, we have

$$kl \leq \delta(P^k R) \leq \frac{7 \cdot 6}{2}(Z(PQRS) - 1) \leq 84l - 21,$$

and hence $k \leq 83$.

Then to finish the proof we need to examine all the cases when a subsum of (25) is zero. We have checked all them by a case by case study. Since all our arguments only use elementary facts about divisibility and there are a lot of cases we only detail the more interesting ones.

If there is some subsum of (25) equals to zero always there exists one subsum with a minimal number of monomials. This minimal number of monomials can be two, three or four. We illustrate our proof by choosing some examples of each of these situations.

(I) There is some subsum of (25) with two monomials equals to zero. Due to symmetry of the four letters there are only seven cases. Namely the monomial $P^k R$ joined with each of the seven remainder monomials. We detail two of these cases.

(i) $P^k R + Q^k S = 0$. Since $(P, Q) = 1$ we get $R = aQ^k$ and $S = -aP^k$ for some $0 \neq a \in \mathbb{R}$. Substituting these equalities in (25) we obtain

$$aP^{2k} - aQ^{2k} - a^k Q^{k^2} P + (-a)^k P^{k^2+1} + a^k Q^{k^2+1} - (-a)^k P^{k^2} Q. \tag{26}$$

Due to the facts that $(P, Q) = 1$ and $\delta(PQ) > 0$ it follows that no proper subsum of (26) is equal to zero. Moreover the greater common divisor of all the monomials appearing in (26) is one. So we are in the hypothesis of Theorem 2.3. Let $l = \max\{\delta(P), \delta(Q)\}$. If $\delta(P) = l$ we will get

$$(k^2 + 1)l = \delta(P^{k^2+1}) \leq 10(Z(PQ) - 1) \leq 20l - 1$$

and then $k^2 < 19$; a contradiction with $k > 83$. If $\delta(Q) = l$ we obtain the same contradiction by considering the monomial Q^{k^2+1}.

(ii) $P^k R - R^k P = 0$. This implies $P = R$ or k odd and $P = -R$. Since when k is odd if P, Q, R, S is a solution of (25) $P, Q, -R, -S$ is also a solution, it suffices to consider the case $P = R$. Substituting the above equality in (25) we obtain

$$-P^k S - Q^k P + Q^k S + S^k P + P^k Q - S^k Q. \tag{27}$$

Here $(P, Q) = (P, S) = 1$. Then, if there are no proper subsumes of (27) equal to zero, we are in the hypotheses of Theorem 2.3. Thus we get a contradiction with $k < 83$, as in the previous cases. The analysis of the cases when a proper subsum of (27) equals to zero is also a very large and tedious analysis of different situations. We omit it because there are not interesting new arguments.

(II) There are some zero subsums, and the minimal length of them is 3. We have to consider 11 cases taking into account the symmetries. We only explain the case

$$P^k R - P^k S - P R^k = 0,$$

because it is the more interesting one. We get $P^{k-1}(R - S) - R^k = 0$. Since $(R, S) = 1$ we get that $R - S = a, R = aB^{k-1}$ and $P = aB^k$ for some $0 \neq a \in \mathbb{R}$ and $B \in \mathbb{R}[t]$. Note that we also have

$$-Q^k(R - S) + R^k Q + S^k(P - Q) = 0 \tag{28}$$

and hence Q divides S^k. So we have $S^k = QH$ for some $H \in \mathbb{R}[t]$. Also we will have

$$P^{k-1} + Q(P^{k-2} + \ldots + PQ^{k-3} + Q^{k-2}) = \frac{P^k - Q^k}{P - Q} = \frac{R^k - S^k}{R - S} = \frac{R^k - S^k}{a} = P^{k-1} - \frac{S^k}{a}.$$

Therefore $H = -a(P^{k-2} + \ldots + PQ^{k-3} + Q^{k-2})$ and then $(Q, H) = 1$. Dividing equation (28) by Q and taking into account that $R - S = a$, we obtain

$$-Q^{k-1}a + R^k + H(P - Q) = 0, \tag{29}$$

and this equation is already under the hypothesis of Theorem 2.3. However we need to control the degree of H. To do this we first claim that $\delta(P) = \delta(Q)$. We have

$$kR^{k-1} - \frac{\sum_{i=2}^k \binom{k}{i}(-a)^i R^{k-i}}{a} = \frac{R^k - (R - a)^k}{R - S} = \frac{P^k - Q^k}{P - Q} = P^{k-1} + \sum_{i=1}^{k-1} P^{k-1-i}Q^i.$$

Note that the degree of the left side of this equality is $(k-1)^2\delta(B)$ and the degree of the right side is $k(k-1)\delta(B)$ if $\delta(P) > \delta(Q)$ while it is $(k-1)\delta(Q) > (k-1)k\delta(B)$ if $\delta(Q) > \delta(P)$. Therefore we must have $\delta(Q) = \delta(P)$ as we have claimed.

On the other hand since $R - S = a$ we get $\delta(R) = \delta(S)$ and therefore $\delta(S^k) = k\delta(R) = k(k-1)\delta(B)$. Thus

$$\delta(H) = \delta(S^k) - \delta(Q) = \delta(S^k) - \delta(P) = k(k-2)\delta(B).$$

Considering (29), since the number of monomials of this equation is $n = 4$, from Theorem 2.3 we get

$$k(k-1)\delta(B) = \delta(HP) \leq 3(Z(QPHR) - 1) = 3(Z(BS) - 1) \leq 3k\delta(B) - 3.$$

We obtain $k(k-1) < 3k$ which gives the desired contradiction.

(III) There are some zero subsums, and the minimal length of them is 4. After symmetries there are 15 cases. Again, we only explain one of them. Assume that

$$P^k R - P^k S - PR^k - Q^k R = 0.$$

Therefore $P^k(R-S) - R(PR^{k-1} + Q^k) = 0$ and since $(P,Q) = (R,S) = 1$ we will have $R = aP^k$ for some $0 \neq a \in \mathbb{R}$. Substituting R and dividing by P^k we obtain

$$aP^k - S - a^k P^{k^2-k+1} - aQ^k. \tag{30}$$

Since $1 = (R,S) = (aP^k, S)$ the above equation is under the hypotheses of the Theorem 2.3. However we need to control the degree of S. To do this notice that in this case we also have $Q^k S + PS^k + QR^k - QS^k = 0$ and hence

$$(Q^k + PS^{k-1})S + Q(R^k - S^k) = 0.$$

Since $(S,R) = (P,Q) = 1$ we get that $S = bQ$ for some $0 \neq b \in \mathbb{R}$ and hence $\delta(Q) = \delta(S)$. Now let $l = \max\{\delta(P), \delta(Q)\}$ and assume for example that $\delta(P) = l$. Then applying Theorem 2.3 to (30) we will have

$$kl = \delta(P^k) \leq 3(Z(PQS) - 1) \leq 9l - 3,$$

which gives the desired contradiction. If $\delta(P) < l$ the result follows by considering the monomial Q^k.

□

Next Theorem gives a complete answer to the question on the number of rational solutions of (19).

Theorem 5.4. *Equation (19) with $k \in (-1,0)$ has at most six rational solutions. Moreover if it has six rational solutions $k = -\frac{1}{2}$. Otherwise it has at most five rational solutions. These upper bounds are both achieved.*

Proof. From Proposition 5.1 we know that two different rational solutions of (19) which are not $0, 1, k$ have the same numerator y. Assume that the differential equation has three different rational solutions z_1, z_2, z_3 with $z_i = \frac{y}{x_i}$ for $i = 1, 2, 3$. First assume that there are two solutions, namely z_1, z_2 such that $|\pi(z_1)| \neq |\pi(z_2)|$. In this case $|\pi(z_3)|$ is different either to $|\pi(z_1)|$ or $|\pi(z_2)|$. So we can assume without loss of generality that $\bar{M} = \frac{\pi(z_2)}{\pi(z_1)}$ and $\bar{L} = \frac{\pi(z_3)}{\pi(z_1)}$ satisfy that $\bar{M}^m \neq 1$ and $\bar{L}^m \neq 1$. Thus applying Proposition 5.2 to z_1, z_2 and z_1, z_3 we know the existence of $P, Q, R, S \in \mathbb{R}[t]$ with $(P,Q) = (R,S) = 1, PQRS \neq 0, \delta(PQ) > 0, \delta(RS) > 0$ such that

$$\begin{aligned}
y =& \frac{n}{n+m}\left(P^{n+m} - \bar{M}Q^{n+m}\right) \\
x_1 =& \frac{n}{n+m}\left(P^{n+m} - \bar{M}Q^{n+m}\right) - \left(P^n - \bar{M}Q^n\right)P^m \\
x_2 =& \frac{n}{n+m}\left(R^{n+m} - \bar{M}S^{n+m}\right) - \left(P^n - \bar{M}Q^n\right)Q^m.
\end{aligned}$$

and

$$\begin{aligned}
y =& \frac{n}{n+m}\left(R^{n+m}-\bar{L}S^{n+m}\right)\\
x_1 =& \frac{n}{n+m}\left(R^{n+m}-\bar{L}S^{n+m}\right)-\left(R^{n}-\bar{L}S^{n}\right)R^{m}\\
x_3 =& \frac{n}{n+m}\left(R^{n+m}-\bar{L}S^{n+m}\right)-\left(R^{n}-\bar{L}S^{n}\right)S^{m}.
\end{aligned}$$

In particular

$$R^{n+m}-\bar{L}S^{n+m}=P^{n+m}-\bar{M}Q^{n+m} \tag{31}$$

and

$$\left(R^{n}-\bar{L}S^{n}\right)R^{m}=\left(P^{n}-\bar{M}Q^{n}\right)P^{m}. \tag{32}$$

Looking at equation $R^{n+m}-\bar{L}S^{n+m}-P^{n+m}+\bar{M}Q^{n+m}=0$, we have that if such polynomials exist, then $n+m$ must satisfy $n+m\le 7$. This is so, because if $n+m\ge 8$, from Theorem 2.2 the equality must decompose into trivial ones and since $R\neq 0\neq S$, $P\neq 0\neq Q$, $(P,Q)=(R,S)=1$ and none of these pairs is constant we have that $R^{n+m}-P^{n+m}=0$, and $\bar{L}S^{n+m}-\bar{M}Q^{n+m}=0$ or $R^{n+m}+\bar{M}Q^{n+m}=0$ and $\bar{L}S^{n+m}+P^{n+m}=0$. In both situations the set of solutions obtained from P,Q and R,S are the same. That is $z_2=z_3$.

Now assume that $n+m\le 7$. Calling $u=\frac{Q}{P}$ and $v=\frac{S}{R}$ from (31) and (32) we deduce that

$$\left(1-\bar{L}\,v^{n+m}\right)\left(1-\bar{M}\,u^{n}\right)-\left(1-\bar{M}\,u^{n+m}\right)\left(1-\bar{L}\,v^{n}\right)=0.$$

Hence the existence of three non-constant rational solutions implies that some of the irreducible components of the above polynomial has a rational parametrization. But it is know that this happens if and only this irreducible component has genus equal to zero. For convenience, we consider the change of coordinates $\bar{u}=\bar{M}^{1/n}\,u$ and $\bar{v}=\bar{L}^{1/n}\,v$ and renaming the variables we will consider

$$F(u,v):=\left(1-L\,v^{n+m}\right)\left(1-u^{n}\right)-\left(1-M\,u^{n+m}\right)\left(1-v^{n}\right),$$

with $n+m\le 7$, $L\neq 0\neq M$ and n,m relatively prime. Note that $L=\bar{L}^{-\frac{m}{n}}$ and $M=\bar{M}^{-\frac{m}{n}}$.

From Propositions 4.1 and 4.2 we see that in our situation (remember that $|\bar{M}|\neq 1\neq|\bar{L}|$) the only case that the genus of some irreducible component of F is zero is when $M^n=L^n$. In this case we get $\bar{M}^{-m}=\bar{L}^{-m}$. Therefore $\bar{L}=\bar{M}$ or m is even and $\bar{L}=-\bar{M}$. In both cases the component of genus zero is $u-v$ which in the original variables gives $\frac{Q}{P}=\frac{S}{R}$ or in the case m even also we can get $\frac{Q}{P}=-\frac{S}{R}$. In all this situations we obtain that $z_2=z_3$. Thus in this case there are only five rational solutions.

Now assume that $|\pi(z_1)|=|\pi(z_2)|=|\pi(z_3)|$ and first suppose that m is odd and $\pi(z_1)=-\pi(z_2)=-\pi(z_3)$. Then applying Proposition 5.2 to the pairs z_1,z_2 and z_2,z_3 it follows that there exist $P,Q,R,S\in\mathbb{R}[t]$ such that $PQRS\neq 0$, $(P,Q)=(R,S)=1$,

$$\begin{aligned}
y =& \frac{n}{n+m}\left(P^{n+m}+Q^{n+m}\right)\\
x_1 =& \frac{n}{n+m}\left(P^{n+m}+Q^{n+m}\right)-\left(P^{n}+Q^{n}\right)P^{m}\\
x_2 =& \frac{n}{n+m}\left(R^{n+m}+S^{n+m}\right)-\left(P^{n}+Q^{n}\right)Q^{m},
\end{aligned}$$

and

$$
\begin{aligned}
y =& \frac{n}{n+m}\left(R^{n+m}+S^{n+m}\right) \\
x_1 =& \frac{n}{n+m}\left(R^{n+m}+S^{n+m}\right)-\left(R^n+S^n\right)R^m \\
x_3 =& \frac{n}{n+m}\left(R^{n+m}+S^{n+m}\right)-\left(R^n+S^n\right)S^m.
\end{aligned}
$$

In particular

$$
R^{n+m}+S^{n+m}=P^{n+m}+Q^{n+m} \tag{33}
$$

and

$$
\left(R^n+S^n\right)R^m=\left(P^n+Q^n\right)P^m. \tag{34}
$$

As before looking at equation $R^{n+m}+S^{n+m}-P^{n+m}-Q^{n+m}=0$, we have that if such polynomials exist, then $n+m$ must satisfy $n+m\le 7$. So we assume that $n+m\le 7$.

Thus from equations (33) and (34) we obtain

$$
\left(1+v^{n+m}\right)\left(1+u^n\right)-\left(1+u^{n+m}\right)\left(1+v^n\right)=0.
$$

As in the previous case, the existence of three non-constant rational solutions implies that some of the irreducible components of the above polynomial has genus equal zero. Again from Proposition 4.2 we see that the only irreducible component of genus zero is $u-v$ which implies that $\frac{Q}{P}=\frac{S}{R}$ and $z_2=z_3$. So again in this case we obtain only five rational solutions.

Lastly consider the case $\pi(z_1)=\pi(z_2)=\pi(z_3)$ or m is even and $\pi(z_1)=-\pi(z_2)=-\pi(z_3)$. In both cases from Proposition 5.2 applied to the pairs (z_1,z_2) and (z_1,z_3) we obtain that there exist $P,Q,R,S\in\mathbb{R}[t]$ such that $PQRS\neq 0$, $(P,Q)=(R,S)=1, \delta(PQ)>0, \delta(RS)>0$,

$$
\begin{aligned}
y =& \frac{n}{n+m}\frac{\left(P^{n+m}-Q^{n+m}\right)}{P-Q}, \\
x_1 =& \frac{n}{n+m}\frac{\left(P^{n+m}-Q^{n+m}\right)}{P-Q}-\frac{\left(P^n-Q^n\right)P^m}{P-Q}, \\
x_2 =& \frac{n}{n+m}\frac{\left(P^{n+m}-Q^{n+m}\right)}{P-Q}-\frac{\left(P^n-Q^n\right)Q^m}{P-Q},
\end{aligned}
$$

and

$$
\begin{aligned}
y =& \frac{n}{n+m}\frac{\left(R^{n+m}-S^{n+m}\right)}{R-S}, \\
x_1 =& \frac{n}{n+m}\frac{\left(R^{n+m}-S^{n+m}\right)}{R-S}-\frac{\left(R^n-S^n\right)R^m}{R-S}, \\
x_3 =& \frac{n}{n+m}\frac{\left(R^{n+m}-S^{n+m}\right)}{R-S}-\frac{\left(R^n-S^n\right)S^m}{R-S}.
\end{aligned}
$$

In particular we have that

$$
(P^{n+m}-Q^{n+m})(R-S)=(R^{n+m}-S^{n+m})(P-Q)
$$

and from Proposition 5.3 it follows that for $n+m \geq 84$ the above equation has no relevant solutions. So we assume that $n+m \leq 83$.

From

$$\frac{(P^{n+m} - Q^{n+m})}{P-Q} = \frac{(R^{n+m} - S^{n+m})}{R-S} \quad \text{and} \quad \frac{(P^n - Q^n)\, P^m}{P-Q} = \frac{(R^n - S^n)\, R^m}{R-S}$$

and putting $Q/P = u$ and $S/R = v$ we obtain

$$(1-u^{n+m})(1-v^n) - (1-v^{n+m})(1-u^n) = 0.$$

Proposition 4.3 shows that for $n+m \leq 83$, and $n > 1$ the only irreducible components with genus zero of the above polynomial are $u = 1, v = 1$ and $u = v$. The two first possibilities give that z_1, z_2 and z_3 are constants. The case $u = v$ forces $z_2 = z_3$ and again we obtain five rational solutions

In the case $n = 1, m = 2$ there is also the component $u + v + 1 = 0$ that has genus zero. In this case we obtain

$$y = \frac{1}{3}\frac{(P^3 - Q^3)}{P-Q}, \quad x_1 = \frac{1}{3}\frac{(P^3 - Q^3)}{P-Q} - P^2, \quad x_2 = \frac{1}{3}\frac{(P^3 - Q^3)}{P-Q} - Q^2$$

and

$$y = \frac{1}{3}\frac{(R^3 - S^3)}{R-S}, \quad x_1 = \frac{1}{3}\frac{(R^3 - S^3)}{R-S} - R^2, \quad x_3 = \frac{1}{3}\frac{(R^3 - S^3)}{R-S} - S^2,$$

which gives the solutions $R = P, S = Q$, or $R = P, S = -(P+Q)$, or $R = -P, S = P+Q$. They give rise to three different solutions with $x_1 = y - P^2$, $x_2 = y - Q^2$, $x_3 = y - (P+Q)^2$. So in this case we can obtain six rational solutions.

In the case $n = 1, m = 3$ there is also the component $1 + u + v + u^2 + uv + v^2 = 0$ that has genus zero. However there are no rational real functions $u, v \in \mathbb{R}(t)$ satisfying this relation.

If $n = 1$, $m > 3$ and $n + m \leq 83$, Proposition 4.3 shows that the only irreducible components of genus zero are again $u = 1, v = 1$ and $u = v$. Thus the result follows as when $n > 1$.

This ends the proof that there are are most six rational solutions.

To get un example with six solutions in the case $k = -\frac{1}{2}$ we simply choose $P(t) = t$ and $Q(t) = 1$ in the corresponding set of equations. Then the equation is

$$3t(t+1)(t^2+t+1)\,\dot{z} = -2(2t+1)(t-1)(t+2)\,z(z-1)(z+\frac{1}{2}).$$

This equation has the solutions $0, 1, -\frac{1}{2}$ and

$$z_1(t) = -\frac{t^2+t+1}{(2t+1)(t-1)}, \quad z_2(t) = \frac{t^2+t+1}{(t+2)(t-1)}, \quad z_3(t) = -\frac{t^2+t+1}{(t+2)(2t+1)}.$$

To get an example with five rational solutions when $k \neq -\frac{1}{2}$, we consider the same P and Q and $k = -\frac{1}{3}$. We get the differential equation

$$4t(t+1)(t^2+1)(t^2+t+1)\,\dot{z} = -3(t-1)(3t^2+2t+1)(t^2+2t+3)\,z(z-1)(z+\frac{1}{3}).$$

For this equation the only rational solutions are $0, 1, -\frac{1}{3}$ and

$$z_1(t) = -\frac{(t+1)(t^2+1)}{(t-1)(3t^2+2t+1)}, \quad z_2(t) = \frac{(t+1)(t^2+1)}{(t-1)(t^2+2t+3)}.$$

□

Now we are ready to set out the main result of this section.

Proof of Theorem B. Assume that equation (3) has $x_1, x_2, x_3 \in \mathbb{R}[t]$ three different solutions which are collinear. Assume also that x_2 is between x_1 and x_3. Then the change $y = x - x_2$ transforms (3) in

$$q(t)\,\dot{y} = p_3(t)\,y^3 + \widetilde{p}_2(t)\,y^2 + \widetilde{p}_1(t)\,y, \tag{35}$$

for some $\widetilde{p}_2(t), \widetilde{p}_1(t) \in \mathbb{R}[t]$. Notice that equation (35) has the collinear solutions $y_1 = x_1 - x_2, y_2 = 0, y_3 = x_3 - x_2 = ky_1$, for some $k < 0$.

If $x_2 = \frac{1}{2}(x_1 + x_3)$ then a simple computation shows that $k = -1$ and $\widetilde{p}_2(t) = 0$. So in this case the result follows from Theorem A.

If $x_2 \neq \frac{1}{2}(x_1 + x_3)$ then $k \neq -1$. We consider the change $z(t) := \frac{y(t)}{\bar{y}_1(t)}$ that transforms equation (35) in $q(t)\,\dot{z} = p(t)z(z-1)(z-k)$ for some $p(t) \in \mathbb{R}[t]$. Note that we can assume that $k \in (-1, 0)$. If this is not the case it suffices to consider the change $z(t) := \frac{y(t)}{\bar{y}_2(t)}$ instead $z(t) := \frac{y(t)}{\bar{y}_1(t)}$ and we obtain equation (19) with $k \in (-1, 0)$. Thus the polynomial solutions of the original equation are transformed in rational solutions of equation (19). Hence, from Theorem 5.4 we obtain that our equation has at most six polynomial solutions. To get an example with this number of polynomial solutions it suffices to modify the example given in the proof of Theorem 5.4. Consider the change of variable $\omega(t) = z(t)(t+2)(2t+1)(t-1)$ where $(t+2)(2t+1)(t-1)$ is the least common multiple of $d_1(t), d_2(t)$ and $d_3(t)$, the respective denominators of $z_1(t), z_2(t)$ and $z_3(t)$ in the mentioned example. We get

$$Q(t)\,\dot{\omega} = P_3(t)\,\omega^3 + P_2(t)\,\omega^2 + P_1(t)\,\omega,$$

where

$$\begin{aligned}
Q(t) &= 3(2t+1)(t-1)t(t+1)(t+2)(t^2+t+1),\\
P_3(t) &= -2, \qquad\quad P_2(t) = (2t+1)(t-1)(t+2),\\
P_1(t) &= 22t^6 + 66t^5 + 60t^4 + 10t^3 - 3t^2 + 3t + 4.
\end{aligned}$$

The above equation has the solutions

$$\begin{aligned}
\omega_1 &= 0, & \omega_2 &= (2t+1)(t-1)(t+2),\\
\omega_3 &= -\frac{1}{2}(2t+1)(t-1)(t+2), & \omega_4 &= -(t+2)\left(t^2+t+1\right),\\
\omega_5 &= (2t+1)\left(t^2+t+1\right), & \omega_6 &= -(t-1)\left(t^2+t+1\right).
\end{aligned}$$

Using the same approach one can construct examples with five polynomial solutions when $k \in \mathbb{Q} \cap (-1, 0)$ and $k \neq -1/2$. □

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