Pergamon

Topology Vol. 37, No. 2, pp. 261–277, 1998

0040-9383/97 $19.00 + 0.00

PII: S0040-9383(97)00021-9

A POINCARÉ–HOPF THEOREM FOR NONCOMPACT MANIFOLDS†

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(*Received* 27 *January* 1997; *in revised form* 29 *April* 1997)

We provide the natural extension, from the dynamical point of view, of the Poincaré–Hopf theorem to noncompact manifolds. On the other hand, given a compact set K being an attractor for a flow generated by a $\mathscr{C}^1$ tangent vector field X on an n-manifold, we prove that the Euler characteristic of its region of attraction $\mathscr{A}$, $\chi(\mathscr{A})$, is defined and satisfies $\mathrm{Ind}_{\mathscr{A}}(X) = (-1)^n \chi(\mathscr{A})$. Finally we prove that $\chi(\mathscr{A}) = \chi(K)$ when K is an euclidean neighbourhood retract being asymptotically stable and invariant.

1. INTRODUCTION

The Poincaré–Hopf theorem (see [2], [8] or [10] for instance) asserts that when a $\mathscr{C}^1$ tangent vector field X on a compact $\mathscr{C}^2$ manifold M is pointing outward at ∂M then

$$\mathrm{Ind}(X) = \chi(M)$$

where $\chi(M)$ and $\mathrm{Ind}(X)$ denote respectively the Euler characteristic of M and the index of X.

Until now there have been many generalizations of this result dropping the restriction that X should point outward and allowing more general boundary conditions. In this direction we can mention for instance the works of Gottlieb [7], Morse [11] and Pugh [12].

A different approach to the problem of generalizing the Poincaré–Hopf theorem is to consider noncompact manifolds. This paper is devoted to give its natural extension to manifolds not being necessarily compact.

An equivalent version of the Poincaré–Hopf theorem asserts that for a tangent vector field X on a compact n-dimensional manifold M vanishing nowhere on ∂M, the relation

$$\mathrm{Ind}(X) = (-1)^n \chi(M) \tag{1}$$

is satisfied if X is never pointing outward at ∂M.

When M is compact the condition that X is never pointing outward at ∂M means dynamically that for every $x_0 \in M$ the unique solution of the initial value problem

$$\dot{x} = X(x)$$

$$x(0) = x_0$$

has a nonempty ω-limit set. Considering the closure of the union of all the ω-limit sets, the mentioned condition yields the existence of a global compact attractor for the flow associated to the above differential equation. Conversely, it is clear that a tangent vector

†All the authors are partially supported by the DGICYT under grant PB93-0860.

field on a manifold M never points outward along ∂M if there is a global compact attractor for its associated flow. Thus, the Poincaré–Hopf theorem asserts, from the dynamical point of view, that relation (1) is satisfied if there exists a global compact attractor for the flow generated by the tangent vector field.

In particular the Poincaré–Hopf theorem shows that for a tangent vector field on a compact manifold M vanishing nowhere on ∂M, the existence of a global compact attractor determines its index. In this setting it is natural to investigate the case in which the compact global attractor is generated by a tangent vector field X on an n-dimensional manifold M not being necessarily compact. The first problem that arises is that when we deal with a tangent vector field on a manifold not necessarily compact, neither the index of the vector field nor the Euler characteristic of the manifold are a priori defined. In fact the definition of the index of X does not represent any problem because, in our case (that is X nonvanishing at ∂M and generating a compact global attractor), the critical points form a compact set not intersecting with ∂M. However, since we will consider also vector fields generating compact attractors not being global, we have adopted the following notation. Given any subset B of M with nonempty interior such that the critical points of X inside B form a compact set Z satisfying $Z \subset \mathrm{Int}(B)\backslash\partial M$, we define $\mathrm{Ind}_B(X)$ to be $\mathrm{Ind}(X|_N)$, where N is any compact n-manifold $Z \subset N\backslash\partial N$ and $N \subset B$. It is to be noted that if the number of critical points of X inside B is finite then $\mathrm{Ind}_B(X)$ is equal the sum of the local index of X at these critical points and that if B is itself a compact n-manifold then $\mathrm{Ind}_B(X) = \mathrm{Ind}(X|_B)$.

We present the following result that provides a generalization of the Poincaré–Hopf theorem to manifolds not necessarily compact.

THEOREM A. *Let M be a $\mathscr{C}^2$ n-dimensional manifold and let X be a $\mathscr{C}^1$ tangent vector field on M vanishing nowhere on ∂M. If there exists a compact global attractor for the flow generated by X then the Euler characteristic of M is defined and satisfies*

$$\mathrm{Ind}_M(X) = (-1)^n \chi(M).$$

Now, Theorem A is a generalization of the Poincaré–Hopf theorem because, when the manifold is compact, $\mathrm{Ind}_M(X) = \mathrm{Ind}(X)$ and the condition that the vector field is never pointing outward along ∂M is equivalent to requiring the existence of a global compact attractor. Since this equivalence is obviously not true for noncompact manifolds it may seem that the existence of a global compact attractor is a condition that becomes too strong for noncompact manifolds (in the sense that a weaker condition may yield the relation between the index and the Euler characteristic). However it does not. This is so because if it is only required, for example, that each solution has nonempty ω-limit set (and this implies that the vector field is not pointing outward along ∂M, if any) then the mentioned relation does not hold anymore. Even more, it neither holds if it is required that each ω-limit set is nonempty and compact (in the literature the vector fields which generate this kind of flow are called bounded vector fields). Indeed, in [4] is given an example of a $\mathscr{C}^\infty$ bounded vector field on $\mathbb{R}^3$, with finitely many critical points and satisfying that the sum of its local index at all the critical points is 0 (and hence $\mathrm{Ind}_{\mathbb{R}^3}(X) = 0$ is not equal to $-\chi(\mathbb{R}^3) = -1$).

Some results about bounded vector fields can be found in [3–5]. It is to be noted that Theorem A is proved in [4] by using very different tools in the special case that M is $\mathbb{R}^n$ and X has finitely many critical points.

In fact we have proved a result that deals with more general situations. Thus, in Theorem B is considered the case of an attractor compact set not being necessarily global. Here the notion of the stabilizer of an attractor compact set K, denoted by $\widetilde{K}$ is introduced.

It is defined to be the set of points of M such that its α-limit has nonempty intersection with K. This notion becomes very useful when $K \cap \partial M = \emptyset$ because then $\widetilde{K}$ is an asymptotically stable invariant compact set with the same region of attraction as K (see Proposition 4.4). Moreover $K = \widetilde{K}$ if and only if K is asymptotically stable and invariant (see Corollary 4.5). In the statement of Theorem B, ENR stands for euclidean neighbourhood retract.

THEOREM B. *Let K be an attractor compact set for the flow generated by a $\mathscr{C}^1$ tangent vector field X on a $\mathscr{C}^2$ n-dimensional manifold M and let $\mathscr{A}$ denote its region of attraction. Then the Euler characteristic of $\mathscr{A}$ is defined and satisfies*

$$\mathrm{Ind}_{\mathscr{A}}(X) = (-1)^n \chi(\mathscr{A})$$

if X is nonvanishing on $K \cap \partial M$. In case that $K \cap \partial M = \emptyset$ the following additional relations are satisfied:

(a) $\chi(\widetilde{K}) = \chi(\mathscr{A})$ *if $\widetilde{K}$ is ENR.*
(b) $\chi(K) = \chi(\mathscr{A})$ *if K is ENR, asymptotically stable and invariant.*

Notice that when X has finitely many critical points in K Theorem B shows that the local index of X at all these critical points equals $(-1)^n \chi(\mathscr{A})$. This is so because K contains all the critical points of X inside $\mathscr{A}$. Notice also that in fact Theorem A follows immediately from applying Theorem B because the region of attraction of a global attractor is by definition the whole manifold.

It is to be noted that the hypothesis in (b) of Theorem B cannot be weakened. This is so because if K is a compact ENR being only attractor and invariant then the relation $\chi(K) = \chi(\mathscr{A})$ does not hold anymore. After proving Theorem B we will show an example in which this relation is not satisfied.

We shall now present some interesting qualitative consequences of these theorems. For instance, Theorem A implies that a global compact attractor must contain at least one critical point if $\chi(M) \neq 0$. This explains why a periodic orbit can be a global attractor in $S^1 \times \mathbb{R}^n$ but it cannot in $\mathbb{R}^n$. In $\mathbb{R}^2$ this is obvious since a periodic orbit in the plane contains at least one critical point in its interior. On the other hand Theorem B shows that an asymptotically stable critical point in $\mathbb{R}^n$ has always local index equal to $(-1)^n$. This property was also proved by Thews in [13].

2. DEFINITIONS AND NOTATION

Let M denote a $\mathscr{C}^2$ n-dimensional manifold and let X be a $\mathscr{C}^1$ tangent vector field on M. For each $x_0 \in M$ we will denote the unique solution of the initial value problem

$$\begin{aligned} \dot{x} &= X(x) \\ x(0) &= x_0 \end{aligned} \tag{2}$$

by $\varphi(x_0, t)$ and its maximal interval of definition by $J(x_0)$.

In the sequel given a nonempty set Q of M we will denote its topological boundary by $\partial_M Q$, its closure by $\overline{Q}$, its interior by $\mathrm{Int}(Q)$ and its complement by $M \backslash Q$.

Definition 2.1. The ω-limit set (respectively α-limit set) of $x_0 \in M$, denoted here by $\omega(x_0)$ (respectively $\alpha(x_0)$), is the set of points $y \in M$ such that there exists $t_n \nearrow +\infty$ (respectively $t_n \searrow -\infty$) as $n \to \infty$ satisfying $\varphi(x_0, t_n) \to y$ as $n \to \infty$.

Notice that if $\omega(x_0) \neq \emptyset$ (respectively $\alpha(x_0) \neq \emptyset$) then by definition $J(x_0)$ contains $\mathbb{R}^+$ (respectively $\mathbb{R}^-$). On the other hand it is well known that $\omega(x_0)$ is a closed set and that if $y \in \omega(x_0)$ then $J(y) = \mathbb{R}$ and $\varphi(y,t) \in \omega(x_0)$ for all $t \in \mathbb{R}$. Finally, the ω-limit set of $\varphi(x_0,t)$ is the same for any $t \in J(x_0)$. Clearly any α-limit set satisfies similar properties.

Definition 2.2. Let K be a nonempty compact subset of M. The set of points $x \in M$ satisfying $\omega(x) \neq \emptyset$ and $\omega(x) \subset K$ is called the *region of attraction* of K and in what follows it will be denoted by $\mathscr{A}$.

The region of attraction $\mathscr{A}$ of a compact set may be empty but, when it is not, notice that if $p \in \mathscr{A}$ then $\varphi(p,t) \in \mathscr{A}$ for all $t \in J(p)$ and that by definition $\mathbb{R}^+ \subset J(p)$.

Definition 2.3. Given a compact set K with region of attraction $\mathscr{A}$ we will say that K is an *attractor* if there is an open neighbourhood of K inside $\mathscr{A}$. That is, if $K \subset \mathrm{Int}(\mathscr{A})$. A compact set is said to be a *global attractor* when its region of attraction is M.

It is easy to see that the region of attraction of an attractor compact set is open when M is a boundaryless manifold or when X never points outward at ∂M.

Definition 2.4. Given an attractor compact set K we define the *stabilizer* of K, denoted by $\widetilde{K}$, as the set of points $p \in M$ such that $\alpha(p) \cap K \neq \emptyset$.

Definition 2.5. Let K be a nonempty compact subset of M. We shall say that K is *stable* if for each open neighbourhood U of K there is an open neighbourhood $V \subset U$ such that for all $x \in V$ we have $\mathbb{R}^+ \subset J(x)$ and $\varphi(x,t) \in U$ for all $t \geqslant 0$.

Definition 2.6. We shall say that a compact set is *asymptotically stable* when it is attractor and stable simultaneously.

For more details about these definitions the reader is referred to [1].

Definition 2.7. If $x_0 \notin \partial M$ is an isolated critical point of X, one defines the *local index of X at x_0* as follows. Select a coordinate neighbourhood U of x_0, homeomorphic to an open n-disk, which contains no other critical points of X. Within U choose an $(n-1)$-sphere about x. At each point of this sphere the associated vector of X must be nonzero. Transferring this into $\mathbb{R}^n$ and normalizing the vectors defines a map from S^{n-1} to S^{n-1}. The degree of this map is the local index of X at x_0 (see [10] for more details).

Definition 2.8. Let N be a compact manifold and let X be a continuous tangent vector field on N vanishing nowhere on ∂N. The *index* of X, denoted here by $\mathrm{Ind}(X)$, is defined as follows. Take any continuous tangent vector field Y on N, close to X (relative to the $\mathscr{C}^0$-topology) and having finitely many critical points, none of them in ∂N. Then $\mathrm{Ind}(X)$ is the sum of the local index of Y at all its critical points (see [9] for details).

Definition 2.9. If B is a subset of M with nonempty interior such that the critical points of X inside B form a compact set Z satisfying $Z \subset \mathrm{Int}(B)$ and $Z \cap \partial M = \emptyset$ then we define $\mathrm{Ind}_B(X)$ to be $\mathrm{Ind}(X|_N)$, where N is any compact n-manifold with $Z \subset N \backslash \partial N$ and $N \subset B$. For a proof of the consistence of this definition the reader is referred to [8], where $\mathrm{Ind}_B(X)$ is defined exactly in the same way but assuming that B is also open in M. Note that when

X has finitely many critical points in B then $\mathrm{Ind}_B(X)$ equals to the sum of the local index of X at these critical points.

It must be pointed out that this definition is only a trivial extension of the one appearing in [8]. We extend it because the region of attraction of an attractor compact set is not necessarily open in M when $\partial M \neq \emptyset$. Anyway if B and X fulfil the conditions of Definition 2.9 then $\mathrm{Ind}_B(X)$ is no more than the index, according to [8], of X at $\mathrm{Int}(B)$.

Definition 2.10. A topological space K is called an *euclidean neighbourhood retract* (ENR) if there exists a positive integer k and $Y \subset \mathbb{R}^k$, Y being homeomorphic with K, such that there is an open set U, $Y \subset U \subset \mathbb{R}^k$ and Y is a retract of U. For instance any finite CW-complex is an ENR (see Example E.8. in [2]).

Definition 2.11. Given a topological space A in what follows $\chi(A)$ will denote, provided that the number is defined, the *Euler characteristic* of A. For instance the Euler characteristic of any compact ENR is defined (see Section V.4.11 in [6]).

We conclude this section by recalling some well known facts that will be frequently used henceforth. If M is a boundaryless manifold then the set $\Omega = \{(x,t): x \in M,\ t \in J(x)\}$ is open in $M \times \mathbb{R}$ and the flow

$$\varphi : \Omega \to M$$
$$(x,t) \mapsto \varphi(x,t)$$

is a $\mathscr{C}^1$ map. If $\partial M \neq \emptyset$ and X is tangent to ∂M everything is as before.

The preceding results are not always true when X has general boundary conditions. However some of them can be used if we first embed M as a closed submanifold of a $\mathscr{C}^2$ n-dimensional manifold $\widehat{M}$ without boundary, such as the double of M, and then extend X to a $\mathscr{C}^1$ tangent vector field on $\widehat{M}$. By using this extension it is easy to see for example that (2) has continuity with respect to initial conditions in the following case:

Remark 2.12. If $p_0 \in M$ and $t_0 \in J(p_0)$ are such that $\varphi(p_0,t) \in M \backslash \partial M$ for all t between 0 and t_0 then for each open neighbourhood V of $\varphi(p_0,t_0)$ there is an open neighbourhood U of p_0 such that if $p \in U$ then $\varphi(p,t_0) \in V$.

Notice that if $\varphi(p_0,t) \in \partial M$ for some t between 0 and t_0 then we cannot even assert the existence of some open neighbourhood U of p_0 such that $t_0 \in J(p)$ for all $p \in U$.

On the other hand, when we deal with the region of attraction $\mathscr{A}$ of a compact set there is also some sort of continuity with respect to initial conditions of (2) in the following sense (recall that $\mathbb{R}^+ \subset J(p)$ for all $p \in \mathscr{A}$):

Remark 2.13. If $p_0 \in \mathscr{A}$ and $t_0 \geqslant 0$ then for each open neighbourhood V of $\varphi(p_0,t_0)$ there is an open neighbourhood U of p_0 such that $\varphi(p,t_0) \in V$ for all $p \in U \cap \mathscr{A}$.

3. THE ASYMPTOTICALLY STABLE CASE

This section is entirely devoted to prove the next theorem.

THEOREM 3.1. *Let K be an asymptotically stable invariant compact set for the flow generated by a $\mathscr{C}^1$ tangent vector field X on a $\mathscr{C}^2$ n-dimensional manifold M and let $\mathscr{A}$*

denote its region of attraction. If $K \cap \partial M = \emptyset$ then the Euler characteristic of $\mathscr{A}$ is defined and satisfies the following relations:

(a) $\mathrm{Ind}_{\mathscr{A}}(X) = (-1)^n \chi(\mathscr{A})$.
(b) *If K is an ENR then* $\chi(K) = \chi(\mathscr{A})$.

We will show first some relations between an attractor compact set and its stabilizer. In particular Lemma 3.3 provides a criterion that turns out to be a very useful tool when we deal with an asymptotically stable compact set.

Remark 3.2. Notice that if A is a compact set with $A \cap \partial M = \emptyset$ and $p \in A$ is such that $\varphi(p,t) \in A$ for all $t \in \mathbb{R}^- \cap J(p)$ then $\alpha(p)$ is a nonempty subset of A. Thus, if K is an attractor compact set with $K \cap \partial M = \emptyset$ then $\widetilde{K}$ contains all the negatively invariant subsets of K. In particular $K \subset \widetilde{K}$ when K is an attractor compact set negatively invariant and $K \cap \partial M = \emptyset$.

LEMMA 3.3. *Let K be an attractor positively invariant compact set with $K \cap \partial M = \emptyset$. Then $\widetilde{K} \subset K$ if and only if K is stable.*

Proof. Assume that $\widetilde{K} \subset K$ and let V be an open neighbourhood of K. In order to show that K is stable we must prove that there is an open neighbourhood U of K satisfying that if $p \in U$ then $\mathbb{R}^+ \subset J(p)$ and $\varphi(p,t) \in V$ for all $t \geqslant 0$. It is clear that there is no loss of generality in assuming that $\overline{V}$ is compact and inside $\mathscr{A} \backslash \partial M$.

Consider any point $y \in \partial_M V$. Then $\widetilde{K} \subset K \subset V$ implies $y \notin \widetilde{K}$ and this means that $\alpha(y) \cap K = \emptyset$. Then there exists $t < 0$ such that $\varphi(y,t) \notin \overline{V}$. Otherwise, since $\overline{V}$ is a compact set inside $M \backslash \partial M$, $\alpha(y)$ would be a nonempty invariant compact subset of $\overline{V} \backslash K$ and this obviously contradicts $\overline{V} \subset \mathscr{A}$. Thus, using that $\overline{V} \subset M \backslash \partial M$, we can choose $\tau(y) < 0$ such that $\varphi(y, \tau(y)) \notin \overline{V}$ and $\varphi(y,t) \in M \backslash \partial M$ for all $t \in [\tau(y), 0]$. By Remark 2.12 we can take $B(y)$ as an open neighbourhood of y such that if $p \in B(y)$ then $\varphi(p, \tau(y)) \notin \overline{V}$.

Since $\partial_M V$ is compact there exist $p_1, p_2, \ldots, p_k \in \partial_M V$ such that

$$\partial_M V \subset \bigcup_{j=1}^{k} B(p_j). \tag{3}$$

We define $U = V \backslash \bigcup_{j=1}^{k} N(p_j)$ where, for $j = 1, 2, \ldots, k$,

$$N(p_j) = \{\varphi(p,t)\colon\ p \in \overline{B(p_j)} \cap \partial_M V,\ t \in [\tau(p_j), 0]\}.$$

Then U is open and, since K is positively invariant, $K \subset U$. Moreover it is clear that $\mathbb{R}^+ \subset J(x)$ for all $x \in U$ since $U \subset V \subset \mathscr{A}$. Next we will show by contradiction that U satisfies the required condition. Assume that $x_0 \in U$ is such that $\varphi(x_0, t) \notin V$ for some $t \geqslant 0$. Define

$$t_0 = \inf\{t \geqslant 0\colon\ \varphi(x_0, t) \notin V\}.$$

Then $\varphi(x_0, t_0) \in \partial_M V$ and, since $x_0 \in U \subset V$, $t_0 > 0$. Saying $y_0 = \varphi(x_0, t_0)$, it is clear that (3) implies $y_0 \in B(p_{j_0}) \cap \partial_M V$ for some $j_0 \in \{1, 2, \ldots, k\}$. Notice that by construction we have that $\varphi(y_0, \tau(p_{j_0})) \notin \overline{V}$. Thus, since

$$\varphi(y_0, t) \in \overline{V} \quad \text{for all } t \in [-t_0, 0]$$

it follows that $\tau(p_{j_0})<-t_0$. Then by definition $x_0\in N(p_{j_0})$ because $x_0=\varphi(y_0,-t_0)$ with $y_0\in B(p_{j_0})\cap\partial_M V$ and $\tau(p_{j_0})<-t_0<0$. Therefore $x_0\notin U$. This is a contradiction since we took $x_0\in U$.

Finally we will show that if the relation $\widetilde{K}\subset K$ is not satisfied then K is not stable. So assume that there is $x_0\notin K$ with $\alpha(x_0)\cap K\neq\emptyset$. Since K is a closed set we can take an open neighbourhood V of K such that $x_0\notin V$. In this case there is no open neighbourhood U of K such that if $p\in U$ then $\varphi(p,t)\in V$ for all $t\in\mathbb{R}^+\cap J(p)$. This is so because $\alpha(x_0)\cap K\neq\emptyset$ implies that there exist some $y_0\in K$ and $t_n\searrow-\infty$ satisfying that $\varphi(x_0,t_n)\to y_0$. ■

We will need some results concerning isolated invariant sets and Lyapunov functions. A compact set K of M is called *isolated invariant set* if it is invariant and there is a compact neighbourhood N of K such that K is the largest invariant set in N. In this case N is called an *isolating neighbourhood* for K. A continuous function $V:\Omega\to\mathbb{R}$, where Ω is an open set of M, is called a *generalized Lyapunov function* for the flow φ if

$$\dot{V}(x)=\lim_{t\to 0}\frac{V(\varphi(x,t))-V(x)}{t}$$

exists and is continuous at every point $x\in\Omega$. Clearly when V is a $\mathscr{C}^1$ function then $\dot{V}$ always exists and satisfies

$$\dot{V}(x)=(DV)_x X(x).$$

That is, $\dot{V}(x)$ is the derivative of V at x in the direction $X(x)$.

The following theorem has been proved by Wilson and Yorke in [14]. In its statement we shall need some new notation. Let N be an isolating neighbourhood for the isolated invariant set K and let Ω be an open subset of N. Then Ω_+ (respectively Ω_-) denotes the set of all $x\in\Omega$ for which $\varphi(x,t)\in N$ for all $t\geqslant 0$ (respectively $t\leqslant 0$).

THEOREM 3.4. *Let K be an isolated invariant set for the $\mathscr{C}^1$ flow φ and let N be an isolating neighbourhood for K. Then there is an open neighbourhood Ω of K in N and there are generalized Lyapunov functions $V_+:\Omega\to[0,+\infty)$ and $V_-:\Omega\to(-\infty,0]$ with the properties*:

(a) $V_+(x)=0$ *if and only if* $x\in\Omega_+$,
(b) $\dot{V}_+(x)>0$ *if* $x\in\Omega\backslash\Omega_+$,
(c) $V_+|_{\Omega\backslash\Omega_+}$ *is a* $\mathscr{C}^2$ *function.*

V_- satisfies similar properties with respect to Ω_-.

The next result is an easy application of the above theorem. However it must be pointed out that it will be assumed that M is a boundaryless manifold in order to assure the required smoothness of the flow (recall the concluding observations of Section 2).

COROLLARY 3.5. *Assume that M is a boundaryless manifold and that K is an asymptotically stable invariant compact set of M. Let $\mathscr{A}$ denote its region of attraction and let N be a compact neighbourhood of K in $\mathscr{A}$. Then there exists an open neighbourhood Ω of K in N and a generalized Lyapunov function $V:\Omega\to(-\infty,0]$ satisfying the following properties*:

(a) $V(x)=0$ *if and only if* $x\in K$,
(b) $\dot{V}(x)>0$ *if* $x\in\Omega\backslash K$,
(c) $V|_{\Omega\backslash K}$ *is a* $\mathscr{C}^2$ *function.*

Proof. Let E be any negatively invariant subset of N and consider any $p \in E$. Since N is a compact set it follows that $\alpha(p)$ is a nonempty invariant compact subset of $\overline{E}$. Due to $\overline{E} \subset N \subset \mathscr{A}$ we conclude that $\alpha(p) \cap K \neq \emptyset$. Then, making use of Lemma 3.3, $p \in K$. Thus K contains all the negatively invariant subsets of N. In particular this shows that K is an isolated invariant set and that N is an isolating neighbourhood. Therefore, using Theorem 3.4, in order to prove Corollary 3.5 it suffices to show that in this case $\Omega_- = K$. The relation $K \subset \Omega_-$ is obvious since K is invariant. On the other hand $\Omega_- \subset K$, because Ω_- is clearly a negatively invariant subset of N and we have seen that K contains all of them. Therefore Corollary 3.5 is proved. ■

We are now able to construct a family of smooth compact manifolds satisfying very pleasant properties.

Proposition 3.6. *Let K be a compact set asymptotically stable and invariant for the flow generated by a $\mathscr{C}^1$ tangent vector field X on a $\mathscr{C}^2$ n-dimensional manifold M and let $\mathscr{A}$ denote its region of attraction. If $K \cap \partial M = \emptyset$ then there exists a family of $\mathscr{C}^2$ compact n-dimensional manifolds $\{S_k\}_{k \in \mathbb{N}}$ that for each $k \in \mathbb{N}$ satisfies:*

(a) *$K \subset S_k \backslash \partial S_k$ and $S_k \subset \mathscr{A} \backslash \partial M$,*
(b) *$S_{k+1} \subset S_k \backslash \partial S_k$ and $\bigcap_{k=1}^{\infty} S_k = K$,*
(c) *The vector field X points inward at all the boundary points of S_k.*

Proof. We will first prove it assuming that M is a boundaryless manifold. In this case we can apply Corollary 3.5 and consider the open neighbourhood Ω of K, with a compact vicinity inside $\mathscr{A}$, and the generalized Lyapunov function $V : \Omega \to (-\infty, 0]$ that we get. By a compactness argument it follows that we can take $\varepsilon > 0$ small enough satisfying that $V^{-1}(-\varepsilon)$ is nonempty and that the closure of $V^{-1}([-\varepsilon, +\infty))$ is inside Ω. For each $k \in \mathbb{N}$ we define

$$S_k = V^{-1}([-\varepsilon/k, +\infty)).$$

Fix any $k \in \mathbb{N}$. Since $V|_{\Omega \backslash K}$ is a $\mathscr{C}^2$ function and $V^{-1}(-\varepsilon/k)$ is a nonempty subset of $\Omega \backslash K$, in order to prove that S_k is a $\mathscr{C}^2$ n-manifold with boundary $\partial S_k = V^{-1}(-\varepsilon/k)$ it is enough to show that $-\varepsilon/k$ is a regular value. But clearly this fact follows from

$$\dot{V}(x) = (DV)_x X(x) \neq 0 \tag{4}$$

for all $x \in \Omega \backslash K$. On the other hand S_k is closed in Ω because in fact $S_k = V^{-1}([-\varepsilon/k, 0])$. Thus, since $\overline{S}_k \subset \Omega$, we conclude that S_k is closed. Now, that S_k is compact and inside $\mathscr{A}$ it is due to the fact that Ω has a compact vicinity inside $\mathscr{A}$. Finally notice that $V^{-1}(0) = K$ implies $K \subset S_k \backslash \partial S_k$. This proves (a), and property (b) follows easily from the definition of the sets S_k.

In addition, (4) implies that X is never tangent to ∂S_k since V is constant in ∂S_k. Thus if we show that S_k is a positively invariant set with respect to the flow φ then (c) would follow. Consider any $x_0 \in \partial S_k$ and let U be an open neighbourhood of x_0 in $\Omega \backslash K$. Take $\delta > 0$ such that $\varphi(x_0, t) \in U$ for all $t \in [0, \delta)$. Then

$$V(\varphi(x_0, t)) \geqslant V(x_0) = -\varepsilon/k$$

for all $t \in [0, \delta)$ since $\dot{V}(x) > 0$ for all $x \in \Omega \backslash K$. Thus for each $x \in \partial S_k$ there exists $\delta > 0$ such that $\varphi(x, t) \in S_k$ for all $t \in [0, \delta)$. This shows that S_k is a positively invariant set because $\partial_M S_k$ coincides with ∂S_k. Therefore Proposition 3.6 is true when M is a boundaryless manifold.

Now assume that M is a manifold with boundary. Embed M as a closed submanifold of a $\mathscr{C}^2$ n-dimensional manifold $\widehat{M}$ without boundary and let $\widehat{X}$ be a $\mathscr{C}^1$ tangent vector field on $\widehat{M}$ satisfying

$$\widehat{X}|_M = X. \tag{5}$$

Notice then that K is asymptotically stable and invariant with respect to the flow generated by $\widehat{X}$. This is so because $K \cap \partial M = \emptyset$ and its region of attraction, say $\widehat{\mathscr{A}}$, contains $\mathscr{A}$. Since we have already proved Proposition 3.6 for boundaryless manifolds we can assert that there is a family of compact $\mathscr{C}^2$ n-manifolds $\{\widehat{S}_k\}_{k\in\mathbb{N}}$ satisfying (a), (b), and (c) with respect to $\widehat{X}$ and $\widehat{\mathscr{A}}$.

Since $K \subset \mathrm{Int}(\mathscr{A})\backslash\partial M$, using that $\bigcap_{k=1}^{\infty} \widehat{S}_k = K$ and that $\widehat{S}_{k+1} \subset \widehat{S}_k$ for all $k \in \mathbb{N}$, it follows that there exists $k_0 \in \mathbb{N}$ such that $\widehat{S}_k \subset \mathscr{A}\backslash\partial M$ for all $k > k_0$. For each $k \in \mathbb{N}$ we define $S_k = \widehat{S}_{k+k_0}$. It is clear that (5) shows that $\{S_k\}_{k\in\mathbb{N}}$ satisfies the required conditions with respect to X and $\mathscr{A}$. Therefore Proposition 3.6 is also true for manifolds with boundary.

■

The next result will further clarify the structure of each manifold S_k in relation to the attractor set and its region of attraction.

LEMMA 3.7. *Let $\mathscr{A}$ be the region of attraction of a compact set K asymptotically stable and invariant and let $\{S_k\}_{k\in\mathbb{N}}$ be the family of compact manifolds that we get from applying Proposition* 3.6. *Then for each $k \in \mathbb{N}$ the following properties are satisfied*:

(a) *∂S_k is a strong deformation retract of $S_k\backslash K$.*
(b) *S_k is a strong deformation retract of $\mathscr{A}$.*

Proof. Fix any $k \in \mathbb{N}$. First of all recall that $K \subset S_k\backslash\partial S_k$ and that $S_k \subset \mathscr{A}\backslash\partial M$.

Consider any point $x_0 \in S_k\backslash K$. By applying (c) in Proposition 3.6 we have that $\varphi(x_0,t) \notin \partial S_k$ for all $t>0$. On the other hand notice that $\varphi(x_0,t) \notin S_k$ for some $t<0$. Otherwise, using that S_k is a compact subset of $\mathscr{A}\backslash\partial M$, it would easily follow that $\alpha(x_0)\cap K \neq \emptyset$ and this contradicts Lemma 3.3 since $x_0 \notin K$. Thus, making use of (c) in Proposition 3.6 again, there exists a unique $\tau(x_0) \leqslant 0$ such that $\varphi(x_0,\tau(x_0)) \in \partial S_k$. Moreover notice that $\varphi(x_0,t) \in S_k\backslash K$ for all $t \in [\tau(x_0), 0]$.

Consider now any point $x_0 \in \mathscr{A}\backslash S_k$. Since S_k is a neighbourhood of K and $\omega(x_0) \subset K$ it follows that $\varphi(x_0,t) \in S_k\backslash\partial S_k$ for some $t>0$. Thus, using (c) in Proposition 3.6, there exists a unique $\tau(x_0)>0$ such that $\varphi(x_0,\tau(x_0)) \in \partial S_k$. On the other hand, by making use of (c) in Proposition 3.6 again, it follows that $\varphi(x_0,t) \notin \partial S_k$ for all $t \in \mathbb{R}^- \cap J(x_0)$.

In brief we have shown that for each $x \in \mathscr{A}\backslash K$ there exists a unique $\tau(x) \in J(x)$ such that $\varphi(x,\tau(x)) \in \partial S_k$ and also that the following properties are satisfied:

(1) If $x \in \mathscr{A}\backslash S_k$ then $\tau(x)>0$ and $\varphi(x,t) \in \mathscr{A}\backslash S_k$ for all $t \in [0,\tau(x))$.
(2) If $x \in S_k\backslash K$ then $\tau(x) \leqslant 0$ and $\varphi(x,t) \in S_k\backslash K$ for all $t \in [\tau(x), 0]$.
(3) $\tau(x) = 0$ if and only if $x \in \partial S_k$.

Next it will be proved that the mapping that applies $\tau(x)$ to each point $x \in \mathscr{A}\backslash K$ is continuous. Consider any $p_0 \in \mathscr{A}\backslash K$ and any $\varepsilon>0$. We must find an open neighbourhood V of p_0 such that $|\tau(p) - \tau(p_0)| < \varepsilon$ for all $p \in V \cap [\mathscr{A}\backslash K]$.

We will consider first the case $\tau(p_0) \leqslant 0$. That is, when $p_0 \in S_k$. Due to $S_k \subset M\backslash\partial M$, taking $\varepsilon>0$ smaller, we can assume without loss of generality that $\varphi(p_0,t) \in M\backslash\partial M$ for all $t \in [\tau(p_0)-\varepsilon, \tau(p_0)]$. Let U' be an open neighbourhood of $\varphi(p_0,\tau(p_0)+\varepsilon)$ in $S_k\backslash\partial S_k$. Since $\varphi(p_0,t) \in S_k \subset M\backslash\partial M$ for all t between 0 and $\tau(p_0)+\varepsilon$, by making use of Remark 2.12, we can take an open neighbourhood V' of p_0 such that if $p \in V'$ then $\varphi(p, \tau(p_0)+\varepsilon) \in U'$. This easily implies that $\tau(p) < \tau(p_0) + \varepsilon$ for all $p \in V'$. Let U'' be an open neighbourhood of $\varphi(p_0, \tau(p_0) - \varepsilon)$ in $M \backslash S_k$. Since $\varphi(p_0, t) \in M \backslash \partial M$ for all $t \in [\tau(p_0) - \varepsilon, 0]$, by using Remark 2.12, we can take an open neighbourhood V'' of p_0 such that if $p \in V''$ then $\varphi(p, \tau(p_0) - \varepsilon) \in U''$. This shows $\tau(p) > \tau(p_0) - \varepsilon$ for all $p \in V''$. Hence, taking $V = V' \cap V''$ it follows that $|\tau(p) - \tau(p_0)| < \varepsilon$ for all $p \in V$.

Consider now the case $\tau(p_0) > 0$. This means $p_0 \in \mathscr{A} \backslash S_k$. It is clear that there is no loss of generality in assuming that $\tau(p_0) > \varepsilon$. Let U' be an open neighbourhood of $\varphi(p_0, \tau(p_0)+\varepsilon)$ in $S_k \backslash \partial S_k$. Since $\tau(p_0) + \varepsilon > 0$, Remark 2.13 shows that we can take an open neighbourhood V' of p_0 such that if $p \in V' \cap \mathscr{A}$ then $\varphi(p, \tau(p_0)+\varepsilon) \in U'$. This shows that $\tau(p) < \tau(p_0) + \varepsilon$ for all $p \in V' \cap \mathscr{A}$. Similarly, let U'' be an open neighbourhood of $\varphi(p_0, \tau(p_0) - \varepsilon)$ in $M \backslash S_k$. Due to $\tau(p_0) - \varepsilon > 0$, by Remark 2.13 we can take an open neighbourhood V'' of p_0 such that if $p \in V'' \cap \mathscr{A}$ then $\varphi(p, \tau(p_0) - \varepsilon) \in U''$. This shows $\tau(p) > \tau(p_0) - \varepsilon$ for all $p \in V'' \cap \mathscr{A}$. Thus, taking $V = V' \cap V''$ it follows that $|\tau(p) - \tau(p_0)| < \varepsilon$ for all $p \in V \cap \mathscr{A}$.

From (2) it follows clearly that the homotopy

$$\begin{aligned} H_1 : S_k \backslash K \times [0,1] &\to S_k \backslash K \\ (p,t) &\mapsto \varphi(p, t\tau(p)) \end{aligned}$$

is well defined. Moreover, the continuity of τ and Remark 2.12 imply that H_1 is also continuous. Thus, by making use of (3), it shows that ∂S_k is a strong deformation retract of $S_k \backslash K$. Hence (a) is proved. On the other hand it is clear that the mapping

$$\begin{aligned} \tau' : \mathscr{A} &\to [0, +\infty) \\ p &\mapsto \begin{cases} 0 & \text{if } p \in S_k \\ \tau(p) & \text{if } p \in \mathscr{A} \backslash S_k \end{cases} \end{aligned}$$

is continuous and, by making use of (1), that the homotopy

$$\begin{aligned} H_2 : \mathscr{A} \times [0,1] &\to \mathscr{A} \\ (p,t) &\mapsto \varphi(p, t\tau'(p)) \end{aligned}$$

is well defined. Moreover Remark 2.13 and the continuity of τ' imply that H_2 is also continuous. Finally H_2 shows, by using the definition of τ', that S_k is a strong deformation retract of $\mathscr{A}$. This proves (b). ■

We have already developed the tools that will allow us to prove the main result of this section.

Proof of Theorem 3.1. Consider the family of compact $\mathscr{C}^2$ n-manifolds $\{S_k\}_{k \in \mathbb{N}}$ that we get from applying Proposition 3.6. Fix any $k \in \mathbb{N}$. By (b) in Lemma 3.7, S_k is a strong deformation retract of $\mathscr{A}$. Consequently $\chi(\mathscr{A})$ is defined and satisfies the relation

$$\chi(\mathscr{A}) = \chi(S_k). \tag{6}$$

Notice that $X|_{S_k}$ is obviously a tangent vector field on S_k since M and S_k have the same dimension. On the other hand notice also that it is nonvanishing on ∂S_k since, by (a) in Proposition 3.6, $\partial S_k \subset \mathscr{A} \backslash K$. Thus, by the Poincaré–Hopf theorem, we can assert that

$$\mathrm{Ind}(X|_{S_k}) = (-1)^n \chi(S_k)$$

since, by (c) in Proposition 3.6, X points inward along ∂S_k. Substitution of (6) in the above expression shows that

$$\mathrm{Ind}_{\mathscr{A}}(X) = (-1)^n \chi(\mathscr{A})$$

because K contains all the critical points of X inside $\mathscr{A}$ and, by (a) in Proposition 3.6, $K \subset S_k \backslash \partial S_k$ and $S_k \subset \mathscr{A}$. This proves (a).

Recall now that, by (a) in Lemma 3.7, ∂S_k is a strong deformation retract of $S_k \backslash K$. Therefore $\chi(S_k \backslash K)$ is defined and equals to $\chi(\partial S_k)$. Since S_k is a compact n-manifold, applying also VIII 8.8 in [6], we conclude that

$$\chi(S_k \backslash K) = \chi(\partial S_k) = (1 + (-1)^{n+1})\, \chi(S_k). \tag{7}$$

Assume now that K is an ENR. Then, according to VIII 8.6 in [6], the relation

$$\chi(S_k) = \chi(S_k \backslash K) + (-1)^n\, \chi(K)$$

must be satisfied. Substitution of (7) in the above expression produces

$$\chi(S_k) = \chi(K). \tag{8}$$

Combination of (6) and (8) shows that $\chi(\mathscr{A}) = \chi(K)$ and therefore (b) is proved. ■

4. THE REGION OF ATTRACTION

This section is completely devoted to show Theorem B. We will first prove two technical results that will turn out to be very useful when we deal with a region of attraction.

LEMMA 4.1. *Let $\mathscr{A}$ be the region of attraction of a compact set. If $p \in \mathrm{Int}(\mathscr{A}) \backslash \partial M$ then $\varphi(p,t) \in \mathrm{Int}(\mathscr{A}) \backslash \partial M$ for all $t \geqslant 0$.*

Proof. Embed M as a closed submanifold of a $\mathscr{C}^2$ n-dimensional manifold $\widehat{M}$ without boundary and consider a $\mathscr{C}^1$ tangent vector field $\widehat{X}$ on $\widehat{M}$ satisfying

$$\widehat{X}|_M = X. \tag{9}$$

For each $x_0 \in \widehat{M}$ denote by $\widehat{\varphi}(x_0, t)$ the unique solution of the initial value problem

$$\begin{aligned} \dot{x} &= \widehat{X}(x) \\ x(0) &= x_0. \end{aligned} \tag{10}$$

Consider $p_0 \in \mathrm{Int}(\mathscr{A}) \backslash \partial M$ and $t_0 \geqslant 0$. Clearly we can take an open set $\widehat{V}$ of $\widehat{M}$ with $p_0 \in \widehat{V} \subset \mathscr{A}$. Notice that if $p \in \widehat{V}$ then $\varphi(p,t) \in \mathscr{A} \subset M$ for all $t \geqslant 0$. Therefore, from (9), if $p \in \widehat{V}$ then

$$\varphi(p,t) = \widehat{\varphi}(p,t) \in \mathscr{A} \qquad \text{for all } t \in [0, t_0].$$

Due to the continuity with respect to initial conditions of (10) there exists an open set $\widehat{U}$ of $\widehat{M}$ containing $\widehat{\varphi}(p_0, t_0)$ such that if $p \in \widehat{U}$ then $\widehat{\varphi}(p, -t_0) \in \widehat{V}$. Since $\widehat{U}$ is an open set of $\widehat{M}$ containing $\widehat{\varphi}(p_0, t_0) = \varphi(p_0, t_0)$, notice that in order to prove that $\varphi(p_0, t_0) \in \mathrm{Int}(\mathscr{A}) \backslash \partial M$ it is enough to show $\widehat{U} \subset \mathscr{A}$. But this is clear because if $y \in \widehat{U}$ then $y = \widehat{\varphi}(x, t_0)$ for some $x \in \widehat{V}$ and, due to $\widehat{V} \subset \mathscr{A}$, $\widehat{\varphi}(x, t_0) = \varphi(x, t_0) \in \mathscr{A}$. ■

LEMMA 4.2. *Let K be an attractor compact set with region of attraction $\mathscr{A}$ and let Q_1 be any compact neighbourhood of K in $\mathscr{A}$. Then there exists a compact neighbourhood Q_2 of K in $\mathscr{A}$ such that if $p \in Q_1$ then $\varphi(p,t) \in Q_2$ for all $t \geqslant 0$. Moreover if $K \cap \partial M = \emptyset$, taking $Q_1 \subset \mathrm{Int}(\mathscr{A}) \backslash \partial M$ then $Q_2 \subset \mathrm{Int}(\mathscr{A}) \backslash \partial M$.*

Proof. Consider any $y \in \partial_M Q_1$. Then there is $\tau(y) > 0$ such that $\varphi(y, \tau(y)) \in \mathrm{Int}(Q_1)$ since $\omega(y) \subset K$ and $K \subset \mathrm{Int}(Q_1)$. By Remark 2.13 we can take $U(y)$ being an open neighbourhood of y such that if $p \in \mathscr{A} \cap U(y)$ then $\varphi(p, \tau(y)) \in \mathrm{Int}(Q_1)$. We define

$$M(y) = \{\varphi(p,t) \colon p \in \overline{U(y)} \cap \partial_M Q_1,\ t \in [0, \tau(y)]\} \cup Q_1.$$

Notice that $M(y)$ is a compact neighbourhood of K in $\mathscr{A}$ because $Q_1 \subset \mathscr{A}$ and if $p \in \mathscr{A}$ then $\varphi(p,t) \in \mathscr{A}$ for all $t \geqslant 0$.

Since $\partial_M Q_1$ is compact we can take $p_1, p_2, \ldots, p_k \in \partial_M Q_1$ with $\partial_M Q_1 \subset \bigcup_{j=1}^{k} U(p_j)$. We define

$$Q_2 = \bigcup_{j=1}^{k} M(p_j).$$

Then Q_2 is a compact neighbourhood of K in $\mathscr{A}$ and, using that any orbit leaving Q_1 in positive time has a point in $\partial_M Q_1$, it is easy to see that by construction Q_2 satisfies the required condition.

Assume now that $K \cap \partial M = \emptyset$ and that we have chosen Q_1 being a compact neighbourhood of K in $\mathrm{Int}(\mathscr{A}) \backslash \partial M$. In this case, by making use of Lemma 4.1, it follows that $Q_1 \subset M(y) \subset \mathrm{Int}(\mathscr{A}) \backslash \partial M$ for all $y \in \partial_M Q_1$. Therefore Q_2 is a compact neighbourhood of K in $\mathrm{Int}(\mathscr{A}) \backslash \partial M$. ■

The next result shows that in order to prove that $\chi(\mathscr{A})$ is defined and satisfies $\mathrm{Ind}_{\mathscr{A}}(X) = (-1)^n \chi(\mathscr{A})$, it can be supposed without loss of generality that K does not intersect ∂M.

Proposition 4.3. *Let K_1 be an attractor compact set for the flow generated by a $\mathscr{C}^1$ tangent vector field X_1 on M and let $\mathscr{A}$ denote its region of attraction. If $K_1 \cap \partial M$ contains no critical point of X_1 then there exists a $\mathscr{C}^1$ tangent vector field X_2 on M and a compact set K_2 such that*:

(a) *X_1 and X_2 have the same critical points and both vector fields coincide in a neighbourhood of them.*
(b) *K_2 is an attractor compact set for the flow generated by X_2 and its region of attraction is $\mathscr{A}$.*
(c) *$K_2 \cap \partial M = \emptyset$.*

Proof. It is obvious that there is nothing to be done when $K_1 \cap \partial M = \emptyset$. So assume that $K_1 \cap \partial M \neq \emptyset$. Since X_1 has no critical points in $K_1 \cap \partial M$ we can choose an open neighbourhood U of $K_1 \cap \partial M$ in $\mathrm{Int}(\mathscr{A})$ such that its closure $\overline{U}$ is compact and X_1 has no critical points in $\overline{U}$.

Let Y be a $\mathscr{C}^1$ tangent vector field on M pointing inward at ∂M and nonvanishing in $\overline{U}$. Let $\rho : M \to [0,1]$ be a $\mathscr{C}^1$ map satisfying

$$\rho^{-1}(0) = M \backslash U \quad \text{and} \quad \rho^{-1}(1) = K_1 \cap \partial M.$$

Let us suppose that M has been endowed with a Riemannian metric. Denoting its corresponding norm by $\|\ \|$ and saying

$$a = \inf\{\|X(p)\| \colon p \in \overline{U}\} \quad \text{and} \quad b = \sup\{\|Y(p)\| \colon p \in \overline{U}\}$$

we define for each $p \in M$

$$X_2(p) = \frac{a}{2b}\rho(p)\,Y(p) + X_1(p).$$

Then X_2 is a $\mathscr{C}^1$ tangent vector field on M that satisfies

$$X_1|_{M\setminus U} = X_2|_{M\setminus U} \tag{11}$$

and it is easy to check that X_2 has no critical points in $\overline{U}$. Hence, this shows (a).

Notice that X_1 at $\mathscr{A}\cap\partial M$ must point inward to M or be tangent to ∂M. Since Y points inward to M at ∂M and $\rho\geqslant 0$, it is clear then that X_2 at $\mathscr{A}\cap\partial M$ points inward to M or is tangent to ∂M. Moreover X_2 points inward to M at $U\cap\partial M$ because $\rho^{-1}(0)=M\setminus U$.

Given $x_0\in M$, for each $i\in\{1,2\}$ we denote the unique solution of the initial value problem

$$\dot{x} = X_i(x)$$

$$x(0) = x_0$$

by $\varphi_i(x_0,t)$, its maximal interval of definition by $J_i(x_0)$ and its ω-limit set by $\omega_i(x_0)$.

Let Q_1 be a compact neighbourhood of $K_1\cup\overline{U}$ in $\mathscr{A}$. By Lemma 4.2 there exists a compact neighbourhood Q_2 of $K_1\cup\overline{U}$ in $\mathscr{A}$ such that if $p\in Q_1$ then $\varphi_1(p,t)\in Q_2$ for all $t\geqslant 0$. We claim that if $p\in\mathscr{A}$ then $\omega_2(p)$ is nonempty and inside Q_2.

In order to prove the claim we will show first that for all $p_0\in\mathscr{A}\setminus Q_1$ there exists $t_0\in\mathbb{R}^+\cap J_2(p_0)$ such that $\varphi_2(p_0,t_0)\in Q_1$. Since $p_0\in\mathscr{A}$ it follows that $\omega_1(p_0)\subset K_1$. Hence, using that $K_1\subset\text{Int}(Q_1)$, there exists some $t>0$ such that $\varphi_1(p_0,t)\in Q_1$. Due to $\overline{U}\subset\text{Int}(Q_1)$ and $p_0\notin Q_1$ we can take $t_0>0$ satisfying $\varphi_1(p_0,t)\notin U$ for all $t\in[0,t_0]$ and that $\varphi_1(p_0,t_0)\in Q_1$. From (11) it follows then

$$\varphi_1(p_0,t)=\varphi_2(p_0,t) \quad \text{for all } t\in[0,t_0]$$

and therefore $\varphi_2(p_0,t_0)\in Q_1$.

It is obvious then that we must only prove the claim when $p_0\in Q_1$. If we show that $\varphi_2(p_0,t)\in Q_2$ for all $t\in\mathbb{R}^+\cap J_2(p_0)$ then, using that X_2 at $\mathscr{A}\cap\partial M$ points inward to M or is tangent to ∂M, it will follow that $\omega_2(p_0)$ is a nonempty subset of Q_2. We will assume that it is false and we will get a contradiction. So assume that $\varphi_2(p_0,t_0)\notin Q_2$ for some $t_0>0$. Since $\overline{U}\subset Q_1\subset Q_2$ it is clear that there exists $t_0'\in[0,t_0)$ such that $\varphi_2(p_0,t_0')\in Q_1$ and that $\varphi_2(p_0,t)\notin U$ for all $t\in[t_0',t_0]$. Then from (11) it follows that $\varphi_2(p_0,t)=\varphi_1(p_0,t)$ for all $t\in[t_0',t_0]$. This contradicts the choosing of Q_2 since $\varphi_1(p_0,t_0')\in Q_1$, $\varphi_1(p_0,t_0)\notin Q_2$ and $t_0>t_0'$. Hence the claim is proved.

Recall that if $p\in\omega_2(p_0)$ then $J_2(p)=\mathbb{R}$ and $\varphi_2(p,t)\in\omega_2(p_0)$ for all $t\in\mathbb{R}$. Consequently if we say

$$W=\bigcup_{p\in\mathscr{A}}\omega_2(p)$$

then $W\subset Q_2$ and for each $p\in W$ it follows that $J_2(p)=\mathbb{R}$ and that $\varphi_2(p,t)\in W$ for all $t\in\mathbb{R}$. We define K_2 being the closure of W. Then K_2 is a compact set inside $Q_2\subset\mathscr{A}$. Moreover notice that the region of attraction of K_2, with respect to the flow generated by X_2, is $\mathscr{A}$. That it contains $\mathscr{A}$ is obvious from the definition of $K_2=\overline{W}$ and that it is exactly $\mathscr{A}$ follows easily from (11).

Our next objective is showing that $K_2\subset\text{Int}(\mathscr{A})\setminus\partial M$. This will prove that K_2 is an attractor and that it does not intersect the boundary ∂M.

We assert now that for each $p_0\in\mathscr{A}\cap[\partial_{\mathrm{M}}\mathscr{A}\cup\partial M]$ there exists $\tau(p_0)>0$ such that $\varphi_2(p_0,\tau(p_0))\in\text{Int}(\mathscr{A})\setminus\partial M$. Its existence is obvious when

$$p_0\in U\cap\mathscr{A}\cap[\partial_{\mathrm{M}}\mathscr{A}\cup\partial M]=U\cap\partial M$$

since $X_2(p_0)$ points inward to M and $U \subset \text{Int}(\mathscr{A})$. Hence it is enough to consider when

$$p_0 \in \mathscr{A} \cap [\partial_M \mathscr{A} \cup \partial M] \backslash U.$$

Notice that in this case it is not possible that $\varphi_2(p_0, t) \in [\partial_M \mathscr{A} \cup \partial M] \backslash U$ for all $t > 0$ because according to (11) it would follow $\omega_1(p_0) \subset [\partial_M \mathscr{A} \cup \partial M] \backslash U$. This is clearly a contradiction because $\omega_1(p_0) \subset \text{Int}(\mathscr{A})$ and on the other hand if $\omega_1(p_0) \subset \partial M$ then $\omega_1(p_0) \subset K_1 \cap \partial M \subset U$. Therefore there exists $t' > 0$ such that $\varphi_2(p_0, t') \notin [\partial_M \mathscr{A} \cup \partial M] \backslash U$. If $\varphi_2(p_0, t') \notin \partial_M \mathscr{A} \cup \partial M$ we choose $\tau(p_0) = t'$ and if $\varphi_2(p_0, t') \in U \subset \text{Int}(\mathscr{A})$ we choose $\tau(p_0) = \tau(\varphi_2(p_0, t'))$ in case that $\varphi_2(p_0, t') \in U \cap \partial M$ and $\tau(p_0) = t'$ otherwise. Hence the assertion is true.

For each $p \in \mathscr{A} \cap [\partial_M \mathscr{A} \cup \partial M]$, by using Remark 2.13, we can define $U(p)$ as an open neighbourhood of p such that if $y \in \mathscr{A} \cap U(p)$ then $\varphi_2(y, \tau(p)) \in \text{Int}(\mathscr{A}) \backslash \partial M$.

Since $Q_2 \subset \mathscr{A}$ is compact, so it is $Q_2 \cap [\partial_M \mathscr{A} \cup \partial M]$ and hence there exist $p_1, p_2, \dots, p_k$ in $Q_2 \cap [\partial_M \mathscr{A} \cup \partial M]$ satisfying that

$$Q_2 \cap [\partial_M \mathscr{A} \cup \partial M] \subset \bigcup_{j=1}^{k} U(p_j).$$

We define $T = \max\{\tau(p_j): j = 1, 2, \dots, k\}$. Then, since $T > 0$ and $\mathscr{A}$ is the region of attraction of the compact set K_2 with respect to the flow generated by X_2, by making use of Lemma 4.1 we can assert that

$$\{\varphi_2(p, T): \ p \in Q_2\} \subset \text{Int}(\mathscr{A}) \backslash \partial M. \tag{12}$$

Since $\{\varphi_2(p, T): \ p \in Q_2\}$ is compact and $W \subset Q_2$, by using (12) we conclude that

$$\overline{\{\varphi_2(p, T): \ p \in W\}} \subset \text{Int}(\mathscr{A}) \backslash \partial M. \tag{13}$$

On the other hand, making use that for each $p \in W$ we have $J_2(p) = \mathbb{R}$ and $\varphi_2(p, t) \in W$ for all $t \in \mathbb{R}$, it follows that

$$W = \{\varphi_2(p, T): \ p \in W\}. \tag{14}$$

The combination of (13) and (14) shows that K_2, the closure of W, is inside $\text{Int}(\mathscr{A}) \backslash \partial M$. This shows $K_2 \cap \partial M = \emptyset$ and consequently (c) is proved. On the other hand K_2 is an attractor compact set due to the fact that $K_2 \subset \text{Int}(\mathscr{A})$. Then (b) is also proved since we had already showed that its region of attraction is $\mathscr{A}$. ■

Once we have proved the following result we will be in position to show Theorem B. It shows that the stabilizer of an attractor compact set is an asymptotically stable invariant compact set with the same region of attraction.

PROPOSITION 4.4. *Let K be an attractor compact set with $K \cap \partial M = \emptyset$ and let $\mathscr{A}$ denote its region of attraction. Then the following properties are satisfied:*

(a) $\widetilde{K} \cap \partial M = \emptyset$.
(b) $\widetilde{K}$ *is an asymptotically stable invariant compact set with region of attraction* $\mathscr{A}$.

Proof. Notice first of all that $\widetilde{K}$ is obviously invariant because for each $x_0 \in M$ the α-limit set of $\varphi(x_0, t)$ is the same for all $t \in J(x_0)$. Due to $K \cap \partial M = \emptyset$ we can take a compact neighbourhood Q_1 of K in $\text{Int}(\mathscr{A}) \backslash \partial M$. By making use of Lemma 4.2, there exists a compact neighbourhood Q_2 of K in $\mathrm{Int}(\mathscr{A})\backslash\partial M$ such that if $p\in Q_1$ then $\varphi(p,t)\in Q_2$ for all $t\geqslant 0$. Therefore it is clear that $p\notin Q_2$ implies $p\notin\widetilde{K}$. This shows

$$\widetilde{K}\subset Q_2\subset \mathrm{Int}(\mathscr{A})\backslash\partial M \tag{15}$$

and hence $\widetilde{K}\cap\partial M=\emptyset$. Thus, (a) is proved.

We will prove that $\widetilde{K}$ is closed by showing that $M\backslash\widetilde{K}$ is open. Consider any $p_0\notin\widetilde{K}$. If $p_0\notin Q_2$ from (15) it follows that we can take an open neighbourhood U of p_0 such that $U\subset M\backslash\widetilde{K}$. Thus, it suffices to consider when $p_0\in Q_2$. Notice that $\alpha(p_0)\cap K=\emptyset$ implies that it exists $t_0\leqslant 0$ such that $\varphi(p_0,t_0)\notin Q_2$. Otherwise, since Q_2 is a compact set inside $M\backslash\partial M$, $\alpha(p_0)$ would be a nonempty invariant compact subset of $Q_2\backslash K$ and this clearly contradicts $Q_2\subset\mathscr{A}$. Since $p_0\in Q_2$ and $Q_2\subset M\backslash\partial M$ we can take $t_0'\in[t_0,0)$ such that $\varphi(p_0,t)\in M\backslash\partial M$ for all $t\in[t_0',0]$ and $\varphi(p_0,t_0')\notin Q_2$. Let V be an open neighbourhood of $\varphi(p_0,t_0')$ in $M\backslash Q_2$. Notice then that from (15) it follows $V\subset M\backslash\widetilde{K}$. According to Remark 2.12 we can take U as an open neighbourhood of p_0 such that if $p\in U$ then $\varphi(p,t_0')\in V$. Since $\widetilde{K}$ is invariant and $V\subset M\backslash\widetilde{K}$ it is clear that $U\subset M\backslash\widetilde{K}$. Hence $M\backslash\widetilde{K}$ is open.

We can assert now that $\widetilde{K}$ is compact since we have just showed that it is closed and from (15) it has a compact neighbourhood.

We will prove next that $\widetilde{K}$ is an attractor with region of attraction $\mathscr{A}$. Consider any $p_0\in\mathscr{A}$. Since K is an attractor with region of attraction $\mathscr{A}$ it follows that $\omega(p_0)\subset K$. Due to the invariance of $\omega(p_0)$ and $K\subset M\backslash\partial M$, we can assert by using Remark 3.2 that $\omega(p_0)\subset\widetilde{K}$. This shows that the region of attraction of $\widetilde{K}$ contains $\mathscr{A}$ and the fact that it is exactly $\mathscr{A}$ is obvious. Now we conclude that $\widetilde{K}$ is an attractor by using that, according to (15), $\widetilde{K}\subset\mathrm{Int}(\mathscr{A})$.

Since we have already proved that $\widetilde{K}$ is an attractor invariant compact set satisfying $\widetilde{K}\cap\partial M=\emptyset$, we can make use of Lemma 3.3 to show that it is stable. Let $p_0\in M$ be such that $\alpha(p_0)\cap\widetilde{K}\neq\emptyset$. Then $\alpha(p_0)\cap\widetilde{K}$ is a nonempty invariant compact set and, making use of (15), this implies

$$[\alpha(p_0)\cap\widetilde{K}]\cap K\neq\emptyset.$$

Hence $\alpha(p_0)\cap K\neq\emptyset$, and this means that $p_0\in\widetilde{K}$. By applying Lemma 3.3 we conclude that $\widetilde{K}$ is stable.

In brief, we have showed that $\widetilde{K}$ is an asymptotically stable invariant compact set with region of attraction $\mathscr{A}$. Hence (b) is proved. ■

COROLLARY 4.5. *Let K be an attractor compact set with $K\cap\partial M=\emptyset$. Then K is asymptotically stable and invariant if and only if $K=\widetilde{K}$.*

Proof. If K is an asymptotically stable invariant compact set with $K\cap\partial M=\emptyset$ then $K=\widetilde{K}$ follows from the combination of Remark 3.2 and Lemma 3.3. The implication in the other direction follows from applying Proposition 4.4. ■

Proof of Theorem B. We will prove first that if X is nonvanishing on $K\cap\partial M$ then $\chi(\mathscr{A})$ is defined and satisfies $\mathrm{Ind}_{\mathscr{A}}(X)=(-1)^n\chi(\mathscr{A})$. In this case, using Proposition 4.3, we may assume without restriction that $K\cap\partial M=\emptyset$. Then, from Proposition 4.4, $\widetilde{K}$ is an asymptotically stable invariant compact set with region of attraction $\mathscr{A}$ and $\widetilde{K}\cap\partial M=\emptyset$. Now the result follows from applying Theorem 3.1 to $\widetilde{K}$.

Assume finally that K does not intersect ∂M. Then, by Proposition 4.4, $\widetilde{K}$ is an asymptotically stable invariant compact set with region of attraction $\mathscr{A}$ and $\widetilde{K}\cap\partial M=\emptyset$. Thus, if

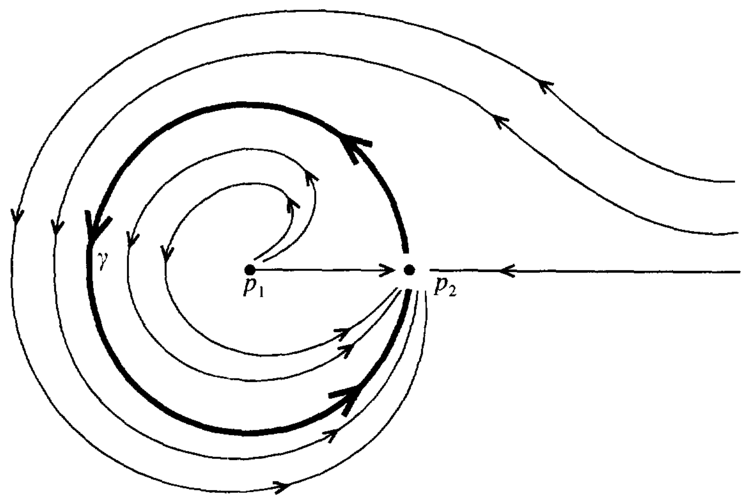

Fig. 1.

$\widetilde{K}$ is an ENR then the relation $\chi(\mathscr{A})=\chi(\widetilde{K})$ follows from applying (b) of Theorem 3.1 to $\widetilde{K}$. This proves (a). On the other hand, if K is asymptotically stable and invariant then Corollary 4.5 shows that $K=\widetilde{K}$. Therefore (b) follows from (a). ■

In view of (b) in Theorem B one may wonder whether it is true that for an attractor compact set K with region of attraction $\mathscr{A}$, the relation $\chi(K)=\chi(\mathscr{A})$ is satisfied if K is an ENR with $K \cap \partial M = \emptyset$. It is not even satisfied when K is also invariant, and we will show it by means of the following example.

The figure below is the phase portrait of a $\mathscr{C}^1$ vector field in $\mathbb{R}^2$ with two critical points, p_1 and p_2. It is clear that $K=\{p_2\}$ is an attractor invariant compact set with region of attraction $\mathscr{A}=\mathbb{R}^2\backslash\{p_1\}$. Here K is an ENR with $\chi(K)=1$ but $\chi(\mathscr{A})=0$. On the other hand notice that $\widetilde{K}$ is $\{p_2\}\cup\gamma$, an asymptotically stable invariant compact set with region of attraction $\mathscr{A}$. Moreover it is to be noted that in this example we can check all the relations given in Theorem B because $\widetilde{K}$ is an ENR and $\mathrm{Ind}_{\mathscr{A}}(X)$ is easy to compute. Thus the local index of the vector field at p_2 is 0, $\chi(\widetilde{K})=0$ and $\chi(\mathscr{A})=0$.

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