ON THE RELATION BETWEEN INDEX AND MULTIPLICITY

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Abstract

This paper is mainly devoted to the study of the index of a map at a zero, and the index of a polynomial map over $\mathbb{R}^n$. For semi-quasi-homogeneous maps we prove that the index at a zero coincides with the index at this zero of its quasi-homogeneous part. For a class of polynomial maps with finite zero set we provide a method which makes easier the computation of its index over $\mathbb{R}^n$. Finally we relate the index and the multiplicity.

1. *Notation and statement of the results*

Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a continuous map such that 0 is isolated in $f^{-1}(0)$. Then the index $\operatorname{ind}_0[f]$ of f at zero is defined as follows: choose a ball B_ε about 0 in $\mathbb{R}^n$ so small that $f^{-1}(0)\cap B_\varepsilon=\{0\}$ and let S_ε be its boundary $(n-1)$-sphere. Choose an orientation of each copy of $\mathbb{R}^n$. Then the index of f at zero is the degree of the mapping $(f/\|f\|):S_\varepsilon\to S$, the unit sphere, where the spheres are oriented as $(n-1)$-spheres in $\mathbb{R}^n$. If f is differentiable, this degree can be computed as the sum of the signs of the Jacobian of f at all the f-preimages near 0 of a regular value of f near 0.

If f is a smooth (that is a $\mathbb{C}^\infty$) map, then consider the germ f_0 of f at 0, and the local ring $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$ of f_0 at 0, where $\mathscr{C}_0^\infty(\mathbb{R}^n)$ is the ring of germs at 0 of smooth real-valued functions on $\mathbb{R}^n$, and (f_0) is the ideal generated by the components of f_0. The multiplicity $\mu_0[f]$ of f at 0 is defined by $\mu_0[f]=\dim_{\mathbb{R}}[\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)]$ and we say that f is a finite map germ if $\mu_0[f]<\infty$. It is known that $\mu_0[f]$ is the number of complex f-preimages near 0 of a regular value of f near 0.

Given a map $g:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$, where $g=(g_1,\ldots,g_n)$ with each g_i a homogeneous polynomial such that 0 is isolated in $g^{-1}(0)$, it is well known that $\mu_0[g]=\prod_{i=1}^n d_i$, where d_i is the degree of each g_i.

On the other hand any smooth function $f_i:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ can be written as $f_i=g_i+G_i$, where g_i is the first non-zero jet of f_i. Hence, any smooth map $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ can be written as $f=g+G$. It is also known that $\mu_0[f]=\mu_0[g]$ if 0 is isolated in $g^{-1}(0)$. Sometimes the above construction provides a g such that 0 is not isolated in $g^{-1}(0)$, but a suitable selection of weights associated with any variable (see the definitions in the sequel) makes possible a different decomposition $f=g'+G'$ satisfying $\mu_0[f]=\mu_0[g']$.

We begin this paper by giving a similar property but one concerning indices instead of multiplicities. In order to enunciate the result, we need some preliminary definitions.

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We say that f is a quasi-homogeneous map with weight $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and quasi-degree $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ if

$$f_i(\lambda^{a_1} x_1, \lambda^{a_2} x_2, \ldots, \lambda^{a_n} x_n) = \lambda^{d_i} f_i(x_1, x_2, \ldots, x_n) \tag{1}$$

for each $i = 1, 2, \ldots, n$ and all $\lambda > 0$. When $a_i = 1$ for $i = 1, 2, \ldots, n$, then f is a homogeneous map. A function f_i satisfying (1) is called a quasi-homogeneous function with weight $\mathbf{a} = (a_1, \ldots, a_n)$ and quasi-degree d_i. Note that any monomial $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ is a quasi-homogeneous function with arbitrary weight $\mathbf{a} = (a_1, \ldots, a_n)$ and quasi-degree $a_1 r_1 + a_2 r_2 + \cdots + a_n r_n$. For fixed $\mathbf{a}$ we say that a smooth function has quasi-order m if all monomials in its Taylor expression have quasi-degree greater than or equal to m.

We also recall the concept of a semi-quasi-homogeneous map (see [**4**]). We say that f is a semi-quasi-homogeneous map with weight $\mathbf{a}$ and quasi-degree $\mathbf{d}$ if $f = g + G$ with g a quasi-homogeneous map with weight $\mathbf{a}$ and quasi-degree $\mathbf{d}$ such that 0 is isolated in $g^{-1}(0)$, and each component G_i of G has quasi-order greater than d_i.

THEOREM 1.1. *Let $f = g + G$ be a semi-quasi-homogeneous map. Then* 0 *is isolated in $f^{-1}(0)$ and*

$$\operatorname{ind}_0[f] = \operatorname{ind}_0[g].$$

Here we assume that f is a polynomial map such that all its zeros are isolated. Then its zero set is finite and we define ind_f by

$$\operatorname{ind}_f = \sum_{\{a : f(a) = 0\}} \operatorname{ind}_a[f].$$

We shall give a result similar to Theorem 1.1 which is useful in computing ind_f.

For a fixed weight $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ we say that a polynomial map $f = (f_1, \ldots, f_n)$ has quasi-degree $\mathbf{d} = (d_1, \ldots, d_n)$ if each f_i has a monomial of quasi-degree d_i and all its other monomials have quasi-degree less than or equal to d_i.

THEOREM 1.2. *Let $f = g + G$ be a polynomial map. Suppose that G is a quasi-homogeneous map with weight* $\mathbf{a}$ *and quasi-degree* $\mathbf{d}$, *with* 0 *isolated in $G^{-1}(0)$ and suppose that g has quasi-degree less than* $\mathbf{d}$. *Then the zero set of f is finite and* $\operatorname{ind}_f = \operatorname{ind}_0[G]$.

Theorems 1.1 and 1.2 will be proved in Section 3. Section 2 contains statements of the general results that we need to prove our assertions.

Section 4 is devoted to giving bounds for $\operatorname{ind}_0[f]$ for semi-quasi-homogeneous maps (see Theorem 4.1) and for ind_f for some polynomial maps (see Theorem 4.3). These bounds generalize results of Khovanskii, see [**6**].

In the last part of the paper we study the relation between index and multiplicity. In [**5**] the authors prove that

$$|\operatorname{ind}_0[f]| \leqslant (\mu_0[f])^{1-1/n}. \tag{2}$$

They also give an algebraic method of computing the index of a finite map germ. They prove that the index of f at zero can be computed as the signature of a certain symmetric bilineal form defined on the local ring $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$.

It turns out that

$$\operatorname{ind}_0[f] \equiv \mu_0[f] \pmod 2. \tag{3}$$

The following question arises: for fixed n and given $\mu = \mu_0[f]$, which values can the index of f at zero take?

For the case $n = 2$, we get a full answer: the number $\operatorname{ind}_0[f]$ is not subject to any restrictions other than (2) and (3) (see Theorem 5.1). As far as we know, the above result was previously known only when $\mu = k^2$ for some $k \in \mathbb{N}$, see [**5**, Proposition 2.4].

If $n > 2$, we find that the bound given in (2) is not the best possible, although we present an example which shows that the order of the exponential growth cannot be improved (see Proposition 5.2).

Finally in the Appendix we study the function $\pi_n(\mathbf{1}, \mathbf{d})$ introduced in Section 4.

2. *Preliminary results*

The next two propositions give the properties of index and multiplicity that we need to prove our results.

PROPOSITION 2.1 (see [**1**, **4**]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a finite map germ. Then we have the following.*

(i) *The multiplicity of f at zero does not depend on the selection of coordinates.*

(ii) *Let $f = (f_1, f_2, \dots, f_n)$ and $f_i = f_i^{k_i}$ + higher order terms. Then $\mu_0[f] \geqslant \prod_{i=1}^n k_i$ and $\mu_0[f] = \prod_{i=1}^n k_i$ if and only if the system $f_i^{k_i} = 0$ for $i = 1, \dots, n$ has only the trivial solution in $\mathbb{C}^n$ (here $f_i^{k_i}$ is the homogeneous part of f_i of degree k_i).*

(iii) *If for some $i \in \{1, \dots, n\}$, the function f_i can be described as $f_i = g_{i_1} \cdot g_{i_2}$, where $g_{i_1}(0) = g_{i_2}(0) = 0$, then $\mu_0[f] = \mu_0[g_1] + \mu_0[g_2]$ where $g_1 = (f_1, \dots, g_{i_1}, \dots, f_n)$ and $g_2 = (f_1, \dots, g_{i_2}, \dots, f_n)$.*

(iv) *Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ also be a finite map germ. Then $\mu_0[f \circ g] = \mu_0[f]\mu_0[g]$.*

(v) *If $g_i = f_i + \sum_{j<i} A_j^i f_j$, then $\mu_0[f] = \mu_0[g]$.*

(vi) *If for some $i \in \{1, \dots, n\}$, the function f_i can be described as $f_i = hg_i$ with $h(0) \neq 0$, then $\mu_0[f] = \mu_0[g]$ where $g = (f_1, \dots, g_i, \dots, f_n)$.*

PROPOSITION 2.2 (see [**7**]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a continuous map such that 0 is isolated in $f^{-1}(0)$. Then the following hold.*

(i) *The index $\operatorname{ind}_0[f]$ of f at zero does not depend on the selection of coordinates.*

(ii) *Assume that g is also a continuous map such that 0 is isolated in $g^{-1}(0)$. Let B be a ball about 0 so small that $f^{-1}(0) \cap B = \{0\}$ and $g^{-1}(0) \cap B = \{0\}$. If f and g are homotopic on the boundary of B, ∂B (that is, there is a continuous homotopy $H(t, x) : [0, 1] \times \overline{B} \to \mathbb{R}^n$ between f and g, such that $H(t, x) \neq 0$ for all $x \in \partial B$) then $\operatorname{ind}_0[f] = \operatorname{ind}_0[g]$.*

(iii) *Let B be a neighbourhood of $0 \in \mathbb{R}^n$ such that $f(x) \neq 0$ for each non-zero $x \in B$. Let f_ε be a one-parameter family, smooth with respect to ε, such that $f_0 = f$. Then for ε small enough, the sum of the indices at the zeros of f_ε equals the index of f at zero.*

In order to compare the numbers $\mu_0[f]$ and $\operatorname{ind}_0[f]$ we give some results from [**5**].

THEOREM 2.3 [**5**]. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a finite map germ. Let I be an ideal of $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$ which is maximal with respect to the property $I^2 = 0$. Then*

$$|\operatorname{ind}_0[f]| = \mu_0[f] - 2\dim_{\mathbb{R}} I.$$

THEOREM 2.4 [**5**]. *Let* $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a finite map germ. Then*
(i) $|\mathrm{ind}_0[f]| \leqslant (\mu_0[f])^{1-1/n}$,
(ii) $\mathrm{ind}_0[f] \equiv \mu_0[f] \pmod 2$.

The following results concern quasi-homogeneous maps (see [**4**]). Let f be a quasi-homogeneous map with weight **a** and quasi-degree **d**. Let $P_f(t)$ be a polynomial of degree $\sum_{s=1}^{n}(d_s - a_s) = d$ such that $P_f(t) = \sum_{i=0}^{d} \delta_i t^i$, where δ_i is the number of monomials of degree i which appear in any basis of $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$. This polynomial is called the Poincaré polynomial associated with f.

The key result about quasi-homogeneous maps is the following.

THEOREM 2.5 [**4**]. *Let f be a quasi-homogeneous map with weight* **a** *and quasi-degree* **d**. *Then its Poincaré polynomial can be computed as*

$$P_f(t) = P_{\mathbf{a},\mathbf{d}}(t) = \prod_{s=1}^{n} \frac{t^{d_s} - 1}{t^{a_s} - 1}.$$

Notice that from the above theorem it is automatically known how many monomials of each degree appear in any basis of $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$.

COROLLARY 2.6 [**4**]. *Let f be a semi-quasi-homogeneous finite map germ with weight* **a** *and quasi-degree* **d**. *Then we have the following.*
(i) *The multiplicity of f at zero can be computed as*

$$\mu_0[f] = \sum_{i=1}^{d} \sigma_i = P_f(1) = \prod_{s=1}^{n} \frac{d_s}{a_s}.$$

(ii) *Any basis of the local ring of f at zero has exactly one monomial of quasi-degree $d = \sum_{s=1}^{n}(d_s - a_s)$, and any monomial of quasi-degree greater than d is zero in* $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$.
(iii) *The Poincaré polynomial of f is recurrent, that is, $\sigma_i = \sigma_{d-i}$, and so*

$$\mu_0[f] = \frac{(-1)^{d+1} - 1}{2} \sigma_{E(d/2)} + 2 \sum_{i=1}^{E(d/2)} \sigma_i.$$

Finally we recall the Poincaré definition of the index for $n = 2$ (see [**2**]).

Let C be a simple closed curve of $\mathbb{R}^2$, let f be thought of as a vector field defined on a simply connected open region of $\mathbb{R}^2$ which contains the curve C and let r be some straight line in the (x, y)-plane. Suppose that there exist only finitely many points M_k (for $k = 1, 2, \ldots, n$) on C at which the vector $f(M)$ is parallel to r. Let M be a point describing the curve in the counterclockwise sense, and let p (respectively q) be the number of points of M_k at which the vector $f(M)$ passes through the direction of r in the counterclockwise (respectively clockwise) sense. Points M_k at which the vector field $f(M)$ assumes the direction of r while moving, say, in the clockwise sense and then begins to move in the opposite sense (or vice versa) are not counted. Then, the index of C, $i(C)$, is defined by $i(C) = (p - q)/2$. If we have a zero M of f, we define the index $\mathrm{ind}_M[f]$ of f at M by $\mathrm{ind}_M[f] = i(C)$, where C is a simple closed curve on which there are no zeros of f and which is such that it surrounds only the point M.

3. *Proof of the main results*

We prove only Theorem 1.1. The proof of Theorem 1.2 follows by using similar arguments.

Proof of Theorem 1.1. Let $h_t(x)$ be defined as $h_t(x) = g(x) + tG(x)$, for $t \in [0, 1]$. We claim that in a neighbourhood of 0 the inequality $h_t(x) \neq 0$ holds for all t. This shows that

(i) 0 is isolated in $f^{-1}(0)$,

(ii) f and g are homotopic in the boundary of a ball of sufficiently small radius.

From (i), (ii) and Proposition 2.2(ii) Theorem 1.1 follows.

In order to prove the claim, assume that $h_t(x) = 0$ arbitrarily near to the origin. Then there exist two sequences, $\{x_m\}$ tending to zero and $\{t_m\}$ with $t_m \in [0, 1]$ such that $h_{t_m}(x_m) = 0$, that is, $h_{i_{t_m}}(x_m) = 0$ for $i = 1, 2, \dots, n$.

Since 0 is isolated in $g^{-1}(0)$, there exists a subsequence of $\{x_m\}$ (let us call it also $\{x_m\}$) and a subindex $i \in \{1, 2, \dots, n\}$ such that $g_i(x_m) \neq 0$. Without loss of generality we can assume that $i = 1$. By dividing the equation $h_{1_{t_m}}(x_m) = 0$ by $g_1(x_m)$ we obtain

$$1 + t_m \frac{G_1(x_m)}{g_1(x_m)} = 0.$$

Now given a point x such that $\sum_{i=1}^n x_i^{(2a_1 \cdots a_n)/a_i} = r^{2a_1 \cdots a_n}$, we consider the point u with $u_i = x_i / r^{a_i}$, so that $u \in S \cong \mathbb{S}^{n-1}$ with $S = \{u \in \mathbb{R}^n : \sum_{i=1}^n u_i^{(2a_1 \cdots a_n)/a_i} = 1\}$.

Given the sequence $\{x_m\} = \{(x_{1_m}, \dots, x_{n_m})\}$ we consider the corresponding sequence $\{u_m\} = \{(u_{1_m}, \dots, u_{n_m})\}$ contained in S. Then there exists a convergent subsequence of $\{u_m\}$ (let us call it also $\{u_m\}$) with limit u^*. We claim that $g_1(u^*) = 0$. If not, since $|t_m| \leqslant 1$ and $G_1(r^a u_m)/r^{d_1} \to 0$ as $r \to 0$, the expression

$$1 + t_m \frac{G_1(r^a u_m)}{r^{d_1} g_1(u_m)},$$

has limit 1, and we get a contradiction.

We consider g_2. Then either there exists m_0 such that $g_2(x_m) = 0$ for $m > m_0$ or there exists a subsequence of $\{x_m\}$ with $g_2(x_m) \neq 0$ for all m. In the first case we have that $g_2(u^*) = 0$. In the second case we apply the above process and we again have that $g_2(u^*) = 0$.

Doing the same with the other components of g we can assert that there exists a point u^* with $g(u^*) = 0$. Since g is a quasi-homogeneous function, we see that $g_i(t^a u^*) = g_i(t^{a_1} u_1^*, \dots, t^{a_n} u_n^*) = t^{d_i} g_i(u^*) = 0$ for all i, that is, $g = 0$ on the curve $t^a u^*$ and 0 is not isolated in $g^{-1}(0)$.

It is easy to give examples that show that Theorem 1.1 cannot be extended to the case that g is non-quasi-homogeneous. Consider $\mathbf{a} = (1, 1)$, and $g = (y^2, y - x^4)$ that has index 0 at 0. On the other hand $g + (-x^3 y, 0)$ or $g + (-x^5, 0)$ have index -1 at the origin.

Remark 3.1. Notice that the statement equivalent to Theorem 1.1, with multiplicity instead of index, can be proved by reducing the problem to the homogeneous case. It suffices to compose the map with $(x_1, \dots, x_n) \to (x_1^{a_1}, \dots, x_n^{a_n})$ and apply Proposition 2.1(iv).

This approach does not work when we are interested in the index, because there is no result similar to Proposition 2.1(iv) concerning indices instead of multiplicities.

To end this section we give an example in which the choice of a suitable weight is useful in order to study the index of a point. Let $f(x,y)$ be defined by

$$f(x,y) = (y^2 - x^3 + 2x^2 y,\ y^4 + x^3 y^2 - x^6 + 4x^3 y^3).$$

Then $f = g + G$ where $g(x,y) = (y^2 - x^3,\ y^4 + x^3 y^2 - x^6)$ and $G(x,y) = (2x^2 y,\ 4x^3 y^3)$. Here g is a quasi-homogeneous map with weight $\mathbf{a} = (2,3)$ and quasi-degree $\mathbf{d} = (6,12)$ such that 0 is isolated in $g^{-1}(0)$. The map G has quasi-degree $\mathbf{d}' = (7,15)$ which is greater than $\mathbf{d}$. So by Proposition 2.1, $\mu_0[f] = \mu_0[g]$ and by Theorem 1.1, $\mathrm{ind}_0[f] = \mathrm{ind}_0[g]$.

By Corollary 2.6(i) we have that

$$\mu_0[g] = \prod_{s=1}^{2} \frac{d_s}{a_s} = 12.$$

On the other hand since $\sum d_s - \sum a_s = 13 \not\equiv 0 \pmod 2$, Theorem 4.1 (see Section 4) implies that $\mathrm{ind}_0[g] = 0$. Notice that taking the weight $\mathbf{a} = (1,1)$, the map g would be $g(x,y) = (y^2, y^4)$ and 0 would not be isolated in $g^{-1}(0)$.

4. *Bounds for the indices*

Given $n \in \mathbb{N}$ and $\mathbf{a}$, $\mathbf{d}$ in $\mathbb{N}^n$ we define

$$\pi_n(\mathbf{a},\mathbf{d}) = \frac{1-(-1)^{d+1}}{2}\,\sigma_{E(d/2)},$$

where E denotes the integer part function, $d = \sum_{i=1}^{n} (d_i - a_i)$ and the σ_s are the coefficients of the following polynomial associated with $\mathbf{a}$, $\mathbf{d}$:

$$p_{\mathbf{a},\mathbf{d}}(t) = \prod_{i=1}^{n} \frac{t^{d_i}-1}{t^{a_i}-1} = \sum_{i=1}^{d} \sigma_i t^i.$$

It is proved in [**4**] that the above polynomial coincides with the Poincaré polynomial (see Theorem 2.5). In the Appendix we give an expression for $\pi_n(\mathbf{1},\mathbf{d})$ when $n = 2$ and $n = 3$, and present some properties of this function. Here we notice that if $\sum_i d_i \not\equiv \sum_{i=1}^{n} a_i \pmod 2$, then $\pi_n(\mathbf{a},\mathbf{d}) = 0$. We have the next result.

THEOREM 4.1. *Let f be a semi-quasi-homogeneous map with weight* $\mathbf{a}$ *and quasi-degree* $\mathbf{d}$. *Then*

(i) $|\mathrm{ind}_0[f]| \leqslant \pi_n(\mathbf{a},\mathbf{d})$,

(ii) $\mathrm{ind}_0[f] \equiv \prod_{s=1}^{n} d_s/a_s \pmod 2$.

Proof. Since f is semi-quasi-homogeneous the hypothesis of Corollary 2.6 holds. Let J be the ideal of $\mathscr{C}_0^\infty(\mathbb{R}^n)/(f_0)$ which is spanned by the monomials of quasi-degree greater than $\frac{1}{2}d = \frac{1}{2}\sum_{s=1}^{n}(d_s - a_s)$. Then, from the definition of the Poincaré polynomial associated with f, $P_f(t) = \sum_{i=0}^{d} \sigma_i t^i$, it is clear that

$$\dim J \geqslant \mu_0[f] - \sum_{i=0}^{E(d/2)} \sigma_i.$$

By Corollary 2.6(ii) we know that $J^2 = 0$ and, by applying Theorem 2.3 and the above inequality, we get

$$|\mathrm{ind}_0[f]| \leqslant \mu_0[f] - 2\dim J \leqslant 2\sum_{i=0}^{E(d/2)} \sigma_i - \mu_0[f].$$

By Corollary 2.6(iii) we obtain that

$$|\mathrm{ind}_0[f]| \leqslant \frac{1-(-1)^{d+1}}{2}\sigma_{E(d/2)} = \pi_n(\mathbf{a}, \mathbf{d}).$$

Part (ii) of the theorem follows directly from (3) and Corollary 2.6.

The inequality (i) of Theorem 4.1, when $\mathbf{a} = \mathbf{1}$ (that is, $a_i = 1$ for all $i = 1, \dots, n$) and f is homogeneous, was proved by Arnold in [**3**] and called by him the Petrovskii–Oleinik inequality. Also for the same cases, Khovanskii in [**6**] gives a more general proof and presents examples of f with multiplicity $\mu_0[f]$ and index $\mathrm{ind}_0[f]$ satisfying (i) and (ii) of Theorem 4.1.

Given $n \in \mathbb{N}$ and $\mathbf{d} \in \mathbb{N}^n$, let d be defined as $d = \sum_{i=1}^{n}(d_i - 1)$. If d is an odd number then we define

$$O_n(d) = \sigma_{(d-1)/2},$$

where $(\prod_{i=1}^{n}(t^{d_i} - 1))/(t-1)^n = \sum_{i=0}^{d} \sigma_i t^i$. The next result was proved by Khovanskii.

THEOREM 4.2 [**6**]. *Let $f = (f_1, f_2, \dots, f_n)$ be a polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$, where the degree of f_i equals d_i, such that all the zeros of f are isolated. Then the following hold.*

(i) *If all the zeros of f are finite and simple, then* $|\mathrm{ind}_f| \leqslant \pi_n(\mathbf{1}, \mathbf{d})$.

(ii) *Let d be defined by $d = \sum_{i=1}^{n}(d_i - 1)$. If $d \equiv 0 \pmod 2$, then $|\mathrm{ind}_f| \leqslant \pi_n(\mathbf{1}, \mathbf{d})$. If $d \not\equiv 0 \pmod 2$, then $|\mathrm{ind}_f| \leqslant O_n(\mathbf{d})$.*

Next we give a generalization of Theorem 4.2(i), by considering quasi-homogeneous maps.

THEOREM 4.3. *Let f be a polynomial map such that $f = g + G$, where G is a quasi-homogeneous map with weight $\mathbf{a}$ and quasi-degree $\mathbf{d}$ such that 0 is isolated in $G^{-1}(0)$ and g has quasi-degree less than $\mathbf{d}$. Then*

(i) $|\mathrm{ind}_f| \leqslant \pi_n(\mathbf{a}, \mathbf{d})$,

(ii) $\mathrm{ind}_f \equiv \prod_{i=1}^{n} d_s/a_s \pmod 2$.

Proof. By using Theorem 1.2 we have that $\mathrm{ind}_f = \mathrm{ind}_0[G]$. Applying Theorem 4.1 to G the result follows.

In some cases the bound given in Theorem 4.2(ii) can be improved by applying Theorem 4.3. Consider the map $f = (x - x^3, y + x^3)$. By using Theorem 4.2 we get that $|\mathrm{ind}_f| \leqslant O_2(3, 4) = 3$. On the other hand if we consider the weight $\mathbf{a} = (1, 4)$, Theorem 4.3 implies that $|\mathrm{ind}_f| \leqslant \pi_2((1, 4), (3, 4)) = 1$. In fact it is easy to prove that $\mathrm{ind}_f = -1$.

5. *On the relation between index and multiplicity*

THEOREM 5.1. *For each $\mu \in \mathbb{N}$ and $i \in \mathbb{Z}$ satisfying $|i| \leqslant \sqrt{\mu}$ and $i \equiv \mu \pmod 2$, there exists a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\mu_0[f] = \mu$ and $\mathrm{ind}_0[f] = i$.*

Proof. It is necessary only to see that for all k and m satisfying $k \geqslant m^2$, $k \equiv m \pmod 2$ there exists a vector field f with $\mu_0[f] = k$ and $\mathrm{ind}_0[f] = m$.

Let $P(x,y)$ and $Q(x,y)$ be homogeneous polynomials of degree $m-1$ of the form

$$P(x,y) = (y-p_1 x)\cdot\ldots\cdot(y-p_{m-1}x), \quad Q(x,y) = (y-q_1 x)\cdot\ldots\cdot(y-q_{m-1}x)$$

with $0 < p_1 < q_1 < p_2 < q_2 < \cdots < q_{m-1}$, and let f be defined as

$$f = (xP(x,y), xQ(x,y) + \varepsilon y^{k-m(m-1)}).$$

By using the properties described in Proposition 2.1, we have

$$\begin{aligned}\mu_0[f] &= \mu_0[(x, xQ(x,y)+\varepsilon y^{k-m(m-1)})] + \mu_0[(P(x,y), xQ(x,y)+\varepsilon y^{k-m(m-1)})]\\ &= \mu_0[(x, \varepsilon y^{k-m(m-1)})] + \textstyle\sum_{i=1}^{m-1} \mu_0[(y-p_i x, xQ(x,y)+\varepsilon y^{k-m(m-1)})]\\ &= k-m(m-1)+m(m-1) = k.\end{aligned}$$

In order to see that $\mathrm{ind}_0[f] = m$, we shall use the Poincaré definition of the index (see Section 2). Let $C = \{x^2+y^2 = \delta^2\}$ with δ small enough so that C surrounds only the point 0. We choose the vertical direction as r and we obtain the intersection points of C and $x = 0$, $y = p_i x$ for $i = 1, \ldots, m-1$. Then

$$\begin{aligned} f|_{x=0} &= (0, \varepsilon y^{k-m(m-1)}),\\ f|_{y=p_i x} &= (0, x^m(p_i-q_1)\cdot\ldots\cdot(p_i-q_{m-1}) + \varepsilon(xp_i)^{k-m(m-1)}).\end{aligned}$$

Let ε be taken as

$$\varepsilon = \frac{1}{2}\frac{\min\limits_{j=1,\ldots,m-1}\left|\prod\limits_{i=1}^{m-1}(p_j-q_i)\right|}{\max\limits_{j=1,\ldots,m-1}|p_j|^m}.$$

We claim that the sign of $x^m(p_i-q_1)\cdot\ldots\cdot(p_i-q_{m-1})+\varepsilon(xp_i)^{k-m(m-1)}$ equals the sign of $x^m(p_i-q_1)\cdot\ldots\cdot(p_i-q_{m-1})$. Since $k \geqslant m^2$ we first consider $k > m^2$. Then the exponent $k-m(m-1)$ is greater than m. Since $x^2+y^2 = \delta^2$, it is clear that, taking δ small enough, the sign of $x^m(p_i-q_1)\cdot\ldots\cdot(p_i-q_{m-1})$ will be the same as the sign of $x^m(p_i-q_1)\cdot\ldots\cdot(p_i-q_{m-1})+\varepsilon(xp_i)^{k-m(m-1)}$. If $k = m^2$, we have that $k-m(m-1) = m$ and from the definition of ε we see that $|\varepsilon p_i^m| < |\prod_{j=1}^{m-1}(p_i-q_j)|$ for each $i = 1, \ldots, m-1$. The claim is proved.

From the choice of $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, the second component of f evaluated at $(x, p_i x)$ will alternate its sign, being positive in $(0, \delta)$. By studying the behaviour of f near these points we can see that $\mathrm{ind}_0[f] = m$.

PROPOSITION 5.2. (i) *Let $f: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a finite map germ with $\mu_0[f] = 2^n$. Then $|\mathrm{ind}_0[f]| < \mu_0[f]^{1-1/n}$ for each $n > 2$.*

(ii) *Take n and m to be positive integer numbers such that $n(m-1) \equiv 0 \pmod 2$. Given $\mu = m^n$ there exists a finite map germ such that $\mu_0[f] = \mu$ and $\mathrm{ind}_0[f] = \sum_{i=0}^{n-1} p_i \mu^{i/n}$, where the p_i are non-negative rational numbers, $\sum_{i=0}^{n-1} p_i = 1$ and $p_{n-1} \neq 0$.*

Proof. (i) Let $f = (f_1, f_2, \ldots, f_n)$ and assume that the Taylor expression of f_i begins with terms of order k_i. Let $f_i^{k_i}$ be the homogeneous part of degree k_i of f_i. If $\mu_0[f] = 2^n$, then either $k_i \geqslant 2$ for all $i \in \{1, 2, \ldots, n\}$ or there exists some $i \in \{1, 2, \ldots, n\}$ with $k_i = 1$.

First assume that $k_i \geqslant 2$ for all $i = 1, 2, \ldots, n$. If there exists some $i \in \{1, 2, \ldots, n\}$ with $k_i > 2$, then from Proposition 2.1(ii), $\mu_0[f] > 2^n$. So, $k_i = 2$ for all $i = 1, 2, \ldots, n$.

Applying Proposition 2.1(ii) again we know that the system $f_i^2 = 0$ for $i = 1, 2, \ldots, n$ has only the trivial solution. Therefore we can apply Theorem 1.1 and assert that $\mathrm{ind}_0[f] = \mathrm{ind}_0[f^2]$, where $f^2 = (f_1^2, f_2^2, \ldots, f_n^2)$. From Theorem 4.1 we have that $|\mathrm{ind}_0[f]| \leqslant \pi_n(\mathbf{a}, \mathbf{d})$, with $\mathbf{a} = (1, 1, \ldots, 1)$ and $\mathbf{d} = (2, 2, \ldots, 2)$. By using the definition of $\pi_n(\mathbf{a}, \mathbf{d})$ we have that

$$\pi_n(\mathbf{a}, \mathbf{d}) = \frac{n!}{[(n/2)!]^2} \quad \text{if } n \text{ is even}$$

$$\pi_n(\mathbf{a}, \mathbf{d}) = 0 \quad \text{if } n \text{ is odd.}$$

If n is odd then clearly $\mathrm{ind}_0[f] = 0$ and the result follows. If n is even, $n > 2$, then

$$\frac{n!}{[(n/2)!]^2} < 2^{n-1} = \mu_0[f]^{1-1/n},$$

and the result follows again. Furthermore notice that the number $n!/[(n/2)!]^2$ satisfies

$$\frac{n!}{[(n/2)!]^2} \leqslant 2^{n(n-2)/(n-1)} = (2^n)^{1-1/(n-1)} = \mu^{1-1/(n-1)}.$$

Now assume that there exist some i with $f_i = f_i^1 + f_i^2 + \cdots$ and $f_i^1 \not\equiv 0$. We can suppose that $i = 1$ and that $f_1^1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x^n$ with $a_{11} \neq 0$. Then, near the origin, the equation $f_1(x_1, \ldots, x_n) = 0$ can be written as $x_1 = x_1(x_2, \ldots, x_n)$. So, we can consider the change of coordinates $(x_1(x_2, \ldots, x_n), x_2, \ldots, x_n)$. With that change the map can be written as $F = (x_1, F_2, \ldots, F_n)$. From the invariance properties of the index and multiplicity (Propositions 2.1 and 2.2), we get $\mathrm{ind}_0[f] = \mathrm{ind}_0[F]$ and $\mu_0[f] = \mu_0[F]$. On the other hand it is easy to prove (by taking preimages, for instance) that $\mathrm{ind}_0[F] = \mathrm{ind}_0[G]$ and $\mu_0[F] = \mu_0[G]$ where $G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is defined by

$$G = (F_2|_{x_1=0}, \ldots, F_n|_{x_1=0}).$$

Since we have reduced the dimension of the space, from Theorem 2.4 we obtain the inequality

$$|\mathrm{ind}_0[G]| \leqslant \mu_0[G]^{1-1/(n-1)}.$$

Therefore, $|\mathrm{ind}_0[f]| \leqslant \mu_0[f]^{1-1/(n-1)}$ and (i) is proved.

(ii) Consider the vector field

$$f_i = \prod_{0 \leqslant k < d_1} \left(\frac{d-1}{2} x_i - k \left(\sum_{j=1}^{n} x_j + 1 \right) \right) \quad \text{for } i = 1, 2, \ldots, n,$$

where $d = n(d_1 - 1)$. This vector field was given by Khovanskii **[6]** in order to see that the bound $\pi_n(\mathbf{1}, \mathbf{d})$ for the sum of the indices always is attained. That means that $|\mathrm{ind}_f| = \pi_n(\mathbf{1}, \mathbf{d})$ where $\mathbf{d} = (d_1, d_1, \ldots, d_1)$.

Now let us consider the homogeneous part of maximal degree of f_i, namely,

$$G_i = \prod_{o \leqslant k < d_1} \left(\frac{d-1}{2} x_i - k \sum_{j=1}^{n} x_j \right),$$

and let g be determined by $f = g + G$. It is easy to see that the system $G_i = 0$ for $i = 1, 2, \ldots, n$ has the unique solution $x = 0$, and so 0 is isolated in $G^{-1}(0)$. From Theorem 1.2 we deduce that $|\mathrm{ind}_f| = |\mathrm{ind}_0[G]| = \pi_n(\mathbf{1}, \mathbf{d})$. Now from the Appendix we know that $\pi_n(\mathbf{1},\mathbf{d})$ is a polynomial in d_1 of degree $n-1$, that is, $\pi_n(\mathbf{1},\mathbf{d})=\sum_{i=0}^{n-1}p_i d_1^i$ with $\sum_{i=0}^{n-1}p_i=1$. Furthermore, from Proposition 2.1(ii) we know that $\mu=\mu_0[G]=d_1^n$. Consequently,

$$|\mathrm{ind}_0[G]|=\sum_{i=0}^{n-1}p_i\mu^{i/n},$$

and the result follows.

For values of $\mu_0[f]$ less than 2^n we can improve the bound given in (2) in a natural way.

PROPOSITION 5.3. *Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a finite map germ with $\mu_0[f]=\mu\leqslant 2^n$. Then the following inequalities hold:*

$$|\mathrm{ind}_0[f]|\leqslant\begin{cases}1 & \text{if } 1\leqslant\mu<4,\\ \mu^{1-1/2} & \text{if } 4\leqslant\mu<8,\\ \vdots & \\ \mu^{1-1/(n-1)} & \text{if } 2^{n-1}\leqslant\mu\leqslant 2^n.\end{cases}$$

Proof. Let $f=(f_1,\dots,f_n)$ and write $f_i=f_i^{k_i}+f_i^{k_i+1}+\cdots$, where $f_i^{k_i}$ is the homogeneous part of f_i of degree k_i.

The case $\mu_0[f]=2^n$ has been studied in the proof of Proposition 5.2(i). Assume here that $\mu_0[f]<2^n$. Then there exists some i with $f_i=f_i^1+f_i^2+\cdots$ and $f_i^1\not\equiv 0$. By applying the same argument as in the proof of Proposition 5.2, we see that $f_i^1\not\equiv 0$ implies that $|\mathrm{ind}_0[f]|\leqslant\mu_0[f]^{1-1/(n-1)}$. By iterating that process (if it is necessary), we obtain the desired result.

From Proposition 5.2 we know that the bound $\mu_0[f]^{1-1/n}$ is not always attained. To end this section we give an example in $\mathbb{R}^3$ such that 0 has multiplicity μ and the absolute value of the index of f at 0 is the largest integer less than $\mu^{2/3}$.

Consider the map $f:(\mathbb{R}^3,0)\to(\mathbb{R}^3,0)$ given by

$$f_t(x,y,z)=(2x(x-2y-2z),2y(y-2x-2z),-z(z-2x-2y)(z+4x+4y)+tx(y+z)).$$

First we prove that $\mu_0[f_t]=10$ for any $t\neq 0$ small enough. Notice that f_0 has $\mu_0[f_0]=12$ and by using the formula of [**5**] it is not difficult to show that $\mathrm{ind}_0[f_0]=4$. Consider the map f_t: the system $f_t=0$ has three solutions for $t\neq 0$, $p_1=(-\frac{4t}{27},\ 0,\ -\frac{2t}{27})$, $p_2=(\frac{4t}{135},\ \frac{4t}{135},\ -\frac{2t}{135})$ and the origin. From the definition and properties of multiplicity we obtain that, for $t\neq 0$, $\mu_0[f_t]=10$, $\mu_{p_1}[f_t]=1$ and $\mu_{p_2}[f_t]=1$.

On the other hand, since the determinant of f_t is kt^4 with $k>0$ at p_1 and $k<0$ at p_2, we know that $\mathrm{ind}_{p_1}[f_t]=1$ and $\mathrm{ind}_{p_2}[f_t]=-1$. So, from Proposition 2.2(iii) we have

$$\mathrm{ind}_0[f_t]=\mathrm{ind}_0[f_0]=4,$$

for $t\neq 0$ small enough.

Appendix. The function $\pi_n(\mathbf{1},\mathbf{d})$

The goal of this Appendix is to give some properties of the function $\pi_n(\mathbf{a},\mathbf{d})$ defined in Section 4 when $\mathbf{a}=(1,\dots,1)=\mathbf{1}$. Given $n\in\mathbb{N}$, $\mathbf{d}\in\mathbb{N}^n$ and $k\in\mathbb{Z}$, we define the function $R(n,\mathbf{d},k)$ by the cardinal of

$$\left\{(x_1,x_2,\dots,x_n)\in\mathbb{Z}^n: 0\leqslant x_i<d_i,\ \sum_{i=1}^n x_i=k\right\}.$$

From this definition since $\pi_n(\mathbf{1},\mathbf{d}) = \prod_{i=1}^{n}(1+t+\cdots+t^{d_i-1})$, it is easy to prove that

$$\pi_n(\mathbf{1},\mathbf{d}) = R\left(n,\mathbf{d},\frac{1}{2}\left(\sum_{i=1}^{n} d_i - n\right)\right).$$

By using the above expression it is not difficult to study some properties of $\pi_n(\mathbf{1},\mathbf{d})$ by induction because of the equality

$$R(n,\mathbf{d},k) = \sum_{j=k-d_1}^{k} R(n-1,(d_2,\dots,d_n),j).$$

Before we state our result we introduce the following notation. Let S be any subset of $\{1,2,\dots,n\}$. Define $d_S = \sum_{i\in S} d_i$, and $d_\varnothing = 0$. We define $\mathscr{D} = \bigcup_{S\in\mathscr{P}(\{1,\dots,n\})} d_S$.

Proposition A.1. (i) *Given* $\mathbf{d}\in\mathbb{N}^n$, *we set* $R(n,\mathbf{d},k)$ *to be a continuous piece-wise polynomial of degree* $n-1$ *in the variables* $k, d_1,\dots,d_n$ *with all its non-smooth points in* $\mathscr{D}$. *Furthermore its expression on each interval of* $[0,d_1+\cdots+d_n]\setminus\mathscr{D}$ *depends on the ordering of the points of* $\mathscr{D}$.

(ii) *When* $\frac{1}{2}(\sum_{i=1}^{n} d_i - n)$ *is a natural number, the function* $\pi_n(\mathbf{1},\mathbf{d})$ *is a polynomial of degree* $n-1$ *in the variables* $d_1,\dots,d_n$. *Furthermore, its expression depends on the ordering of the points of* $\mathscr{D}$.

As an illustration of Proposition A.1(ii), and assuming that $d_1\leqslant d_2\leqslant\cdots\leqslant d_n$, we have that $\pi_2(\mathbf{1},\mathbf{d}) = d_1$, and

$$\pi_3(\mathbf{1},\mathbf{d}) = \begin{cases} d_1 d_2 & \text{when } d_1+d_2\leqslant d_3,\\ \frac{1}{4}(1+2(d_1d_2+d_1d_3+d_2d_3)-d_1^2-d_2^2-d_3^2) & \text{when } d_1+d_2\geqslant d_3.\end{cases}$$

When $d_i = d$ for all $i=1,\dots,n$, it follows that $\mathscr{D}=\{0,d,2d,\dots,nd\}$, and then only one ordering is possible. In this case the expression for $\pi_n(\mathbf{1},\mathbf{d})$, for low values of n, is

$$\pi_2(\mathbf{1},\mathbf{d}) = d,$$

$$\pi_3(\mathbf{1},\mathbf{d}) = \begin{cases}\frac{1}{4}(1+3d^2) & \text{when } d \text{ is odd},\\ 0 & \text{when } d \text{ is even},\end{cases}$$

$$\pi_4(\mathbf{1},\mathbf{d}) = \tfrac{1}{3}(d+2d^3),$$

$$\pi_5(\mathbf{1},\mathbf{d}) = \begin{cases}\frac{1}{192}(27+50d^2+115d^4) & \text{when } d \text{ is odd},\\ 0 & \text{when } d \text{ is even},\end{cases}$$

$$\pi_6(\mathbf{1},\mathbf{d}) = \tfrac{1}{20}(4d+5d^3+11d^5),$$

$$\pi_7(\mathbf{1},\mathbf{d}) = \begin{cases}\frac{1}{11520}(1125+1813d^2+2695d^4+5887d^6) & \text{when } d \text{ is odd},\\ 0 & \text{when } d \text{ is even},\end{cases}$$

$$\pi_8(\mathbf{1},\mathbf{d}) = \tfrac{1}{315}(45d+49d^3+70d^5+151d^7).$$

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