ON THE RELATION BETWEEN INDEX AND MULTIPLICITY

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Abstract

This paper is mainly devoted to the study of the index of a map at a zero, and the index of a polynomial map over $\mathbb{R}^n$. For semi-quasi-homogeneous maps we prove that the index at a zero coincides with the index at this zero of its quasi-homogeneous part. For a class of polynomial maps with finite zero set we provide a method which makes easier the computation of its index over $\mathbb{R}^n$. Finally we relate the index and the multiplicity.

1. *Notation and statement of the results*

Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a continuous map such that 0 is isolated in $f^{-1}(0)$. Then the index $\mathrm{ind}_0[f]$ of f at zero is defined as follows: choose a ball B_ε about 0 in $\mathbb{R}^n$ so small that $f^{-1}(0)\cap B_\varepsilon=\{0\}$ and let S_ε be its boundary $(n-1)$-sphere. Choose an orientation of each copy of $\mathbb{R}^n$. Then the index of f at zero is the degree of the mapping $(f/\|f\|):S_\varepsilon\to S$, the unit sphere, where the spheres are oriented as $(n-1)$-spheres in $\mathbb{R}^n$. If f is differentiable, this degree can be computed as the sum of the signs of the Jacobian of f at all the f-preimages near 0 of a regular value of f near 0.

If f is a smooth (that is a $\mathbb{C}^\infty$) map, then consider the germ f_0 of f at 0, and the local ring $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$ of f_0 at 0, where $\mathscr{C}_0^\infty(\mathbb{R}^n)$ is the ring of germs at 0 of smooth real-valued functions on $\mathbb{R}^n$, and (f_0) is the ideal generated by the components of f_0. The multiplicity $\mu_0[f]$ of f at 0 is defined by $\mu_0[f]=\dim_{\mathbb{R}}[\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)]$ and we say that f is a finite map germ if $\mu_0[f]<\infty$. It is known that $\mu_0[f]$ is the number of complex f-preimages near 0 of a regular value of f near 0.

Given a map $g:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$, where $g=(g_1,\dots,g_n)$ with each g_i a homogeneous polynomial such that 0 is isolated in $g^{-1}(0)$, it is well known that $\mu_0[g]=\prod_{i=1}^n d_i$, where d_i is the degree of each g_i.

On the other hand any smooth function $f_i:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ can be written as $f_i=g_i+G_i$, where g_i is the first non-zero jet of f_i. Hence, any smooth map $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ can be written as $f=g+G$. It is also known that $\mu_0[f]=\mu_0[g]$ if 0 is isolated in $g^{-1}(0)$. Sometimes the above construction provides a g such that 0 is not isolated in $g^{-1}(0)$, but a suitable selection of weights associated with any variable (see the definitions in the sequel) makes possible a different decomposition $f=g'+G'$ satisfying $\mu_0[f]=\mu_0[g']$.

We begin this paper by giving a similar property but one concerning indices instead of multiplicities. In order to enunciate the result, we need some preliminary definitions.

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