

# Linearization of planar involutions in $\mathcal{C}^1$

A. Cima · A. Gasull · F. Mañosas · R. Ortega

Received: 13 November 2013 / Accepted: 22 April 2014 / Published online: 10 May 2014  
© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag Berlin Heidelberg 2014

**Abstract** The celebrated Kerékjártó Theorem asserts that planar continuous periodic maps can be continuously linearized. We prove that  $\mathcal{C}^1$ -planar involutions can be  $\mathcal{C}^1$ -linearized.

**Keywords** Kerékjártó theorem · Periodic maps · Linearization · Involution

**Mathematical Subject Classification 2000** Primary 37C15; Secondary 37C05 · 54H20

## 1 Introduction and statement of the main result

A map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called  $m$ -periodic if  $F^m = \text{Id}$ , where  $F^m = F \circ F^{m-1}$ , and  $m$  is the smallest positive natural number with this property. When  $m = 2$ , then it is said that  $F$  is an *involution*.

When there exists a  $\mathcal{C}^k$ -diffeomorphism  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , such that  $\psi \circ F \circ \psi^{-1}$  is a linear map, then it is said that  $F$  is  $\mathcal{C}^k$ -linearizable. In this case, the map  $\psi$  is called a *linearization* of  $F$ . This property is very important because it is not difficult to describe the dynamics of the discrete dynamical system generated by linearizable maps. For instance, planar  $m$ -periodic linearizable maps behave as planar  $m$ -periodic linear maps: They are either symmetries with respect to a “line” or “rotations.”

---

A. Cima · A. Gasull · F. Mañosas (✉)  
Departament de Matemàtiques, Universitat Autònoma de Barcelona, Barcelona, Spain  
e-mail: manyosas@mat.uab.cat

A. Cima  
e-mail: cima@mat.uab.cat

A. Gasull  
e-mail: gasull@mat.uab.cat

R. Ortega  
Departamento de Matemática Aplicada, Universidad de Granada, Granada, Spain  
e-mail: rortega@ugr.es

There is a strong relationship between periodic maps and linearizable maps. For instance, it is well known that when  $n = 1$ , every  $C^k$  periodic map is either the identity, or it is 2-periodic and  $C^k$ -conjugated to the involution  $-\text{Id}$ , see for instance [8]. When  $n = 2$ , the following result holds, see [4] for a simple and nice proof.

**Theorem 1.1** (Kerékjártó Theorem) *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a continuous  $m$ -periodic map. Then  $F$  is  $C^0$ -linearizable.*

The situation changes for  $n \geq 3$ . In [1, 2], Bing shows that for any  $m \geq 2$ , there are continuous  $m$ -periodic maps in  $\mathbb{R}^3$  which are not linearizable. Nevertheless, Montgomery and Bochner give a positive local result proving that for  $C^k$ ,  $k \geq 1$ ,  $m$ -periodic maps having a fixed point are always locally  $C^k$ -linearizable in a neighborhood of this point, see [9] or Theorem 3.1 below. In any case, in [3, 5, 7], it is shown that for  $n \geq 7$ , there are continuous and also differentiable periodic maps on  $\mathbb{R}^n$  without fixed points.

The aim of this paper is to prove the following improvement for planar involutions of the result of Kerékjártó.

**Theorem A** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^1$ -differentiable involution. Then  $F$  is  $C^1$ -linearizable.*

As we will see, our proof uses classical ideas of differential topology together with some ad hoc tricks for extending and gluing non-global diffeomorphisms. The authors thank Professor Sánchez Gabites for suggesting the use of the classification theorem of surfaces for the proof of Lemma 2.5.

## 2 Preliminary results on differential topology

In this paper, unless it is explicitly stated, a differentiable map will mean a map of class  $C^1$ . Also a diffeomorphism will be a  $C^1$ -diffeomorphism.

### 2.1 Results in dimension $n$

We state two results that we will use afterward when  $n = 2$ . The first one asserts that any local diffeomorphism can be extended to be a global diffeomorphism, see [10].

**Theorem 2.1** *Let  $M$  be a differentiable manifold and let  $g : V \rightarrow g(V) \subset M$  be a diffeomorphism defined on a neighborhood  $V$  of a point  $p \in M$ . Then, there exists a diffeomorphism  $f : M \rightarrow M$  such that  $f|_W = g|_W$  for some neighborhood  $W \subset V$  of  $p$ .*

The second one is given in [6] for  $C^\infty$ -manifolds. Here, we state a slightly modified version of the theorem for  $C^1$ -manifolds. We leave the details of this generalization to the reader. Notice that it allows to glue diffeomorphisms that match as a global homeomorphism, only changing them in a neighborhood of the gluing set, but not on the gluing set itself.

**Theorem 2.2** *For each  $i = 0, 1$ , let  $W_i$  be an  $n$ -dimensional  $C^1$ -manifold without boundary which is the union of two closed  $n$ -dimensional submanifolds  $M_i, N_i$  such that*

$$M_i \cap N_i = \partial M_i = \partial N_i = V_i.$$

*Let  $f : W_0 \rightarrow W_1$  be a homeomorphism which maps  $M_0$  and  $N_0$  diffeomorphically onto  $M_1$  and  $N_1$ , respectively. Then, there is a diffeomorphism  $\tilde{f} : W_0 \rightarrow W_1$  such that  $\tilde{f}(M_0) = M_1$ ,  $\tilde{f}(N_0) = N_1$  and  $\tilde{f}|_{V_0} = f|_{V_0}$ . Moreover,  $\tilde{f}$  can be chosen such that it coincides with  $f$  outside a given neighborhood  $Q$  of  $V_0$ .*

## 2.2 Results in the plane

The aim of this subsection is to prove the following local result that will play a key role in our proof of Theorem A.

**Lemma 2.3** *Let  $D \subset \mathbb{R}^2$  be an open and simply connected set such that  $\{0\} \times \mathbb{R} \subset D$ . Then, there exist an open set  $V$  such that  $\{0\} \times \mathbb{R} \subset V \subset D$  and a diffeomorphism  $\psi : D \rightarrow \mathbb{R}^2$  such that  $\psi|_V = \text{Id}$ .*

To prove Lemma 2.3, we introduce two more results. The first one is a direct corollary of the natural generalization for non-compact  $\mathcal{C}^1$ -surfaces of the theorem of classification of  $\mathcal{C}^\infty$ -compact surfaces given in [6].

**Theorem 2.4** *Let  $M$  be a simply connected and non-compact  $\mathcal{C}^1$ - surface such that  $\partial M$  is connected and non-empty. Then,  $M$  is diffeomorphic to  $H = \{(x, y) \in \mathbb{R}^2 : x \geq 1\}$ .*

The second result is a lemma that allows to transform by a diffeomorphism any  $\mathcal{C}^1$ -curve “going from infinity to infinity” into a straight line.

**Lemma 2.5** *Let  $C$  be a closed, connected and non-compact  $\mathcal{C}^1$ -submanifold of  $\mathbb{R}^2$ . Then, there exists a diffeomorphism  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\varphi(C) = \{0\} \times \mathbb{R}$ .*

*Proof* First of all note that  $\mathbb{R}^2 \setminus C$  has two connected components that we will denote by  $C^+$  and  $C^-$ . Denote also by  $C_1$  and  $C_2$  the simply connected and non-compact differentiable surfaces obtained by adding  $C$  to  $C^+$  and  $C^-$ . Applying Theorem 2.4 to  $C_1$  and  $C_2$ , we obtain diffeomorphisms  $\phi_1 : C_1 \rightarrow H_1$  and  $\phi_2 : C_2 \rightarrow H_2$  where  $H_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$  and  $H_2 = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$ . Clearly, the map  $\phi_2 \circ \phi_1^{-1}$  is a diffeomorphism of  $\{0\} \times \mathbb{R}$  into itself. Thus,  $(\phi_2 \circ \phi_1^{-1})(0, y) = (0, \lambda(y))$  for a certain diffeomorphism  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ . Consider the diffeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $h(x, y) = (x, \lambda(y))$  and define  $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as

$$G(x, y) = \begin{cases} (h \circ \phi_1)(x, y), & \text{if } (x, y) \in C_1; \\ \phi_2(x, y), & \text{if } (x, y) \in C_2. \end{cases}$$

Thus applying Theorem 2.2 with  $W_0 = W_1 = \mathbb{R}^2$ ,  $M_0 = C_1$ ,  $N_0 = C_2$ ,  $M_1 = H_1$ ,  $N_1 = H_2$  and  $f = G$ , we obtain the desired diffeomorphism  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .  $\square$

We are ready to prove the main result of this subsection.

*Proof of Lemma 2.3* We consider first the case when there exists  $\epsilon > 0$  such that  $[-\epsilon, \epsilon] \times \mathbb{R} \subset D$ . In this particular case denote by

$$D_+ = \{(x, y) \in D : x > 0\} \quad \text{and} \quad D_\epsilon = \{(x, y) \in D : x \geq \epsilon\}.$$

Since  $D$  is an open and simply connected set, by the Riemann Theorem, there exists a diffeomorphism  $G : D \rightarrow \mathbb{R}^2$ . Set

$$C_+ = G(\{\epsilon\} \times \mathbb{R}).$$

Clearly, we have that  $C_+$  is a closed, connected and non-compact submanifold of  $\mathbb{R}^2$ . Thus by Lemma 2.5, there exists a diffeomorphism

$$\Phi_+ : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{such that} \quad \Phi_+(C_+) = \{\epsilon\} \times \mathbb{R}.$$

Composing  $\Phi_+$  with an appropriate involution, if necessary, we can assume that  $(\Phi_+ \circ G)(D_\epsilon) = \{(x, y) \in \mathbb{R}^2 : x \geq \epsilon\} \doteq H_\epsilon$ . Set

$$\psi_+ = \Phi_+ \circ G.$$

Thus, we have that  $\psi_+(D_\epsilon) = H_\epsilon$  and  $\psi_+(\{\epsilon\} \times \mathbb{R}) = \{\epsilon\} \times \mathbb{R}$ . Therefore,  $\psi_+(\epsilon, y) = (\epsilon, h(y))$  for some diffeomorphism  $h$  of  $\mathbb{R}$ . Let  $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the diffeomorphism defined by  $H(x, y) = (x, h^{-1}(y))$ . Lastly, if we denote by  $\Upsilon_+ = H \circ \psi_+$ , we get that  $\Upsilon_+$  is a diffeomorphism between  $D_\epsilon$  and  $H_\epsilon$  such that  $\Upsilon_+|_{\{\epsilon\} \times \mathbb{R}} = \text{Id}$ . As before, denote by  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x > 0\}$  and consider the map  $T_+ : D_+ \rightarrow \mathbb{R}_+^2$  defined by

$$T_+(z) = \begin{cases} \Upsilon_+(z) & \text{if } x \in D_\epsilon, \\ z & \text{otherwise.} \end{cases}$$

Applying Theorem 2.2 with  $W_\epsilon = D_+$ ,  $W_1 = \mathbb{R}_+^2$ ,  $M_0 = M_1 = (0, \epsilon] \times \mathbb{R}$ ,  $N_0 = D_\epsilon$ ,  $N_1 = H_\epsilon$  and  $f = T_+$ , we obtain a diffeomorphism  $g_+ : D_+ \rightarrow \mathbb{R}_+^2$  such that  $g_+|_{(0, \epsilon/2) \times \mathbb{R}} = \text{Id}$ .

In a similar way, if we denote by  $D_- = \{(x, y) \in D; x < 0\}$ , and  $\mathbb{R}_-^2 = \{(x, y) \in \mathbb{R}^2 : x < 0\}$ , we can construct a diffeomorphism  $g_- : D_- \rightarrow \mathbb{R}_-^2$  such that  $g_-|_{(-\epsilon/2, 0) \times \mathbb{R}} = \text{Id}$ . Clearly, the map  $g : D \rightarrow \mathbb{R}^2$  defined by

$$g(z) = \begin{cases} g_+(z) & \text{if } x \in D_+, \\ g_-(z) & \text{if } x \in D_-, \\ z & \text{otherwise.} \end{cases}$$

is a diffeomorphism and  $g|_{(-\epsilon/2, \epsilon/2) \times \mathbb{R}} = \text{Id}$ . This ends the proof in this particular case.

Next, we will see how to reduce the general case to one that we have already solved.

Consider a differentiable map  $\sigma : \mathbb{R} \rightarrow (0, 1)$  such that  $D_\sigma \doteq \{(x, y) \in \mathbb{R}^2; |x| < \sigma(y)\} \subset D$ . Denote by  $D_{\sigma/3} \doteq \{(x, y) \in \mathbb{R}^2; |x| < \sigma(y)/3\}$ . We want to transform with a diffeomorphism the set  $D_\sigma$  into the vertical strip  $(-1, 1) \times \mathbb{R}$ . Moreover, we want that this diffeomorphism is the identity on  $D_{\sigma/3}$ . To this end, we construct a diffeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of the type  $h(x, y) = (h_y(x), y)$  where  $h_y : \mathbb{R} \rightarrow \mathbb{R}$  is an odd diffeomorphism satisfying  $h_y(x) = x$  if  $0 \leq x \leq \frac{\sigma(y)}{3}$  and  $h_y(\sigma(y)) = 1$ . Then,  $h$  maps diffeomorphically  $D$  onto  $h(D)$ . Moreover,  $h|_{D_{\sigma/3}} = \text{Id}$  and  $h(D) \supset h(D_\sigma) = (-1, 1) \times \mathbb{R}$ . Using the first part of the proof with any  $\epsilon < 1$ , we can assert that there exist a diffeomorphism  $g : h(D) \rightarrow \mathbb{R}^2$  and a neighborhood  $V$  of  $\{0\} \times \mathbb{R}$  such that  $g|_V = \text{Id}$ . We obtain the desired result by considering the diffeomorphism  $g \circ h$  and the neighborhood  $V \cap D_{\sigma/3}$ .  $\square$

The last preliminary result is given in next lemma.

**Lemma 2.6** *Let  $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}$  be continuous maps, such that  $\alpha(y) \neq 0$  for all  $y \in \mathbb{R}$ . Then, there exists a diffeomorphism  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $F|_{\{0\} \times \mathbb{R}} = \text{Id}$  and*

$$(dF)_{(0,y)} = \begin{pmatrix} \alpha(y) & 0 \\ \beta(y) & 1 \end{pmatrix}$$

for all  $y \in \mathbb{R}$ .

*Proof* Set  $R(x, y) = 1 + \beta(x + y) - \beta(y)$  and  $S(x, y) = \alpha(x + y) - \frac{\beta(x+y)(\alpha(x+y) - \alpha(y))}{R(x, y)}$ . We have that  $R(0, y) = 1$  and  $S(0, y) = \alpha(y) \neq 0$  for all  $y \in \mathbb{R}$ . By continuity, there exists an open neighborhood  $V$  of  $\{0\} \times \mathbb{R}$  such that  $R(x, y) \neq 0$  and  $S(x, y) \neq 0$  for all  $(x, y) \in V$ . Moreover, we can choose  $V$  simply connected and satisfying the following

property: If  $(x, y_1)$  and  $(x, y_2)$  belong to  $V$  then  $(x, y) \in V$  for all  $y \in (y_1, y_2)$ . Now consider  $H : V \rightarrow \mathbb{R}^2$  defined as

$$H(x, y) = (H_1(x, y), H_2(x, y)) = \left( \int_y^{y+x} \alpha(s) \, ds, y + \int_y^{y+x} \beta(s) \, ds \right).$$

Clearly,  $H$  is  $\mathcal{C}^1$  and  $H(0, y) = (0, y)$  for all  $y \in \mathbb{R}$ .

We claim that  $H$  restricted to an appropriate open subset of  $V$  is an embedding. To prove this fact, note first that  $\det((dH)_{(x,y)}) = R(x, y)S(x, y) \neq 0$  for all  $(x, y) \in V$ . Then,  $H$  is a local diffeomorphism. Moreover, by the Implicit Function Theorem, since  $\frac{\partial H_2}{\partial y}(0, b) \neq 0$  it follows that for any  $b \in \mathbb{R}$ , there exist an open interval  $I_b$  containing 0 and a differentiable map  $\phi_b : I_b \rightarrow \mathbb{R}$  satisfying the following property: For all  $x \in I_b$ ,  $(x, \phi_b(x)) \in V$  and  $H_2(x, \phi_b(x)) = b$ . We can choose  $I_b$  maximal with respect this property. Since  $\frac{\partial H_2}{\partial y}(x, y) \neq 0$  for all  $(x, y) \in V$ , it follows that  $I_b$  and  $\phi_b$  are uniquely determined and the graph of  $\phi_b(x)$  tends to the boundary of  $V$  when  $x$  tends to the boundary of  $I_b$ .

For any  $b \in \mathbb{R}$  denote by  $J_b$  the graph of  $\phi_b$  and set  $\tilde{W} = \cup_{b \in \mathbb{R}} J_b$ . Now, we claim that  $H$  restricted to  $\tilde{W}$  is globally one to one. To do this note that the equation  $H(x, y) = (a, b)$  with  $(x, y) \in \tilde{W}$  implies that  $(x, y) \in J_b$ . Then calling  $L_b(s) = H_1(s, \phi_b(s))$ , we need to solve the equation  $L_b(s) = a$ . Since

$$\begin{aligned} L'_b(s) &= \frac{\partial H_1}{\partial x}(s, \phi_b(s)) + \frac{\partial H_1}{\partial y}(s, \phi_b(s))\phi'_b(s) \\ &= \frac{\partial H_1}{\partial x}(s, \phi_b(s)) - \frac{\partial H_1}{\partial y} \frac{\frac{\partial H_2}{\partial x}}{\frac{\partial H_2}{\partial y}}(s, \phi_b(s)) = S(s, \phi_b(s)) \neq 0, \end{aligned}$$

it follows that  $L_b$  is monotone and consequently  $H(x, y) = (a, b)$  has at most one solution in  $\tilde{W}$ .

Lastly, we claim that there exists an open neighborhood  $W$  of  $\{0\} \times \mathbb{R}$  contained in  $\tilde{W}$ . For  $b \in \mathbb{R}$ , let  $\bar{W}_b$  be an open neighborhood of  $(0, b)$  in  $V$  such that  $H|_{\bar{W}_b}$  is a diffeomorphism onto  $H(\bar{W}_b)$  and let  $\epsilon > 0$  be such that  $(-\epsilon, \epsilon) \times (b - \epsilon, b + \epsilon) \subset H(\bar{W}_b)$ . Then,  $W_b = H^{-1}((-\epsilon, \epsilon) \times (b - \epsilon, b + \epsilon))$  is open. Note that

$$W_b = \bigcup_{s \in (-\epsilon, \epsilon)} H^{-1}((-\epsilon, \epsilon) \times \{s\}) \subset \bigcup_{s \in (-\epsilon, \epsilon)} J_s \subset \tilde{W}.$$

Therefore, the claim is proved by selecting  $W \subset \cup_{b \in \mathbb{R}} W_b$  with the following properties:  $W$  is open, connected, simply connected and contains  $\{0\} \times \mathbb{R}$ . Thus, we will have that  $H|_W$  is a diffeomorphism onto  $H(W)$ . Therefore,  $H(W)$  is also connected and simply connected. By Lemma 2.3, there exist open sets  $V_1 \subset W$ ,  $V_2 \subset H(W)$  and diffeomorphisms  $\varphi_1 : W \rightarrow \mathbb{R}^2$  and  $\varphi_2 : H(W) \rightarrow \mathbb{R}^2$  such that  $\varphi_1|_{V_1} = \text{Id}$  and  $\varphi_2|_{V_2} = \text{Id}$ . Then,  $F = \varphi_2 \circ H \circ \varphi_1^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a diffeomorphism, and for any  $(x, y) \in V_1 \cap H^{-1}(V_2)$ , we have

$$d(F)_{(x,y)} = d(\varphi_2)_{H \circ \varphi_1^{-1}(x,y)} \circ d(H)_{\varphi_1^{-1}(x,y)} \circ d(\varphi_1^{-1})_{(x,y)} = \text{Id} \circ d(H)_{(x,y)} \circ \text{Id}.$$

In particular, we obtain that

$$d(F)_{(0,y)} = d(H)_{(0,y)} = \begin{pmatrix} \alpha(y) & 0 \\ \beta(y) & 1 \end{pmatrix},$$

for all  $y \in \mathbb{R}$ , as we wanted to prove.  $\square$

### 3 Proof of Theorem A

We will use the classical Kerékjártó Theorem and the Montgomery–Bochner Theorem, see [9]. We also include the proof of the second result because it is very simple and explains what is understood by a locally linearizable map.

**Theorem 3.1** (Montgomery–Bochner Theorem, see [9]). *Let  $\mathcal{U} \subset \mathbb{R}^n$  be an open set and let  $F : \mathcal{U} \rightarrow \mathcal{U}$  be a class  $\mathcal{C}^r$ ,  $r \geq 1$ ,  $m$ -periodic map, having a fixed point  $p \in \mathcal{U}$ . Then, there is a neighborhood of  $p$ , where  $F$  is  $\mathcal{C}^r$ -linearizable and conjugated to the linear map  $L(x) := d(F)_p x$ .*

*Proof* Consider the map from  $\mathcal{U}$  into  $\mathbb{R}^n$ ,  $\psi = \sum_{i=0}^{m-1} L^{-i} \circ F^i$ . Since both,  $F$  and  $L$ , are  $m$ -periodic it holds that  $L \circ \psi = \psi \circ F$ . Moreover, since  $d(\psi)_p = m \text{Id}$ , by applying the Inverse Function Theorem, we get that  $\psi$  is locally invertible and has the same regularity as  $F$ .  $\square$

*Proof of Theorem A* By the Kerékjártó Theorem, the map  $F$  is  $\mathcal{C}^0$  conjugated to a linear involution. Hence, it is conjugated either to  $S(x, y) = (-x, y)$  or to  $-\text{Id}$ . First, we consider the case when  $F$  is  $\mathcal{C}^0$ -conjugated to  $S$ . Let  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the homeomorphism such that  $F \circ g = g \circ S$ . Then, since  $g$  is a homeomorphism, we know that  $L := g(\{0\} \times \mathbb{R})$  is a non-compact, closed and connected topological submanifold of  $\mathbb{R}^2$  which is fixed by  $F$ . We claim that  $L$  is a differentiable submanifold of  $\mathbb{R}^2$ . To do this, we will show that  $L$  is locally the graph of a  $\mathcal{C}^1$  function.

Let  $(a, b) \in L$ . Then,  $(a, b)$  is a fixed point of  $F$ , and by the Montgomery–Bochner Theorem,  $d(F)_{(a,b)}$  is conjugated to  $S$ . Then,  $d(F)_{(a,b)} - \text{Id} \neq 0$ . If we write  $F = (F_1, F_2)$ , this implies that at least one of the functions  $F_1(x, y) - x$  and  $F_2(x, y) - y$  has nonzero gradient at  $(a, b)$ . Assume for instance that  $\frac{\partial(F_1(x, y) - x)}{\partial x}(a, b) \neq 0$ . By the Implicit Function Theorem, there exist neighborhoods  $V$  of  $(a, b)$  and  $W$  of  $b$  and a  $\mathcal{C}^1$ -map  $\psi : W \rightarrow \mathbb{R}$  such that  $L \cap W = \{(\psi(t), t) : t \in W\}$ . This proves the claim.

By Lemma 2.5, there exists a diffeomorphism  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\varphi(L) = \{0\} \times \mathbb{R}$ . Therefore,  $\tilde{F} = \varphi \circ F \circ \varphi^{-1}$  is a  $\mathcal{C}^1$ -involution that has  $\{0\} \times \mathbb{R}$  as a line of fixed points. Then,  $\tilde{F}(0, y) = (0, y)$ . Thus,  $d(\tilde{F})_{(0,y)} = \begin{pmatrix} A(y) & 0 \\ B(y) & 1 \end{pmatrix}$  for some  $A, B : \mathbb{R} \rightarrow \mathbb{R}$  continuous.

Moreover, since  $d(\tilde{F})_{(0,y)}$  must be conjugated to  $S$ , it follows that  $A(y) = -1$  for all  $y \in \mathbb{R}$ .

Now, using Lemma 2.6, we choose  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a diffeomorphism such that  $\phi|_{\{0\} \times \mathbb{R}^2} = \text{Id}$  and

$$d(\phi)_{(0,y)} = \begin{pmatrix} 1 & 0 \\ B(y)/2 & 1 \end{pmatrix}.$$

Lastly define

$$\Phi(x, y) = \begin{cases} \phi(x, y) & \text{if } x \geq 0, \\ \tilde{F}(\phi(S(x, y))) & \text{otherwise.} \end{cases}$$

which is  $\mathcal{C}^1$  because

$$\begin{aligned} \lim_{x \rightarrow 0^+} d(\Phi)_{(x,y)} &= \begin{pmatrix} 1 & 0 \\ -B(y)/2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ B(y) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B(y)/2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \lim_{x \rightarrow 0^-} d(\Phi)_{(x,y)}. \end{aligned}$$

Since  $\det(d(\phi)_{(0,y)}) = 1$ , it follows that  $\phi$  preserves orientation. In addition, we know that all points on the line  $x = 0$  are fixed and then  $\phi(\{x, y\} \in \mathbb{R}^2 : x \geq 0) = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ . Thus, we obtain that  $\Phi$  is a diffeomorphism. Computing directly  $\Phi^{-1}$  we have

$$\Phi^{-1}(x, y) = \begin{cases} \phi^{-1}(x, y) & \text{if } x \geq 0, \\ S(\phi^{-1}(\tilde{F}(x, y))) & \text{otherwise.} \end{cases}$$

Finally, again a direct computation gives that  $\Phi^{-1} \circ \tilde{F} \circ \Phi = S$ . Since  $\tilde{F} = \varphi \circ F \circ \varphi^{-1}$ , the map  $\Phi^{-1} \circ \varphi$  is the desired  $\mathcal{C}^1$ -conjugation. This ends the proof for this case.

Now, we consider the case when  $F$  is  $\mathcal{C}^0$ -conjugated to  $-\text{Id}$ . Then,  $F$  has a unique fixed point  $p$ . By the proof of Theorem 3.1, the map  $\text{Id} - F$  conjugates  $F$  to  $-\text{Id}$  in a neighborhood  $W$  of  $p$ . By Theorem 2.1, the embedding  $(\text{Id} - F)|_V$  can be extended to be a global diffeomorphism  $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\pi|_V = (\text{Id} - F)|_V$  for some neighborhood  $V \subset W$  of  $p$ . Since  $F$  is topologically conjugated to  $-\text{Id}$ , we can select  $V$  so that  $F(V) \subset V$ . Consider now  $\tilde{F} = \pi \circ F \circ \pi^{-1}$ . The map  $\tilde{F}$  has 0 as a fixed point and  $\tilde{F}|_{\pi(V)} = -\text{Id}$ . Let  $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the homeomorphism such that  $\gamma^{-1} \circ \tilde{F} \circ \gamma = -\text{Id}$  and consider  $L = \gamma(\{0\} \times \mathbb{R})$ . Then,  $L$  is a connected, closed and non-compact topological submanifold of  $\mathbb{R}^2$  invariant by  $\tilde{F}$ . Our next objective will be to modify  $L$  for obtaining a  $\mathcal{C}^1$  submanifold with the same properties.

Let  $r > 0$  be such that  $B_r = \{x \in \mathbb{R}^2 : |x| < r\} \subset \pi(V)$  and set  $t_0 = \max\{t \in \mathbb{R} : |\gamma(0, t)| = r\}$ . Then,  $L_1 = \gamma(\{0\} \times (t_0, \infty))$  does not intersect  $B_r$ . Since  $\tilde{F}|_{B_r} = -\text{Id}$ , it follows that  $\tilde{F}(L_1) = \gamma(\{0\} \times (-t_0, -\infty))$  neither cuts  $B_r$ . Set  $L_0 = \{t\gamma(0, t_0) : t \in [-1, 1]\}$  and

$$\tilde{L} = L_1 \cup L_0 \cup \tilde{F}(L_1).$$

Clearly,  $\tilde{L}$  is also a connected closed and non-compact topological submanifold of  $\mathbb{R}^2$  invariant by  $\tilde{F}$ . Hence, it divides  $\mathbb{R}^2$  in two connected and open regions  $A$  and  $B$  that are permuted by  $\tilde{F}$ . Consider now a differentiable map  $f : (0, \infty) \rightarrow \mathbb{R}^2$  satisfying the following properties:

1.  $f(t) = t\gamma(0, t_0)$  if  $t \leq 1/2$ ,
2.  $f(t) \in A$  for all  $t > 1/2$ ,
3.  $\lim_{t \rightarrow \infty} |f(t)| = \infty$ ,
4.  $f$  is one to one.

Denote by  $M_0 = f((0, \infty))$ . By construction,  $M_0$  is a connected and differentiable submanifold of  $\mathbb{R}^2$  and  $M_0 \cap \tilde{F}(M_0) = \emptyset$ . Thus,  $M = M_0 \cup \tilde{F}(M_0) \cup \{(0, 0)\}$  is a connected, closed and non-compact differentiable submanifold of  $\mathbb{R}^2$  which is invariant by  $\tilde{F}$ . By Lemma 2.5, there exists a diffeomorphism  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\varphi(M) = \{0\} \times \mathbb{R}$ . Therefore, the map

$$\hat{F} = \varphi \circ \tilde{F} \circ \varphi^{-1}$$

is a differentiable involution that has  $\{0\} \times \mathbb{R}$  as an invariant line. Thus,  $\hat{F}(0, y) = (0, g(y))$  for a certain one dimensional differentiable involution  $g : \mathbb{R} \rightarrow \mathbb{R}$ . In this case, the map  $h(y) = y - g(y)$  is a global diffeomorphism that conjugates  $g$  with  $-\text{Id}$ . Therefore, the map  $\tilde{\varphi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $\tilde{\varphi}(x, y) = (x, h(y))$  is a diffeomorphism that conjugates  $\hat{F}$  with an involution  $\tilde{F}$  that satisfies that  $\tilde{F}|_{\{0\} \times \mathbb{R}} = -\text{Id}$ . Therefore

$$d(\tilde{F})_{(0,y)} = \begin{pmatrix} A(y) & 0 \\ B(y) & -1 \end{pmatrix},$$

for some continuous functions  $A$  and  $B$  with  $A(0) = -1$  and  $B(0) = 0$ . Note that since  $A(0) = -1$  and  $\tilde{F}$  is a diffeomorphism, it follows that  $A(y) < 0$  for all  $y \in \mathbb{R}$ . On the other hand, since  $\tilde{F}^2 = \text{Id}$ , we will have

$$d(\bar{F})_{(0,-y)} \circ d(\bar{F})_{(0,y)} = \text{Id},$$

which implies that

$$A(-y) = \frac{1}{A(y)} \quad \text{and} \quad B(-y) = \frac{B(y)}{A(y)}$$

for all  $y \in \mathbb{R}$ .

Consider now the continuous maps  $a, b : \mathbb{R} \rightarrow \mathbb{R}$  defined as follows:

$$a(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ -\frac{1}{A(y)} & \text{otherwise,} \end{cases} \quad \text{and} \quad b(y) = \begin{cases} 0 & \text{if } y \geq 0, \\ -\frac{B(y)}{A(y)} & \text{otherwise.} \end{cases}$$

Direct computations show that

$$a(y) = -A(-y)a(-y) \quad \text{and} \quad b(y) = b(-y) - B(-y)a(-y),$$

for all  $y \in \mathbb{R}$ .

Since  $a(y) \neq 0$  for all  $y \in \mathbb{R}$ , by Lemma 2.6, we can choose a diffeomorphism  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  satisfying that  $\phi|_{\{0\} \times \mathbb{R}} = \text{Id}$  and

$$d(\tilde{\phi})_{(0,y)} = \begin{pmatrix} a(y) & 0 \\ b(y) & 1 \end{pmatrix}.$$

As in the previous case, we define the map

$$\Phi(x, y) = \begin{cases} \phi(x, y) & \text{if } x \geq 0, \\ \bar{F}(\phi(-x, -y)) & \text{otherwise,} \end{cases}$$

satisfying

$$\begin{aligned} \lim_{x \rightarrow 0^+} d(\Phi)_{(x,y)} &= \begin{pmatrix} a(y) & 0 \\ b(y) & 1 \end{pmatrix} = \begin{pmatrix} -A(-y)a(-y) & 0 \\ b(-y) - B(-y)a(-y) & 1 \end{pmatrix} \\ &= \begin{pmatrix} A(-y) & 0 \\ B(-y) & -1 \end{pmatrix} \begin{pmatrix} a(-y) & 0 \\ b(-y) & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \lim_{x \rightarrow 0^-} d(\Phi)_{(x,y)}. \end{aligned}$$

The same considerations as in the previous case show that  $\Phi$  is a  $\mathcal{C}^1$ -diffeomorphism that conjugates  $\bar{F}$  and  $-\text{Id}$ . Since  $\bar{F}$  and  $F$  are  $\mathcal{C}^1$ -conjugated, this fact ends the proof of the theorem.  $\square$

**Acknowledgments** The first and second authors are supported by a MINECO/FEDER Grant No. MTM2008-03437. The third author by a MINECO/FEDER Grant No. MTM2008-01486 and the fourth author by a MINECO/FEDER Grant No. MTM2011-23652. The first three authors are also supported by a CIRIT Grant No. 2009SGR 410.

## References

1. Bing, R.H.: A homeomorphism between the 3-sphere and the sum of two solid horned spheres. *Ann. Math.* **56**, 354–362 (1952)
2. Bing, R.H.: Inequivalent families of periodic homeomorphisms of  $E^3$ . *Ann. Math.* **80**, 78–93 (1964)
3. Conner, P.E., Floyd, E.E.: On the construction of periodic maps without fixed points. *Proc. Am. Math. Soc.* **10**, 354–360 (1959)
4. Constantin, A., Kolev, B.: The theorem of Kerékjártó on periodic homeomorphisms of the disc and the sphere. *Enseign. Math.* **40**, 193–204 (1994)
5. Haynes, R., Kwasik, S., Mast, J., Schultz, R.: Periodic maps on  $\mathbb{R}^7$  without fixed points. *Math. Proc. Cambridge Philos. Soc.* **132**, 131–136 (2002)



6. Hirsch, M.W.: Differential Topology. Springer, New York (1976)
7. Kister, J.M.: Differentiable periodic actions on  $E^8$  without fixed points. *Am. J. Math.* **85**, 316–319 (1963)
8. Kuczma, M., Choczewski, B., Ger, R.: Iterative Functional Equations. *Encyclopedia of Mathematics and its Applications* 32. Cambridge University Press, Cambridge (1990)
9. Montgomery, D., Zippin, L.: Topological Transformation Groups. Interscience, New York (1955)
10. Morse, M.: Differentiable mappings in the Shoenflies theorem. *Compos. Math.* **14**, 83–151 (1959)