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A POINCARÉ–HOPF THEOREM FOR NONCOMPACT MANIFOLDS†

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We provide the natural extension, from the dynamical point of view, of the Poincaré–Hopf theorem to noncompact manifolds. On the other hand, given a compact set K being an attractor for a flow generated by a $\mathscr{C}^1$ tangent vector field X on an n-manifold, we prove that the Euler characteristic of its region of attraction $\mathscr{A}$, $\chi(\mathscr{A})$, is defined and satisfies $\mathrm{Ind}_{\mathscr{A}}(X) = (-1)^n \chi(\mathscr{A})$. Finally we prove that $\chi(\mathscr{A}) = \chi(K)$ when K is an euclidean neighbourhood retract being asymptotically stable and invariant. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The Poincaré–Hopf theorem (see [2], [8] or [10] for instance) asserts that when a $\mathscr{C}^1$ tangent vector field X on a compact $\mathscr{C}^2$ manifold M is pointing outward at ∂M then

$$\mathrm{Ind}(X) = \chi(M)$$

where $\chi(M)$ and $\mathrm{Ind}(X)$ denote respectively the Euler characteristic of M and the index of X.

Until now there have been many generalizations of this result dropping the restriction that X should point outward and allowing more general boundary conditions. In this direction we can mention for instance the works of Gottlieb [7], Morse [11] and Pugh [12].

A different approach to the problem of generalizing the Poincaré–Hopf theorem is to consider noncompact manifolds. This paper is devoted to give its natural extension to manifolds not being necessarily compact.

An equivalent version of the Poincaré–Hopf theorem asserts that for a tangent vector field X on a compact n-dimensional manifold M vanishing nowhere on ∂M, the relation

$$\mathrm{Ind}(X) = (-1)^n \chi(M) \tag{1}$$

is satisfied if X is never pointing outward at ∂M.

When M is compact the condition that X is never pointing outward at ∂M means dynamically that for every $x_0 \in M$ the unique solution of the initial value problem

$$\dot{x} = X(x)$$

$$x(0) = x_0$$

has a nonempty ω-limit set. Considering the closure of the union of all the ω-limit sets, the mentioned condition yields the existence of a global compact attractor for the flow associated to the above differential equation. Conversely, it is clear that a tangent vector

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