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Isochronicity for Several Classes of Hamiltonian Systems[1]

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In this paper we study isochronous centers of analytic Hamiltonian systems giving special attention to the polynomial case. We first revisit the potential systems and we show the connection between isochronicity and involutions. We then study a more general system, namely the ones associated to Hamiltonians of the form $H(x, y) = A(x) + B(x)\,y + C(x)\,y^2$. As an application we classify the cubic polynomial Hamiltonian isochronous centers and we give examples of nontrivial and nonglobal polynomial Hamiltonian isochronous centers. © 1999 Academic Press

1. INTRODUCTION

In this paper we study isochronous centers of analytic Hamiltonian systems giving special attention to the polynomial case.

The problem of characterizing isochronous centers has attracted the attention of several authors. However there are very few families of polynomial differential systems in which a complete classification of the isochronous centers has been found. Quadratic systems were classified by Loud [8] and cubic systems with homogeneous nonlinearities by Pleshkan [11]. Kukles' systems were classified in [3]. Some other results can be found in [1]. Concerning Hamiltonian systems there are also very few results. It is proved in [2] that in the potential case the unique polynomial isochronous center is the linear one. Several authors (see [3, 5, 13]) proved that there are not Hamiltonian systems with homogeneous nonlinearities having an isochronous center at the origin. Apart from few other special cases, the knowledge of polynomial systems with isochronous centers is slight. For some other results on isochronicity we refer the reader to [3, 6, 9, 10] and references there in.

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There is a simple method to generate polynomial Hamiltonian isochronous centers. Take two polynomials P and Q in two variables with $P(0,0) = Q(0,0) = 0$ such that the determinant of the Jacobian of the mapping $(x, y) \mapsto (P(x, y), Q(x, y))$ is constant. A couple of polynomials such that is called a Jacobian pair. It is readily seen that the Hamiltonian system associated to

$$H(x, y) = \frac{P(x, y)^2 + Q(x, y)^2}{2}$$

is linearizable by means of the canonical change of coordinates $u = P(x, y)$, $v = Q(x, y)$ and hence that the origin is an isochronous center. We call trivial isochronous centers the centers constructed with this method. One of the main motivations of this work was the following question.

QUESTION 1. *Are there polynomial Hamiltonian nontrivial isochronous centers?*

In Section 3 we present some examples which give a positive answer to Question 1. On the other hand, Sabatini gives in [12] a relationship between the polynomial Hamiltonian isochronous centers and the Jacobian conjecture in dimension two. He proved that this last conjecture is equivalent to the following.

Conjecture. Any Jacobian pair produces an isochronous global center at the origin.

Here the word global means that every solution of the Hamiltonian system is a periodic orbit surrounding the origin. Related to the above problem it arises in a natural way the following question.

QUESTION 2. *Are there polynomial Hamiltonian isochronous nonglobal centers?*

The examples in Section 3 give also a positive answer to Question 2. Turning now to the problem of characterizing isochronicity, the lowest degree polynomial Hamiltonian family which is not yet completely classified is the family of cubic Hamiltonian systems. This type of systems have been studied in [9]. In that paper the authors proved that any cubic Hamiltonian isochronous center which is Darboux linearizable is a trivial isochronous center in the sense mentioned before. In the last part of this paper we complete the classification proving that every cubic Hamiltonian isochronous center is a trivial isochronous center.

We now recall briefly some general notions that will be used frequently and we introduce the notation used henceforth.

For any center p of a planar differential system, the largest neighbourhood of p which is entirely covered by periodic orbits is called the *period annulus* of p and we will denote it by $\mathscr{P}$. A center is said to be a *global center* when its period annulus is the whole plane. The function which associates to any periodic orbit γ in $\mathscr{P}$ its period is called the *period function*. The center is called an *isochronous center* when the period function is constant. It is well known that only nondegenerate centers can be isochronous. Moreover a center of an analytic differential system is isochronous if and only if there exists an analytic change of coordinates transforming the initial system to the linear center

$$\begin{cases} \dot{x} = -ky, \\ \dot{y} = kx. \end{cases}$$

It can be shown also that an isochronous center has not finite critical points in the boundary of its period annulus. When the differential system is analytic this implies that the period annulus of an isochronous center is unbounded.

This paper deals with Hamiltonian systems, i.e., with differential systems of the form

$$\begin{cases} \dot{x} = -H_y(x, y), \\ \dot{y} = H_x(x, y), \end{cases} \tag{1}$$

where H is an analytic function on $\mathbb{R}^2$. The solutions of these systems are contained in the level curves $\{H(x, y) = h,\ h \in \mathbb{R}\}$. From now on we will assume that $H(0, 0) = 0$ and that the system (1) has a nondegenerate center at the origin.

One can show (see [5] for instance) that $H(z) \neq 0$ for every point $z \in \mathscr{P}$ different from the origin. Thus we will assume, without loss of generality, that $H(z) > 0$ for all $z \in \mathscr{P} \backslash \{(0, 0)\}$. In this case $H(\mathscr{P}) = [0, h_0)$, where $h_0 \in \mathbb{R}^+ \cup \{+\infty\}$. We will use this notation all over the paper. It can be shown that when h_0 is finite then the boundary of the period annulus is contained in the energy level $\{H(x, y) = h_0\}$.

In addition one can prove (see [5]) that the set of all the periodic orbits in the period annulus can be parametrized by the energy. Thus, for each $h \in (0, h_0)$ we will denote the periodic orbit in $\mathscr{P}$ of energy level h by γ_h. This allows us to consider the period function over $(0, h_0)$ instead of the original period function which is defined over the set of periodic orbits contained in the period annulus. Therefore in the sequel we will talk about the period function $T(h)$ which gives the period of the periodic orbit with energy $h \in (0, h_0)$. It can be proved that $T(h)$ can be extended analytically to $h = 0$ when the origin is a nondegenerate center.

The paper is organized as follows. In Section 2 we revisit the potential systems and we give a new characterization of isochronicity which is more geometric than Urabe's criterion (see Theorem A). As a consequence we show that every analytic isochronous center in this family is global. We also show the strong connection between isochronous centers and involutions on the real line (see Theorem B). In Section 3 we study the family of Hamiltonian systems which are quadratic with respect to one of the variables. That is, Hamiltonian systems associated to $H(x, y) = A(x) + B(x)\, y + C(x)\, y^2$. It is to be noted that this family generalizes the potential Hamiltonian systems. We give a isochronicity criterion in the analytic case (see Theorem C) and a complete classification of the isochronous global centers in the polynomial case (see Theorem D). We also give a method to construct nontrivial and nonglobal polynomial Hamiltonian isochronous centers. Finally in Section 4 we classify all the cubic polynomial Hamiltonian isochronous centers (see Theorem E).

2. THE POTENTIAL CASE

In this section we consider Hamiltonian systems of the form

$$H(x, y) = \frac{y^2}{2} + V(x),$$

where V is an analytic function defined on $\mathbb{R}$ with a nondegenerate relative minimum at the origin. This kind of systems arise from conservative second order scalar differential equations of the form

$$\ddot{x} + f(x) = 0, \qquad \text{where} \quad f = V'.$$

The isochronous centers of these Hamiltonian systems are already characterized by Urabe's Theorem (see [14, 15]) even in the case that V is $\mathscr{C}^1$ and defined only in a neighbourhood of the origin. However our approach will give a simple geometric interpretation of the isochronicity condition and, since we suppose that V is analytic, the proofs will be elementary. Moreover the fact that V is defined not only in a neighbourhood of the origin but in $\mathbb{R}$ will allow us to give a new result. In addition we show a strong connection between isochronous centers and analytic involutions. This connection provides a method to construct isochronous centers and a very simple proof of some classical results.

We define

$$(x_{\mathrm{I}}, x_{\mathrm{S}}) = \{x \in \mathbb{R} \colon \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \mathscr{P}\}.$$

That is, $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is the projection of the period annulus to the x-axis. Since by definition the origin is the unique critical point in $\mathscr{P}$ it easily follows that V' is negative (respectively positive) on the interval $(x_{\mathrm{I}}, 0)$ (respectively on $(0, x_{\mathrm{S}})$). Moreover if $\partial\mathscr{P}$ is contained in the level curve $H=h_0$ then $V((x_{\mathrm{I}}, x_{\mathrm{S}}))=[\,0, h_0)$. Denoting by V_- (respectively V_+) the restriction of V on $(x_{\mathrm{I}}, 0)$ (respectively on $(0, x_{\mathrm{S}})$), for each $h\in(0, h_0)$ we define

$$\ell(h)=V_+^{-1}(h)-V_-^{-1}(h).$$

In Fig. 1 is given a geometric interpretation of this definition. The first result that we will prove shows that this length characterizes the isochronous centers. However we must first prove the following easy fact.

LEMMA 2.1. *Let* $\phi\colon [a, b)\, \mathbb{R}$ *be analytic and let* $\psi\colon (a, b)\times(a, b)\to (0, +\infty]$ *satisfy*

$$\int_a^x \phi(z)\,\psi(x, z)\,dz=k_1 \qquad \textit{and} \qquad \int_a^x \psi(x, z)\,dz=k_2$$

for all $x\in(a, b)$. *Then* $\phi(z)=k_1/k_2$ *for all* $z\in(a, b)$.

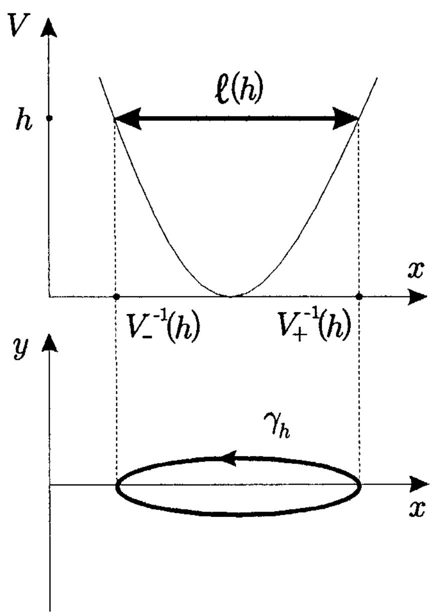

FIG. 1. Interpretation of $\ell(h)$ in terms of γ_h.

Proof. One can easily verify that both conditions imply that

$$\int_a^x \left(\phi(z)-\frac{k_1}{k_2}\right)\psi(x,z)\,dz=0 \qquad \text{for all} \quad x\in(a,b). \tag{2}$$

If $\phi\not\equiv k_1/k_2$ then the analyticity of ϕ implies that there exists $\varepsilon>0$ such that $\phi(z)\neq k_1/k_2$ for all $z\in(a,a+\varepsilon)$. We can assume without loss of generality that

$$\phi(z)>\frac{k_1}{k_2} \qquad \text{for all} \quad z\in(a,a+\varepsilon).$$

Therefore, since ψ is a strictly positive function, this shows that (2) cannot hold for any $x\in(a,a+\varepsilon)$. ∎

THEOREM A. *The origin is an isochronous center of period ω if and only if*

$$\ell(h)=\frac{\omega}{\pi}\sqrt{2h} \qquad \textit{for all} \quad h\in(0,h_0).$$

Proof. Consider any $h\in(0,\sqrt{h_0})$. The period of the periodic orbit γ_{h^2} is given by

$$\begin{aligned} T(h^2)&=2\int_{V_-^{-1}(h^2)}^{V_+^{-1}(h^2)}\frac{dx}{\sqrt{2(h^2-V(x))}}\\ &=\sqrt{2}\int_0^{V_+^{-1}(h^2)}\frac{dx}{\sqrt{h^2-V(x)}}-\sqrt{2}\int_0^{V_-^{-1}(h^2)}\frac{dx}{\sqrt{h^2-V(x)}}. \end{aligned}$$

The change of coordinates $x=V_+^{-1}(u^2)$ and $x=V_-^{-1}(u^2)$ in the first and second integral above respectively yield

$$T(h^2)=\sqrt{2}\int_0^h\frac{(V_+^{-1}(u^2)-V_-^{-1}(u^2))'}{\sqrt{h^2-u^2}}\,du.$$

Thus, we have showed that for all $h\in(0,\sqrt{h_0})$ it holds

$$T(h^2)=\sqrt{2}\int_0^h\frac{(\ell(u^2))'}{\sqrt{h^2-u^2}}\,du. \tag{3}$$

We claim that $u\mapsto\ell(u^2)$ is analytic on $[0,\sqrt{h_0})$. To prove this we define

$$g(x)=\operatorname{sgn}(x)\sqrt{V(x)}=x\sqrt{\frac{V(x)}{x^2}}.$$

Since $V(0)=V'(0)=0$, $V''(0)>0$ and $V(x)\neq 0$ for all $x\in(x_{\mathrm{I}}, x_{\mathrm{S}})\backslash\{0\}$ it follows that g is analytic on $(x_{\mathrm{I}}, x_{\mathrm{S}})$. Moreover, using also that $V'(x)\neq 0$ for all $x\in(x_{\mathrm{I}}, x_{\mathrm{S}})\backslash\{0\}$ and that $V((x_{\mathrm{I}}, x_{\mathrm{S}}))=[0, h_0)$, one can verify that its inverse g^{-1} is well defined and analytic on $(-\sqrt{h_0}, \sqrt{h_0})$. Now the claim follows from the fact that

$$\operatorname{sgn}(u)\,\ell(u^2)=g^{-1}(u)-g^{-1}(-u) \qquad \text{for all} \quad u\in(-\sqrt{h_0}, \sqrt{h_0}).$$

Assume now the the origin is an isochronous center of period ω. Then (3) implies that

$$\int_0^h \frac{(\ell(u^2))'}{\sqrt{h^2-u^2}}\,du=\frac{\omega}{\sqrt{2}} \qquad \text{for all} \quad h\in(0, \sqrt{h_0})$$

and on the other hand a computation shows that

$$\int_0^h \frac{du}{\sqrt{h^2-u^2}}=\frac{\pi}{2} \qquad \text{for all} \quad h>0.$$

In this situation by applying Lemma 2.1 we can assert that

$$\frac{d}{du}\,\ell(u^2)=\frac{\sqrt{2}\,\omega}{\pi} \qquad \text{for all} \quad u\in(0, \sqrt{h_0}).$$

Therefore we have showed the necessity of the condition

$$\ell(h)=\frac{\omega}{\pi}\sqrt{2h} \qquad \text{for all} \quad h\in(0, h_0).$$

The sufficiency of this condition is readily seen from (3). ∎

It is clear that Theorem A imposes severe restrictions on the shape of the period annulus in case that the origin is isochronous. In particular, using that the differential equation is defined in the whole plane, it implies the following result.

COROLLARY 2.2. *If the origin is an isochronous center then its period annulus is the whole plane.*

Proof. It is easy to see that if $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is bounded then the period annulus is also bounded. In this case there exists a critical point in $\partial\mathscr{P}$ and then the center can not be isochronous. Thus, if the center is isochronous then $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is unbounded and consequently $\ell(h)\to+\infty$ when $h\nearrow h_0$. Now, using Theorem A we conclude that $h_0=+\infty$ and hence that $\mathscr{P}$ is the whole plane. ∎

Next we will show the strong relation between involutions and isochronicity, and how this relation can be used to prove easily two classical results.

DEFINITION 2.3. We will say that a function σ is a *strict involution* if it is an analytic function on $\mathbb{R}$ different from the identity satisfying $\sigma(0)=0$ and $\sigma(\sigma(x))=x$ for all $x\in\mathbb{R}$.

LEMMA 2.4. *If σ is a strict involution then $\sigma'(x)<0$ for all $x\in\mathbb{R}$ and $Id-\sigma$ is an analytic diffeomorphism on $\mathbb{R}$.*

Proof. Notice first that $\sigma'(x)\neq 0$ for all $x\in\mathbb{R}$ since $\sigma^2=Id$. We will assume that $\sigma'(x)>0$ for all $x\in\mathbb{R}$ and we will get a contradiction. Since σ is different from the identity we can take $x_0\in\mathbb{R}$ such that $\sigma(x_0)\neq x_0$. Let us assume that $x_0<\sigma(x_0)$ (the other case is similar). Then, since σ is increasing, $\sigma(x_0)<\sigma^2(x_0)=x_0$ and this contradicts that $x_0<\sigma(x_0)$. Thus $\sigma'(x)<0$ for all $x\in\mathbb{R}$. Finally, using also that $|\sigma(x)|\to+\infty$ when $|x|\to+\infty$ we conclude that $Id-\sigma$ is a diffeomorphism. ∎

THEOREM B. *The origin is an isochronous center of period ω if and only if there exists a strict involution σ such that*

$$V(x)=\frac{\pi^2}{2\omega^2}(x-\sigma(x))^2 \qquad \textit{for all} \quad x\in\mathbb{R}.$$

Proof. Assume that the origin is an isochronous center of period ω. In this case Corollary 2.2 shows that the center is global. Therefore $h_0=+\infty$ and $g(x)=\operatorname{sgn}(x)\sqrt{V(x)}$ is an analytic diffeomorphism on $\mathbb{R}$. By applying Theorem A we can assert that

$$\ell(h)=\frac{\omega}{\pi}\sqrt{2h} \qquad \text{for all} \quad h>0.$$

Now, using that $\operatorname{sgn}(h)\,\ell(h^2)=g^{-1}(h)-g^{-1}(-h)$, we conclude

$$\frac{\sqrt{2}\,\omega}{\pi}h=g^{-1}(h)-g^{-1}(-h) \qquad \text{for all} \quad h\in\mathbb{R}.$$

Evaluating the above expression at $h=g(x)$ and using that $g(x)^2=V(x)$ we get

$$V(x)=\frac{\pi^2}{2\omega^2}(x-g^{-1}(-g(x)))^2 \qquad \text{for all} \quad x\in\mathbb{R}. \tag{4}$$

If we define $\sigma = g^{-1} \circ (-g)$ then one can verify that $\sigma^2 = Id$. Since $\sigma'(0) = -1$ and $\sigma(0) = 0$ this shows that σ is a strict involution. Thus (4) proves the necessity of the condition.

Let us prove now the sufficiency of the condition. First of all notice that by Lemma 2.4 the function $Id - \sigma$ is a diffeomorphism on $\mathbb{R}$. Since $\sigma(0) = 0$, this implies that the origin is a global center of the Hamiltonian system associated to $H(x, y) = (y^2/2) + V(x)$.

Fix any $\tilde{h} > 0$ and take any $\tilde{x} \in \mathbb{R}$ such that $V(\tilde{x}) = \tilde{h}$. Then, using that it holds $V(x) = V(\sigma(x))$ for all $x \in \mathbb{R}$, it follows

$$\ell(\tilde{h}) = \operatorname{sgn}(\tilde{x})(\tilde{x} - \sigma(\tilde{x})).$$

Finally we conclude that

$$\ell(\tilde{h}) = \sqrt{(\tilde{x} - \sigma(\tilde{x}))^2} = \frac{\omega}{\pi} \sqrt{2V(\tilde{x})} = \frac{\omega}{\pi} \sqrt{2\tilde{h}}$$

and this proves, applying Theorem A, the sufficiency of the condition. ∎

Remark 2.5. In the proof of Theorem B we have shown that if the origin is an isochronous center then its associated involution σ is determined by $V(x) = V(\sigma(x))$.

The following result was originally proved by Urabe in [14].

COROLLARY 2.6. *If V is even and the origin is an isochronous center then $V(x) = kx^2$ for some $k > 0$.*

Proof. Remark 2.5 shows that if the origin is an isochronous center and V is even then its associated involution is $\sigma = -Id$. Then Theorem B gives the result. ∎

The next result can be found in [2] proved by using different tools.

COROLLARY 2.7. *If V is a polynomial and the origin is an isochronous center then $V(x) = kx^2$ for some $k > 0$.*

Proof. If the origin is an isochronous center then by Theorem B there exists $k > 0$ such that $\sigma(x) = x - k \operatorname{sgn}(x) \sqrt{V(x)}$ is a strict involution.

Now if V is a polynomial of degree n we have that σ has order $n/2$ at the infinity. On the other hand, since $\sigma^2 = Id$, it follows that $n^2/4 = 1$. This shows that the degree of V is 2. That $V(x) = kx^2$ follows from the fact that the origin must be a nondegenerate center. ∎

Finally Theorem 2.8 shows, roughly speaking, the size of the set of strict involutions, denoted by $Inv(\mathbb{R})$. In its statement $Diff(\mathbb{R})$ and $Diff_{-}(\mathbb{R})$

denote respectively the set of analytic diffeomorphisms on $\mathbb{R}$ and the set of analytic odd diffeomorphisms on $\mathbb{R}$. Notice also that the combination of Theorem B and Theorem 2.8 shows how it can be constructed any isochronous center.

THEOREM 2.8. *Let σ be an analytic function on $\mathbb{R}$ with $\sigma(0)=0$. Then $\sigma \in Inv(\mathbb{R})$ if and only if there exists $f \in Diff(\mathbb{R})$ with $f(0)=0$ such that $\sigma = f^{-1} \circ (-f)$. Moreover the sets $Inv(\mathbb{R})$ and $Diff(\mathbb{R})/Diff\_(\mathbb{R})$ are in bijective correspondence.*

Proof. Given any $f \in Diff(\mathbb{R})$ with $f(0)=0$ we define $\mathscr{F}(f) = f^{-1} \circ (-f)$. One can easily verify that $\mathscr{F}(f)(0)=0$, $\mathscr{F}(f)'(0)=-1$ and $\mathscr{F}(f)^2 = Id$. This shows that $\mathscr{F}(f) \in Inv(\mathbb{R})$. Consider now some $\sigma \in Inv(\mathbb{R})$. Then by Lemma 2.4 it follows that $Id-\sigma \in Diff(\mathbb{R})$ and a computation shows that $\mathscr{F}(Id-\sigma)=\sigma$. This proves the first part of the result.

Finally we will show that if for some $f, g \in Diff(\mathbb{R})$ it holds $\mathscr{F}(f)=\mathscr{F}(g)$ then $f \circ g^{-1} \in Diff\_(\mathbb{R})$. This will prove the second part of the result. However this is readily seen because a computation shows that $\mathscr{F}(f)=\mathscr{F}(g)$ implies

$$f \circ g^{-1} = -Id \circ (f \circ g^{-1}) \circ (-Id),$$

and this means clearly that $f \circ g^{-1} \in Diff\_(\mathbb{R})$. ∎

Remark 2.9. The assumption that V must be defined in $\mathbb{R}$ could be easily replaced in all over the section by V defined in a neighbourhood of the origin, with the unique obvious exception of Corollary 2.2. That V must be analytic could be replaced by only $\mathscr{C}^1$ if before is done an accurated analysis. More precisely the proof of Theorem A requires a generalization of Lemma 2.1 as the one in [7]. Then all the results hold with slight modifications if before a strict involution is defined to be any $\mathscr{C}^1$ function σ different from the identity such that $\sigma(0)=0$ and $\sigma^2=Id$ in a neighbourhood of the origin.

In this setting we can consider the example that appears in [14] under the point of view of involutions. Indeed, the local diffeomorphism $f(x) = (x^2/2)-x$ provides the involution $\sigma = f \circ (-f^{-1})$, which yields the isochronous center associated to $V(x)=x+1-\sqrt{1+2x}$.

As it has been noted before, in [2] it is proved that the unique isochronous center with V polynomial is the linear one. Perhaps the easiest families that one could study next are those with V being an entire function and those with V being a rational function without real poles. An interesting question for further research is the existence of isochronous centers, different from the linear one, in these families. Theorem B shows that the center is isochronous if and only if $V(x) = k(x - \sigma(x))^2$ for some strict involution σ. Thus, as a first approach we could take σ being itself an entire function or a rational function without real poles and try to construct an isochronous center different from the linear one (that is, different from $V(x) = kx^2$).

In the first case σ extends holomorphically to $\mathbb{C}$ satisfying $\sigma^2(z) = z$ for all $z \in \mathbb{C}$. This implies that σ is an automorphism on $\mathbb{C}$, and hence that it must be of the form $\sigma(z) = az + b$ for some $a, b \in \mathbb{C}$. Now, since $\sigma(\mathbb{R}) = \mathbb{R}$, $\sigma^2(z) = z$ for all $z \in \mathbb{C}$ and $\sigma|_{\mathbb{R}}$ is a strict involution, it follows that $\sigma(z) = -z$ for all $z \in \mathbb{C}$. Thus, taking σ entire it is not possible to construct an isochronous center different from the linear one because in fact $\sigma = -Id$.

On the other hand, if we take σ being a rational function then it extends holomorphically to the Riemman sphere $\hat{\mathbb{C}}$ satisfying $\sigma^2(z) = z$ for all $z \in \hat{\mathbb{C}}$. Again this implies that σ is an automorphism on $\hat{\mathbb{C}}$, and hence that it must be a Moëbius transformation, i.e.,

$$\sigma(z) = \frac{az+b}{cz+d} \quad \text{for some} \quad a, b, c, d \in \mathbb{C} \quad \text{with} \quad ad - bc \neq 0.$$

Since $\sigma(\mathbb{R}) = \mathbb{R}$, $\sigma^2(z) = z$ for all $z \in \hat{\mathbb{C}}$ and $\sigma|_{\mathbb{R}}$ is a strict involution, it follows that

$$\sigma(z) = \frac{az}{cz - a} \quad \text{with} \quad a, c \in \mathbb{R} \backslash \{0\}. \tag{5}$$

Thus we conclude that every rational strict involution is of the form (5). In particular it has a real pole. Therefore in this way it is neither possible to construct an isochronous center different from the linear one because in fact there is not any σ being a rational function without real poles. What it is certainly possible using the strict involution given in (5) is to construct a potential Hamiltonian system defined in a neighbourhood of the origin and having an isochronous center. For instance the examples of rational Hamiltonian systems given in [10] can be constructed in this way.

3. QUADRATIC-LIKE HAMILTONIANS

In this section we consider Hamiltonian systems given by

$$H(x, y) = A(x) + B(x)\, y + C(x)\, y^2,$$

where A, B, and C are analytic functions on $\mathbb{R}$. We note that this family generalizes the one in Section 2, which corresponds to take $B \equiv 0$ and $C \equiv \frac{1}{2}$. Now the associated differential equation is

$$\begin{cases} \dot{x} = -B(x) - 2C(x)\, y, \\ \dot{y} = A'(x) + B'(x)\, y + C'(x)\, y^2, \end{cases} \tag{6}$$

and hence the assumptions that $H(0, 0) = 0$ and that the origin is a nondegenerate center correspond to require

$$A'(0) = A(0) = B(0) = 0 \qquad \text{and} \qquad 2C(0)\, A''(0) - B'(0)^2 > 0. \tag{7}$$

Recall also that we are assuming without loss of generality that

$$H(z) > 0 \qquad \text{for all} \quad z \in \mathscr{P} \quad \text{different from the origin.} \tag{8}$$

From now on we will use the auxiliary function $G = 4AC - B^2$ and, as in Section 2, we define

$$(x_{\mathrm{I}}, x_{\mathrm{S}}) = \{x \in \mathbb{R}\colon \text{there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \mathscr{P}\}.$$

We also define for all $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$ the function

$$g(x) = \frac{\operatorname{sgn}(x)}{2} \sqrt{\frac{G(x)}{C(x)}} = \frac{x}{2} \sqrt{\frac{G(x)}{x^2 C(x)}}.$$

It can be proved (see (a) in Proposition 3.4) that g is analytic on $(x_{\mathrm{I}}, x_{\mathrm{S}})$ and that its inverse g^{-1} is well defined and analytic on $(-\sqrt{h_0}, \sqrt{h_0})$, where h_0 is the energy level of the boundary of the period annulus $\mathscr{P}$.

The first part of the section is devoted to prove the following theorem. In its statement, that $C(x) = O(|x|^k)$ when $|x| \to +\infty$ means that there exists some $l \in (0, +\infty)$ such that

$$\lim_{|x| \to +\infty} \frac{|C(x)|}{|x|^k} = l,$$

and $(d_-(z), d_+(z))$ denotes the maximal interval of definition of the unique solution of (6) passing through $z \in \mathbb{R}^2$.

THEOREM C. *Assume that the origin is a nondegenerate center and that A, B, and C are analytic functions on $\mathbb{R}$. Then the following statements hold*:

(a) *The origin is an isochronous center of period ω if and only if*

$$\int_{g^{-1}(-x)}^{g^{-1}(x)} \frac{ds}{\sqrt{C(s)}} = \frac{2\omega}{\pi} x \qquad \textit{for all} \quad x \in (-\sqrt{h_0}, \sqrt{h_0}).$$

(b) *The origin is an isochronous center of period* ω *if*

$$(G'(x)\,C(x)-G(x)\,C'(x))^2=\frac{16\pi^2}{\omega^2}\,G(x)\,C(x)^2 \qquad \textit{for all} \quad x\in(x_{\mathrm{I}},x_{\mathrm{S}}).$$

Moreover this condition is also necessary when G *and* C *are even functions.*

(c) *If the origin is an isochronous center then either* $x_{\mathrm{I}}=-\infty$ *or* $x_{\mathrm{S}}=+\infty$. *If in addition* $C(x)=O(|x|^k)$ *when* $|x|\to+\infty$ *for some* $k\in\mathbb{R}$, *then* $\mathscr{P}$ *is the whole plane if and only if* $k\leqslant 2$.

(d) *If the origin is an isochronous nonglobal center of period* ω *and* $(x_{\mathrm{I}},x_{\mathrm{S}})=\mathbb{R}$ *then for any* $z\notin\mathscr{P}$

$$d_+(z)-d_-(z)=\frac{\omega}{\pi}\arcsin\sqrt{\frac{h_0}{h}},$$

where $h=H(z)$.

The proof of Theorem C is organized in the following way. Theorem 3.8 shows the isochronicity criterion given in (a). The sufficiency of the condition appearing in (b) is shown in Proposition 3.10, while that this condition is also necessary when G and C are even it is proved in Proposition 3.13. Turning now to the result in (c), Corollary 3.7 shows that $(x_{\mathrm{I}},x_{\mathrm{S}})$ is unbounded in case that the center is isochronous and Proposition 3.9 proves the relation concerning the order of C at infinity. Finally Theorem 3.15 proves (d).

In the second part of the section we focus on the polynomial case and we prove the following.

THEOREM D. *Assume that the origin is a nondegenerated center and that* A, B *and* C *are polynomials. Then the following statements hold*:

(a) *The origin is an isochronous global center of period* ω *if and only if* $C(x)=c$ *with* $c>0$ *and* $G(x)=((2\pi/\omega)\,x)^2$.

(b) *If the origin is an isochronous nonglobal center then* $\deg(G)=\deg(C)\geqslant 4$.

(c) *There are polynomials* A, B *and* C *such that the planar Hamiltonian system given by* $H(x,y)=A(x)+B(x)\,y+C(x)\,y^2$ *has an isochronous nonglobal center at the origin.*

It is to be noted that Theorem C and Theorem D generalize the results appearing in Section 2. The proof of Theorem D is organized as follows. The algebraic characterization of the isochronous global centers in (a) is proved in Theorem 3.18. Turning now to the necessary condition in (b), that $\deg(G)=\deg(C)$ is proved in Lemma 3.16 and that $\deg(C)\geqslant 4$ is proved in Proposition 3.19. We note that the combination of (a) and (b) in Theorem D determines all the isochronous centers with $\deg(C) \leqslant 3$. Finally (c) is shown by means of Example 3.22 and Example 3.23.

First we shall prove a technical result that describes the behaviour of the functions G and C on $(x_{\mathrm{I}}, x_{\mathrm{S}})$.

LEMMA 3.1. *The following relations are satisfied*:

(a) $C(x) > 0$ *for all* $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$.

(b) $G(x) > 0$ *for all* $x \in (x_{\mathrm{I}}, x_{\mathrm{S}}) \backslash \{0\}$, $G(0) = G'(0) = 0$ *and* $G''(0) > 0$.

(c) *If* $x \in (x_{\mathrm{I}}, x_{\mathrm{S}}) \backslash \{0\}$ *then* $(G/C)'(x) \neq 0$. *Moreover* $(G/C)'(0) = 0$ *and* $(G/C)''(0) > 0$.

(d) *If* (x_0, y_0) *is a critical point of* (6) *different from the origin then* $x_0 \notin (x_{\mathrm{I}}, x_{\mathrm{S}})$. *In particular the origin is the unique critical point when* $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$.

Proof. Fix any $\tilde{x} \in (x_{\mathrm{I}}, x_{\mathrm{S}})$. Then there exist y_1 and y_2 with $y_1 \neq y_2$ such that

$$H(\tilde{x}, y_1) = H(\tilde{x}, y_2) = h \qquad \text{for some} \quad h \in (0, h_0)$$

and $(\tilde{x}, y) \in \mathscr{P}$ for all $y \in [y_1, y_2]$. Since y_1 and y_2 are different roots of the equation $H(\tilde{x}, y) = h$ it is clear that $C(\tilde{x}) \neq 0$. From the inequality (7) it follows that $A''(0) \neq 0$ and, using (8), one can show that in fact $A''(0) > 0$. By using again (7) this shows that $C(0) > 0$. Since $\tilde{x}$ is arbitrary this implies that $C(\tilde{x}) > 0$ and hence (a) is proved.

Assume now that $\tilde{x} \neq 0$. If there exists $\bar{y} \in \mathbb{R}$ with $H(\tilde{x}, \bar{y}) = 0$ it is easy to verify making use of $C(\tilde{x}) > 0$ and $H(\tilde{x}, y_1) = H(\tilde{x}, y_2) > 0$, that $\bar{y} \in [y_1, y_2]$. Then $(\tilde{x}, \bar{y}) \in \mathscr{P} \backslash \{(0, 0)\}$ would satisfy $H(\tilde{x}, \bar{y}) = 0$ and this contradicts (8). Thus $A(\tilde{x}) + B(\tilde{x})\, y + C(\tilde{x})\, y^2 \neq 0$ for all $y \in \mathbb{R}$ and this implies, using $C(\tilde{x}) \neq 0$, that $B(\tilde{x})^2 - 4A(\tilde{x})\, C(\tilde{x}) < 0$. Therefore $G(\tilde{x}) > 0$. This shows (b) since the relations $G(0) = G'(0) = 0$ and $G''(0) > 0$ follow readily from (7).

Next we will show that

$$H_x\left(\tilde{x}, \frac{-B(\tilde{x})}{2C(\tilde{x})}\right) \neq 0. \tag{9}$$

First of all we note that $H(\tilde{x}, y_1) = H(\tilde{x}, y_2)$ implies that there exists $\tilde{y} \in [y_1, y_2]$ such that $H_y(\tilde{x}, \tilde{y}) = 0$. Since $(\tilde{x}, \tilde{y}) \in \mathscr{P}$ with $\tilde{x} \neq 0$ this shows that $H_x(\tilde{x}, \tilde{y}) \neq 0$ (otherwise there would be a critical point inside $\mathscr{P}$ different from the origin). This proves (9) because using $C(\tilde{x}) \neq 0$ and $H_y(\tilde{x}, \tilde{y}) = B(\tilde{x}) + 2C(\tilde{x})\, \tilde{y}$ it follows $\tilde{y} = -B(\tilde{x})/2C(\tilde{x})$.

Now a computation shows that

$$H_x\left(\tilde{x}, \frac{-B(\tilde{x})}{2C(\tilde{x})}\right) = -\frac{4A'(\tilde{x})\, C(\tilde{x})^2 - 2B(\tilde{x})\, B'(\tilde{x})\, C(\tilde{x}) + C'(\tilde{x})\, B(\tilde{x})^2}{2C(\tilde{x})^2}$$
$$= -2\left(\frac{G}{C}\right)'(\tilde{x}).$$

Therefore (c) is proved since $(G/C)'(0) = 0$ and $(G/C)''(0) > 0$ follow easily from the last equality in the above expression using that $G(0) = G'(0) = 0$, $G''(0) > 0$ and $C(0) > 0$.

Finally we will prove (d) by contradiction. So assume that (x_0, y_0) is a critical point of (6) different to the origin with $x_0 \in (x_{\mathrm{I}}, x_{\mathrm{S}})$. Then again $H_y(x_0, y_0) = 0$ implies

$$y_0 = \frac{-B(x_0)}{2C(x_0)}. \tag{10}$$

Here we have used that $C(x_0) \neq 0$ due to $x_0 \in (x_{\mathrm{I}}, x_{\mathrm{S}})$. Notice that, from (7), $B(0) = 0$. Consequently (10) shows that $x_0 \neq 0$ because $(x_0, y_0) \neq (0, 0)$. On the other hand (x_0, y_0) must also satisfy $H_y(x_0, y_0) = 0$. Therefore, using (10),

$$H_x\left(x_0, \frac{-B(x_0)}{2C(x_0)}\right) = 0$$

with $x_0 \neq 0$. This is a contradiction because (9) holds for every $\tilde{x} \in (x_{\mathrm{I}}, x_{\mathrm{S}}) \backslash \{0\}$. ∎

Remark 3.2. We note that there is a change of coordinates transforming system (6) into a potential Hamiltonian system. Indeed, at any point (x, y) with $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$, using (a) in Lemma 3.1 we can rewrite the Hamiltonian as

$$H(x, y) = \frac{1}{2}\left(\sqrt{2C(x)}\, y + \frac{B(x)}{\sqrt{2C(x)}}\right)^2 + \frac{1}{4}\left(\frac{G}{C}\right)(x).$$

Now the canonical change of coordinates

$$u = \psi(x) = \int_0^x \frac{ds}{\sqrt{2C(s)}} \qquad \text{and} \qquad v = \sqrt{2C(x)}\, y + \frac{B(x)}{\sqrt{2C(x)}}$$

brings system (6) to the potential system associated to

$$\widetilde{H}(u, v) = \frac{v^2}{2} + \frac{1}{4}\left(\frac{G}{C}\right)(\psi^{-1}(u)). \tag{11}$$

Notice that even in case that $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is unbounded it may happen that $\psi((x_{\mathrm{I}}, x_{\mathrm{S}}))$ is a bounded interval. Thus, the potential system associated to (11) it may be defined only in a vertical strip containing the v-axis. In fact we will not make use of this transformation at any part of the paper.

For each $h \in (0, h_0)$ let γ_h denote the periodic orbit of $\mathscr{P}$ with energy level h and let $T(h)$ denote its period. We define

$$[x_0(h), x_1(h)] = \{x \in \mathbb{R}\text{: there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \gamma_h\}.$$

The results stated in the next lemma will be used frequently henceforth.

LEMMA 3.3. *The following statements hold*:

(a) $\frac{1}{4}(G/C)(x) \to h_0$ *when* $x \searrow x_{\mathrm{I}}$ *or* $x \nearrow x_{\mathrm{S}}$.

(b) *The period of the periodic orbit* γ_h *is given by*

$$T(h) = 2 \int_{x_0(h)}^{x_1(h)} \frac{dx}{\sqrt{B(x)^2 - 4C(x)(A(x)-h)}}.$$

(c) *If* $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$ *then* $(x, -B(x)/2C(x)) \in \mathscr{P}$.

Proof. Notice first of all that

$$x_0(h) \searrow x_{\mathrm{I}} \qquad \text{and} \qquad x_1(h) \nearrow x_{\mathrm{S}} \quad \text{when} \quad h \nearrow h_0. \tag{12}$$

From the equation $A(x) + B(x)\, y + C(x)\, y^2 = h$ it follows that

$$B^2(x) - 4C(x)(A(x) - h) = 4C(x)\, h - G(x)$$

is positive when $x \in (x_0(h), x_1(h))$ and zero when $x = x_0(h)$ or $x = x_1(h)$. This shows in particular that

$$\frac{1}{4}\left(\frac{G}{C}\right)(x_0(h)) = h \qquad \text{and} \qquad \frac{1}{4}\left(\frac{G}{C}\right)(x_1(h)) = h$$

and hence, using (12), that $\frac{1}{4}(G/C)(x) \to h_0$ when $x \searrow x_{\mathrm{I}}$ or $x \nearrow x_{\mathrm{S}}$. This proves (a).

It is also clear that if $(x, y) \in \gamma_h$ then

$$y = \frac{-B(x) \pm \sqrt{B(x)^2 - 4C(x)(A(x)-h)}}{2C(x)} \tag{13}$$

and this implies, using that $\dot{x} = -B(x) - 2C(x)\,y$, that the period of γ_h is given by

$$T(h) = 2 \int_{x_0(h)}^{x_1(h)} \frac{dx}{\sqrt{B(x)^2 - 4C(x)(A(x) - h)}}.$$

This proves (b).

Consider finally any $\bar{x} \in (x_{\mathrm{I}}, x_{\mathrm{S}})$. Then $\bar{x} \in (x_0(h), x_1(h))$ for some periodic orbit γ_h in $\mathscr{P}$. If y_1 and y_2 are the roots of $H(\bar{x}, y) = h$ then it is clear that $(\bar{x}, y) \in \mathscr{P}$ for all $y \in [y_1, y_2]$. In particular, using expression (13), this implies that $(\bar{x}, -B(\bar{x})/2C(\bar{x})) \in \mathscr{P}$. This shows (c) and completes the proof of the lemma. ▮

We are now in position to give another expression of the period function that will be very useful in order to characterize the isochronous centers.

Proposition 3.4. *The following statements hold*:

(a) *If* $g(x) = (\operatorname{sgn}(x)/2)\sqrt{G(x)/C(x)}$ *then* g *is analytic on* $(x_{\mathrm{I}}, x_{\mathrm{S}})$, $g(0) = 0$ *and* $g'(x) > 0$ *for all* $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$. *The inverse function* g^{-1} *is defined and analytic on* $(-\sqrt{h_0}, \sqrt{h_0})$.

(b) *If* $h \in (0, h_0)$ *then the period of the periodic orbit* γ_h *is given by*

$$T(h) = \int_{-\pi/2}^{\pi/2} \frac{(g^{-1})'(\sqrt{h}\sin\theta)}{\sqrt{C(g^{-1}(\sqrt{h}\sin\theta))}}\, d\theta.$$

(c) *If the center is isochronous of period* ω *then* $G''(0) = 8\pi^2/\omega^2$.

Proof. That g is well defined and analytic on $(x_{\mathrm{I}}, x_{\mathrm{S}})$ and that $g(0) = 0$ follow from the combination of (a), (b), and (c) in Lemma 3.1. Since its derivative is given by

$$g'(x) = \frac{\operatorname{sgn}(x)}{4}\sqrt{\frac{G(x)}{C(x)}}\left(\frac{G}{C}\right)'(x),$$

(c) in Lemma 3.1 shows that $g'(x) > 0$ for all $x \in (x_{\mathrm{I}}, x_{\mathrm{S}}) \backslash \{0\}$. On the other hand, using (b) in Lemma 3.1 again, it is easy to verify that

$$g'(0) = \frac{1}{2}\sqrt{\frac{G''(0)}{2C(0)}} \neq 0. \tag{14}$$

That g^{-1} is defined and analytic on $(-\sqrt{h_0}, \sqrt{h_0})$ follows from (a) in Lemma 3.3 and using that $g'(x) > 0$ for all $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$. This proves (a).

Turning now to the result in (b) we make the change of variables $z = g(x)$ in the expression of $T(h)$ given by (b) in Lemma 3.3. Notice that for all $h \in (0, h_0)$ it holds

$$g(x_0(h)) = -\sqrt{h} \qquad \text{and} \qquad g(x_1(h)) = \sqrt{h}$$

since $x_0(h) < 0 < x_1(h)$. Thus, using that $4C(x)\, z^2 = 4C(x)\, A(x) - B(x)^2$, we obtain

$$T(h) = \int_{-\sqrt{h}}^{\sqrt{h}} \frac{dz}{\sqrt{(h - z^2)\, C(g^{-1}(z))}\, g'(g^{-1}(z))}. \tag{15}$$

A final change of variables $z = \sqrt{h} \sin \theta$ produces

$$T(h) = \int_{-\pi/2}^{\pi/2} \frac{(g^{-1})'\,(\sqrt{h} \sin \theta)}{\sqrt{C(g^{-1}(\sqrt{h} \sin \theta))}}\, d\theta.$$

Hence (b) is proved. Finally if the origin is an isochronous center of period ω, the above expression shows

$$\omega = \lim_{h \searrow 0} T(h) = \frac{\pi}{\sqrt{C(0)}\, g'(0)}.$$

Now taking into account (14) it follows $G''(0) = 8\pi^2/\omega^2$, and hence (c) is proved.

Proposition 3.4 can now be applied to prove the following result.

COROLLARY 3.5. *Suppose that the origin is a global center. Then* $T(h) \to 0$ *as* $h \to +\infty$ *if* $\sqrt{C(x)}\, g'(x) \to +\infty$ *as* $|x| \to +\infty$.

Proof. Notice first that when the origin is a global center, $h_0 = +\infty$ and $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$. Using those facts and (a) in Lemma 3.3 we obtain that

$$\lim_{|x| \to +\infty} |g(x)| = +\infty.$$

Since g is analytic on $\mathbb{R}$ by (a) in Proposition 3.4, this shows that if $\theta \neq 0$ then

$$\lim_{h \to +\infty} |g^{-1}(\sqrt{h} \sin \theta)| = +\infty.$$

Consequently, the required condition in the statement implies that

$$F_h(\theta) = \frac{(g^{-1})'(\sqrt{h}\sin\theta)}{\sqrt{C(g^{-1}(\sqrt{h}\sin\theta))}} \to 0 \qquad \text{as} \quad h \to +\infty \tag{16}$$

if $\theta \neq 0$. Using that $C(x) > 0$ for all $x \in \mathbb{R}$ by (a) in Lemma 3.1, and that $g'(x) > 0$ for all $x \in \mathbb{R}$ by (a) in Proposition 3.4, the required condition also implies the existence of $m > 0$ such that

$$0 < \frac{1}{\sqrt{C(x)}\, g'(x)} < m \qquad \text{for all} \quad x \in \mathbb{R}.$$

Thus, for any $h > 0$ it follows that $0 < F_h(\theta) < m$ for all $\theta \in (-\pi/2, \pi/2)$. Now using (16), the expression of the period function given by (b) in Proposition 3.4, and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h \to +\infty} T(h) = \lim_{h \to +\infty} \int_{-\pi/2}^{\pi/2} F_h(\theta)\, d\theta = 0. \quad \blacksquare$$

Next we will prove a technical result that will become particularly useful to describe the shape of the period annulus of an isochronous center in the polynomial case. Notice that in its statement it is implicit that we deal with a nonglobal center.

PROPOSITION 3.6. *$T(h) \to +\infty$ when $h \nearrow h_0$ in case that x_S* (respectively *x_I*) *is finite and $C(x_S) = 0$* (respectively $C(x_I) = 0$). *Moreover this condition is satisfied if there exist a bounded sequence $(x_n)_{n\in\mathbb{N}}$ and an unbounded sequence $(y_n)_{n\in\mathbb{N}}$ such that $(x_n, y_n) \in \partial\mathscr{P}$ for all $n \in \mathbb{N}$.*

Proof. First of all we note that $H(\partial\mathscr{P}) = h_0$ is finite because $\mathscr{P}$ is not the whole plane. If $\partial\mathscr{P}$ contains some critical point it is well known that $T(h) \to +\infty$ as $h \nearrow h_0$. Hence we may assume without loss of generality that $\partial\mathscr{P}$ does not contain any critical point. Consider for example that $x_S < +\infty$ and $C(x_S) = 0$ (the other case is similar). Then, since $G(x_S) \geqslant 0$ by (b) in Lemma 3.1, it follows that $B(x_S) = 0$. Notice that in this case the straight line $x = x_S$ is invariant with respect to the flow generated by (6).

We claim that it holds $C'(x_S) = 0$. The combination of Lemma 3.1 and (a) in Lemma 3.3 implies that $C(x) > 0$ and $B(x)^2 - 4C(x)(A(x) - h_0) > 0$ for all $x \in (x_I, x_S)$. Then, for each $x \in (x_I, x_S)$, the equation $H(x, y) = h_0$ has two different real roots given by

$$y = \frac{-B(x) \pm \sqrt{B(x)^2 - 4C(x)(A(x) - h_0)}}{2C(x)}.$$

Denote them by $y_{h_0}^+(x)$ and $y_{h_0}^-(x)$ taking $y_{h_0}^-(x) < y_{h_0}^+(x)$. It is clear that when $x \in (x_{\mathrm{I}}, x_{\mathrm{S}})$ then $(x, y) \in \partial\mathscr{P}$ if and only if either $y = y_{h_0}^+(x)$ or $y = y_{h_0}^-(x)$.

Since there are not critical points on $\partial\mathscr{P}$, the invariance of the straight line $x = x_{\mathrm{S}}$ implies that $|y_{h_0}^+(x)| \to +\infty$ and $|y_{h_0}^-(x)| \to +\infty$ as $x \nearrow x_{\mathrm{S}}$. Then there are only two possibilities:

(a) $y_{h_0}^+(x) \to +\infty$ (respectively $-\infty$) and $y_{h_0}^-(x) \to +\infty$ (respectively $-\infty$) when $x \nearrow x_{\mathrm{S}}$.

(b) $y_{h_0}^+(x) \to +\infty$ and $y_{h_0}^-(x) \to -\infty$ when $x \nearrow x_{\mathrm{S}}$.

In the first case $|y_{h_0}^+(x) + y_{h_0}^-(x)| \to +\infty$ as $x \nearrow x_{\mathrm{S}}$. Then, taking into account that

$$y_{h_0}^+(x) + y_{h_0}^-(x) = -\frac{B(x)}{C(x)}$$

and that $C(x_{\mathrm{S}}) = B(x_{\mathrm{S}}) = 0$, we conclude that $C'(x_{\mathrm{S}}) = 0$. In the second case we note that $\partial\mathscr{P}$ contains the straight line $x = x_{\mathrm{S}}$. Then $H(x_{\mathrm{S}}, y) = h_0$ for all $y \in \mathbb{R}$. This shows $A(x_{\mathrm{S}}) = h_0$ since $C(x_{\mathrm{S}}) = B(x_{\mathrm{S}}) = 0$. On the other hand notice that in this case $|y_{h_0}^+(x)\, y_{h_0}^-(x)| \to +\infty$ as $x \nearrow x_{\mathrm{S}}$. Therefore, due to

$$y_{h_0}^+(x)\, y_{h_0}^-(x) = \frac{A(x) - h_0}{C(x)}$$

and $C(x_{\mathrm{S}}) = A(x_{\mathrm{S}}) - h_0 = 0$, this implies that $C'(x_{\mathrm{S}}) = 0$. Hence $C'(x_{\mathrm{S}}) = 0$ in both cases and this proves the claim.

We are now in position to finish the proof of the first part. Take any $h \in (0, h_0)$ and consider the periodic orbit γ_h. Using the expression of $T(h)$ given by (b) in Lemma 3.3 and Fatou's Lemma we obtain

$$\lim_{h \nearrow h_0} T(h) \geqslant 2 \int_{x_{\mathrm{I}}}^{x_{\mathrm{S}}} \frac{dx}{\sqrt{B(x)^2 - 4C(x)(A(x) - h_0)}}. \tag{17}$$

Here we have used that $x_0(h) \to x_{\mathrm{I}}$ and $x_1(h) \to x_{\mathrm{S}}$ when $h \nearrow h_0$. Finally from (17) we can conclude that

$$\lim_{h \nearrow h_0} T(h) = +\infty$$

since $C(x_{\mathrm{S}}) = C'(x_{\mathrm{S}}) = B(x_{\mathrm{S}}) = 0$.

Assume now that there exist a bounded sequence $(x_n)_{n\in\mathbb{N}}$ and an unbounded sequence $(y_n)_{n\in\mathbb{N}}$ such that $(x_n, y_n)\in\partial\mathscr{P}$ for all $n\in\mathbb{N}$.

Since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence there exist $\tilde{x}\in\mathbb{R}$ and $(n_k)_{k\in\mathbb{N}}$ such that $x_{n_k}\to\tilde{x}$ as $k\to+\infty$. It is clear that either $\tilde{x}\in(x_{\mathrm{I}}, x_{\mathrm{S}})$, $\tilde{x}=x_{\mathrm{I}}$ or $\tilde{x}=x_{\mathrm{S}}$. On the other hand, $H(x_{n_k}, y_{n_k})=h_0$ for all $k\in\mathbb{N}$ since $(x_{n_k}, y_{n_k})\in\partial\mathscr{P}$ for all $k\in\mathbb{N}$. This implies that $C(\tilde{x})=0$ because $(y_{n_k})_{k\in\mathbb{N}}$ must be an unbounded sequence and

$$y_{n_k}=\frac{-B(x_{n_k})\pm\sqrt{B(x_{n_k})^2-4C(x_{n_k})(A(x_{n_k})-h_0)}}{2C(x_{n_k})}\qquad\text{for all}\quad k\in\mathbb{N}.$$

Thus, due to $C(x)>0$ for all $x\in(x_{\mathrm{I}}, x_{\mathrm{S}})$ by (a) in Lemma 3.1, it follows that either $x_{\mathrm{S}}<+\infty$ with $C(x_{\mathrm{S}})=0$ or $x_{\mathrm{I}}>-\infty$ with $C(x_{\mathrm{I}})=0$. This proves the result. ∎

As a first consequence of Proposition 3.6 we get the following result, which provides some information about the period annulus of an isochronous center.

COROLLARY 3.7. *If* $(x_{\mathrm{I}}, x_{\mathrm{S}})$ *is bounded then* $T(h)\to+\infty$ *as* $h\nearrow h_0$.

Proof. Notice first that if $\mathscr{P}$ is bounded then $T(h)\to+\infty$ when $h\nearrow h_0$ because in this case $\partial\mathscr{P}$ contains some critical point. Hence we can suppose without restriction that $\mathscr{P}$ is not bounded. Then, if $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is bounded, it is clear that there exist a bounded sequence $(x_n)_{n\in\mathbb{N}}$ and an unbounded sequence $(y_n)_{n\in\mathbb{N}}$ such that $(x_n, y_n)\in\partial\mathscr{P}$ for all $n\in\mathbb{N}$. Now Proposition 3.6 shows that $T(h)\to+\infty$ as $h\nearrow h_0$. ∎

Now our next objective is to give necessary and sufficient conditions for isochronicity. To this end we prove the following criterion by making use of the expression of the period function given in (15).

THEOREM 3.8. *The origin is an isochronous center of period* ω *if and only if*

$$\int_{g^{-1}(-x)}^{g^{-1}(x)}\frac{ds}{\sqrt{C(s)}}=\frac{2\omega}{\pi}\,x$$

for all $x\in(-\sqrt{h_0}, \sqrt{h_0})$.

Proof. Take any $h\in(0, h_0)$ and consider the periodic orbit γ_h. A manipulation of the expression of the period function given by (15) shows that

$$T(h)=\int_0^{\sqrt{h}}\left(\frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}}+\frac{(g^{-1})'(-z)}{\sqrt{C(g^{-1}(-z))}}\right)\frac{dz}{\sqrt{h-z^2}}. \tag{18}$$

For each $z \in (-\sqrt{h_0}, \sqrt{h_0})$ denote

$$\phi(z) = \frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}} + \frac{(g^{-1})'(-z)}{\sqrt{C(g^{-1}(-z))}} \qquad \text{and} \qquad \psi(h, z) = \frac{1}{\sqrt{h^2 - z^2}}.$$

It is clear that ϕ is an analytic function on $(-\sqrt{h_0}, \sqrt{h_0})$ and that ψ is strictly positive.

Assume that the origin is an isochronous center of period ω. Then $T(h^2) = \omega$ for all $h \in (0, \sqrt{h_0})$ and, from (18), we have that

$$\int_0^h \phi(z)\, \psi(h, z)\, dz = \omega \qquad \text{for all} \quad h \in (0, \sqrt{h_0}).$$

Moreover one can easily check that

$$\int_0^h \psi(h, z)\, dz = \frac{\pi}{2} \qquad \text{for all} \quad h \in (0, \sqrt{h_0}).$$

Now, applying Lemma 2.1, we conclude that $\phi(z) = 2\omega/\pi$ for all $z \in (0, \sqrt{h_0})$. Since ϕ is an analytic function this implies that

$$\frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}} + \frac{(g^{-1})'(-z)}{\sqrt{C(g^{-1}(-z))}} = \frac{2\omega}{\pi} \qquad \text{for all} \quad z \in (-\sqrt{h_0}, \sqrt{h_0}). \tag{19}$$

Consider any $x \in (-\sqrt{h_0}, \sqrt{h_0})$. Then from (19) we obtain

$$\int_0^x \frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}}\, dz + \int_0^x \frac{(g^{-1})'(-z)}{\sqrt{C(g^{-1}(-z))}}\, dz = \frac{2\omega}{\pi}\, x.$$

The changes of variables $s = g^{-1}(z)$ and $s = g^{-1}(-z)$ in the first and the second integral above respectively yields

$$\int_{g^{-1}(-x)}^{g^{-1}(x)} \frac{ds}{\sqrt{C(s)}} = \frac{2\omega}{\pi}\, x. \tag{20}$$

This proves the necessity of the condition.

Finally let us prove the sufficiency of the condition. If one calculates the derivative with respect to x on both sides of (20), one gets (19). From the substitution of (19) in (18) and a computation it follows that $T(h) = \omega$ for all $h \in (0, h_0)$. Thus if the relation (20) holds for all $x \in (-\sqrt{h_0}, \sqrt{h_0})$ then the origin is an isochronous center of period ω. ∎

The following result will show that when the origin is an isochronous center and the function C has order at the infinity then this order determines if the center is global.

PROPOSITION 3.9. *Suppose that the origin is an isochronous center and that* $C(x) = O(|x|^k)$ *when* $|x| \to +\infty$ *for some* $k \in \mathbb{R}$. *Then the center is global if and only if* $k \leqslant 2$.

Proof. We note that if (x_I, x_S) is bounded then Corollary 3.7 shows that $T(h) \to +\infty$ as $h \nearrow h_0$. Thus, if the origin is an isochronous center then either $x_S = +\infty$ or $x_I = -\infty$. We consider the case $x_S = +\infty$ (the other case is similar).

First of all notice that $\mathscr{P}$ is the whole plane if and only if $h_0 = +\infty$. On the other hand applying Theorem 3.8 we obtain

$$\frac{2\omega}{\pi}\sqrt{h_0} = \lim_{x \nearrow \sqrt{h_0}} \int_{g^{-1}(-x)}^{g^{-1}(x)} \frac{ds}{\sqrt{C(s)}} = \int_{x_I}^{+\infty} \frac{ds}{\sqrt{C(s)}}. \tag{21}$$

Here we use that, by (a) in Lemma 3.3, $g^{-1}(x) \to x_S$ and $g^{-1}(-x) \to x_I$ as $x \nearrow \sqrt{h_0}$.

First we shall prove the result when $x_I > -\infty$. In this case it holds $C(x_I) \neq 0$ otherwise $T(h) \to +\infty$ as $h \nearrow h_0$ by Proposition 3.6. Thus, using also (a) in Lemma 3.1, we have that $C(x) > 0$ for all $x \in [x_I, +\infty)$. Taking into account (21), this shows that $h_0 = +\infty$ if and only if $k \leqslant 2$.

Consider finally that $x_I = -\infty$. Then $C(x) > 0$ for all $x \in \mathbb{R}$ by (a) in Lemma 3.1. Clearly (21) shows that if $C(x) = O(|x|^k)$ when $|x| \to +\infty$ then $h_0 = +\infty$ if and only if $k \leqslant 2$.

From Proposition 3.9 we conclude that in the polynomial case every isochronous nonglobal center must satisfy $\deg(C) \geqslant 3$. In fact this bound will be improved in the last part of this section.

We provide now a sufficient condition for isochronicity in terms of G and C. Notice that this condition is purely algebraic when G and C are polynomials.

PROPOSITION 3.10. *The origin is an isochronous center of period* ω *if*

$$(G'(x)\, C(x) - G(x)\, C'(x))^2 = \frac{16\pi^2}{\omega^2}\, G(x)\, C(x)^2 \qquad \textit{for all} \quad x \in (x_I, x_S).$$

Proof. Denoting $\{(x, y) \in \mathbb{R}^2 : x \in (x_I, x_S)\}$ by $\mathscr{R}$, define for any $(x, y) \in \mathscr{R}$

$$u = \sqrt{2}\, g(x) \qquad \text{and} \qquad v = \frac{B(x) + 2C(x)\, y}{\sqrt{2C(x)}}. \tag{22}$$

Then Lemma 3.1 and Proposition 3.4 show that (22) is an analytic change of coordinates on $\mathscr{R}$. On the other hand one can easily verify that

$$H(x, y) = A(x) + B(x)\, y + C(x)\, y^2 = \frac{u^2+v^2}{2}. \tag{23}$$

Moreover a computation shows that the required condition guarantees that

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{2\pi}{\omega} \qquad \text{for all} \quad (x, y) \in \mathscr{R}. \tag{24}$$

Here we use that $\operatorname{sgn}(G'(x)\, C(x) - G(x)\, C'(x)) = \operatorname{sgn}(x)$ for all $x \in (x_{\mathrm{I}}, x_{\mathrm{S}}) \setminus \{0\}$ by (c) in Lemma 3.1. It is well known (see [12] for instance) that (23) together with (24) imply that the change of coordinates (22) transforms the system

$$\begin{cases} \dot{x} = -H_y(x, y), \\ \dot{y} = H_x(x, y), \end{cases} \qquad \text{into the form} \qquad \begin{cases} \dot{u} = -\dfrac{2\pi}{\omega}\, v, \\[2ex] \dot{v} = \dfrac{2\pi}{\omega}\, u. \end{cases}$$

Now the result follows from the fact that the second system has an isochronous center of period ω at the origin. ∎

Remark 3.11 The proof of Proposition 3.10 shows that

$$u = x\sqrt{\frac{G(x)}{2x^2 C(x)}}, \qquad v = \frac{B(x) + 2C(x)\, y}{\sqrt{2C(x)}}$$

is a linearizing change of coordinates for these systems.

In general, the condition given in Proposition 3.10 is not necessary for isochronicity. This can be shown by means of the following example.

EXAMPLE 3.12. The planar Hamiltonian system given by

$$H(x, y) = (e^{x/2} - 1)^2 + \tfrac{1}{8}x^2 + \tfrac{1}{2}xe^{-x}y + \tfrac{1}{2}e^{-2x}y^2$$

has an isochronous center at the origin and it does not verify the required condition in Proposition 3.10.

One can easily verify making use of Theorem 3.8 that the Hamiltonian system given in Example 3.12 has an isochronous center at the origin. Indeed, in this case

$$C(x)=\frac{1}{2}\,e^{-2x} \qquad \text{and} \qquad \frac{G(x)}{4\,C(x)}=(e^{x/2}-1)^2.$$

Thus $g(x)=e^{x/2}-1$ and $g^{-1}(x)=\ln(x+1)^2$. Therefore it follows that

$$\int_{g^{-1}(-x)}^{g^{-1}(x)} \frac{ds}{\sqrt{C(s)}}=\sqrt{2}\int_{g^{-1}(-x)}^{g^{-1}(x)} e^s\,ds=\sqrt{2}\,(x+1)^2-\sqrt{2}\,(1-x)^2=4\sqrt{2}\,x.$$

On the other hand a computation shows that it does not satisfy the required condition in Proposition 3.10. Nevertheless this sufficient condition turns out to be also necessary in case that G and C are even functions. This is stated in the following proposition.

PROPOSITION 3.13. *Assume that G and C are analytic even functions on $\mathbb{R}$. Then the origin is an isochronous center of period ω if and only if*

$$(G'(x)\,C(x)-G(x)\,C'(x))^2=\frac{16\pi^2}{\omega^2}\,G(x)\,C(x)^2 \qquad \textit{for all} \quad x\in(x_{\mathrm{I}},x_{\mathrm{S}}).$$

Proof. It is clear that we must only prove the necessity because Proposition 3.10 shows the sufficiency. Thus, assume that the origin is an isochronous center of period ω. Then, denoting $\sigma(x)=g^{-1}(-g(x))$ for all $x\in(x_{\mathrm{I}},x_{\mathrm{S}})$, Theorem 3.8 implies that

$$\int_{\sigma(x)}^{x} \frac{ds}{\sqrt{C(s)}}=\frac{2\omega}{\pi}\,g(x) \qquad \text{for all} \quad x\in(x_{\mathrm{I}},x_{\mathrm{S}}).$$

Since G and C are even functions notice that $\sigma(x)=-x$ for all $x\in(x_{\mathrm{I}},x_{\mathrm{S}})$. Using this fact and derivating the above expression with respect to x we find

$$\frac{1}{\sqrt{C(x)}}+\frac{1}{\sqrt{C(-x)}}=\frac{2\omega}{\pi}\,g'(x) \qquad \text{for all} \quad x\in(x_{\mathrm{I}},x_{\mathrm{S}}).$$

Now, using again that C is an even function we get

$$\frac{1}{\sqrt{C(x)}}=\frac{\omega}{\pi}\,g'(x) \qquad \text{for all} \quad x\in(x_{\mathrm{I}},x_{\mathrm{S}}).$$

Finally, taking into account the definition of g and an elementary manipulation yields

$$(G'(x)\, C(x) - G(x)\, C'(x))^2 = \frac{16\pi^2}{\omega^2}\, G(x)\, C(x)^2 \qquad \text{for all} \quad x \in (x_{\mathrm{I}}, x_{\mathrm{S}}).$$

Thus the validity of the proposition is proved. ∎

Next we will show how the isochroniciy of the origin determines the maximal interval of definition of any solution outside the period annulus. This surprising fact is stated precisely in Theorem 3.15, but first we need the next easy result. Here we shall use the following notation. Given any $z_0 \in \mathbb{R}^2$ we denote by

$$\varphi(z_0, t) = (\varphi_1(z_0, t), \varphi_2(z_0, t))$$

the unique solution of (6) with $\varphi(z_0, 0) = z_0$ and by $(d_-(z_0), d_+(z_0))$ its maximal interval of definition.

LEMMA 3.14. *Assume that the origin is a nonglobal center and that* $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$. *Then for any point* $z_0 = (x_0, y_0)$ *outside the period annulus the following holds*:

(a) *If* $y_0 > -B(x_0)/2C(x_0)$ *then* $\varphi_1(z_0, t) \to +\infty$ (*respectively* $-\infty$) *when* $t \nearrow d_+(z_0)$ (*respectively* $t \searrow d_-(z_0)$).

(b) *If* $y_0 < -B(x_0)/2C(x_0)$ *then* $\varphi_1(z_0, t) \to -\infty$ (*respectively* $+\infty$) *when* $t \nearrow d_+(z_0)$ (*respectively* $t \searrow d_-(z_0)$).

Proof. Notice first of all that $\varphi(z_0, t) \notin \mathscr{P}$ for all $t \in (d_-(z_0), d_+(z_0))$. Thus

$$\varphi_2(z_0, t) \neq \frac{-B(\varphi_1(z_0, t))}{2C(\varphi_1(z_0, t))} \qquad \text{for all} \quad t \in (d_-(z_0), d_+(z_0)) \tag{25}$$

by (c) in Lemma 3.3 since $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$. Assume for instance that $y_0 < -B(x_0)/2C(x_0)$. If $H(z_0) = h$ then condition (25) shows that, for all $t \in (d_-(z_0), d_+(z_0))$,

$$B(x)^2 - 4C(x)(A(x) - h)\big|_{x = \varphi_1(z_0, t)} > 0$$

and

$$\varphi_2(z_0, t) = \frac{-B(x) - \sqrt{B(x)^2 - 4C(x)(A(x) - h)}}{2C(x)}\Bigg|_{x = \varphi_1(z_0, t)}. \tag{26}$$

Since $\dot{x} = -B(x) - 2C(x)\, y$, we conclude that

$$\dot{\varphi}_1(z_0, t) < 0 \qquad \text{for all} \quad t \in (d_-(z_0), d_+(z_0)). \tag{27}$$

On the other hand the origin is the unique critical point of (6) by (d) in Lemma 3.1. Thus, any periodic orbit must surround the origin and therefore, taking into account that $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$, any periodic orbit is inside $\mathscr{P}$.

We will prove for instance that $\varphi_1(z_0, t) \to -\infty$ when $t \nearrow d_+(z_0)$. We claim that

$$\|\varphi(z_0, t)\| \to +\infty \qquad \text{as} \quad t \nearrow d_+(z_0). \tag{28}$$

If the claim is false then there exists $K > 0$ such that $\|\varphi(z_0, t)\| < K$ for all $t \in (0, d_+(z_0))$. This implies that the ω-limit of the solution passing through z_0 is nonempty and compact. Since the origin is the unique critical point, according to the Poincaré–Bendixson Theorem there are only two possibilities: the ω-limit contains the origin or it is a periodic orbit. The first one is obviously not possible. The second one, due to the analyticity of the Hamiltonian, implies that $z_0 \notin \mathscr{P}$ belongs to a periodic orbit and this is neither possible because we have shown that any periodic orbit is inside $\mathscr{P}$. Therefore the claim is true.

Next it will be shown that $|\varphi_1(z_0, t)| \to +\infty$ as $t \nearrow d_+(z_0)$. If this is false then

$$\lim_{t \nearrow d_+(z_0)} \varphi_1(z_0, t) = \tilde{x} \qquad \text{for some} \quad \tilde{x} \in \mathbb{R}. \tag{29}$$

However, from (26) then we obtain that

$$\lim_{t \nearrow d_+(z_0)} \varphi_2(z_0, t) = \frac{-B(\tilde{x}) - \sqrt{B(\tilde{x})^2 - 4C(\tilde{x})(A(\tilde{x}) - h)}}{2C(\tilde{x})}. \tag{30}$$

Due to $C(\tilde{x}) \neq 0$, it is clear that the combination of (29) and (30) contradicts (28). Therefore $|\varphi_1(z_0, t)| \to +\infty$ when $t \nearrow d_+(z_0)$. Finally, using (27), we conclude that

$$\lim_{t \nearrow d_+(z_0)} \varphi_1(z_0, t) = -\infty.$$

That $\varphi_1(z_0, t) \to +\infty$ as $t \searrow d_-(z_0)$ can be proved exactly the same way. ∎

THEOREM 3.15. *Assume that the origin is an isochronous nonglobal center of period* ω *and that* $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$. *If* $z_0 \notin \mathscr{P}$ *then*

$$d_+(z_0) - d_-(z_0) = \frac{\omega}{\pi} \arcsin \sqrt{\frac{h_0}{h}},$$

where $H(z_0) = h$.

Proof. Notice first of all that making use of Theorem 3.8 we get

$$\frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}}+\frac{(g^{-1})'(-z)}{\sqrt{C(g^{-1}(-z))}}=\frac{2\omega}{\pi}\qquad\text{for all}\quad z\in(-\sqrt{h_0},\sqrt{h_0})\tag{31}$$

since the origin is an isochronous center of period ω. Moreover $y_0\neq -B(x_0)/2C(x_0)$ by (c) in Lemma 3.3. Suppose for instance that $y_0> -B(x_0)/2C(x_0)$ and say that $H(z_0)=h$. Then

$$\dot{\varphi}_1(z_0,t)=\sqrt{B(x)^2-4C(x)(A(x)-h)}\,|_{x=\varphi_1(z_0,t)}>0\tag{32}$$

for all $t\in(d_-(z_0),d_+(z_0))$. We omit the proof of this fact because it proceeds just like the first part of the proof of Lemma 3.14. Now taking any $t^1,t^2\in(d_-(z_0),d_+(z_0))$ and making use of (32) we conclude that

$$t^1-t^2=\int_{\varphi_1(z_0,t^2)}^{\varphi_1(z_0,t^1)}\frac{dx}{\sqrt{B(x)^2-4C(x)(A(x)-h)}}.$$

Making $t^1\nearrow d_+(z_0)$ and $t^2\searrow d_-(z_0)$ above, (a) in Lemma 3.14 shows that

$$d_+(z_0)-d_-(z_0)=\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{B(x)^2-4C(x)(A(x)-h)}}.$$

The change of variables $z=g(x)$ and a manipulation yields

$$\begin{aligned}d_+(z_0)-d_-(z_0)&=\frac{1}{2}\int_{-\sqrt{h_0}}^{\sqrt{h_0}}\frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}}\frac{dz}{\sqrt{h-z^2}}\\&=\frac{1}{2}\int_0^{\sqrt{h_0}}\left(\frac{(g^{-1})'(z)}{\sqrt{C(g^{-1}(z))}}+\frac{(g^{-1})'(-z)}{\sqrt{C(g^{-1}(-z))}}\right)\frac{dz}{\sqrt{h-z^2}}.\end{aligned}$$

Here we have used that $g(z)\to\sqrt{h_0}$ and $g(-z)\to-\sqrt{h_0}$ when $z\to+\infty$ by (a) in Lemma 3.3. Finally, making use of (31) and a computation shows that

$$d_+(z_0)-d_-(z_0)=\frac{\omega}{\pi}\int_0^{\sqrt{h_0}}\frac{dz}{\sqrt{h-z^2}}=\frac{\omega}{\pi}\arcsin\sqrt{\frac{h_0}{h}}.$$

Therefore the result is proved. ∎

It follows from Theorem 3.15 that when the origin is a nonglobal isochronous center satisfying $(x_{\mathrm{I}},x_{\mathrm{S}})=\mathbb{R}$ then the length of the maximal interval of definition of any solution outside the period annulus it is uniquely determined by its energy level, the energy level of the boundary of $\mathscr{P}$ and the period of the center.

The Polynomial Case

Next we will apply the results obtained until now to the case that A, B, and C are polynomials. However when it is possible we state some result assuming only that C and $G = 4AC - B^2$ are polynomials.

The following lemma follows readily from (a) in Lemma 3.3 using the fact that the period annulus is the whole plane if and only if $h_0 = +\infty$.

LEMMA 3.16. *Assume that G and C are polynomials and that $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is not bounded. The following statements hold*:

(a) *If the origin is a global center then* $\deg(G) > \deg(C)$.

(b) *If the origin is not a global center then* $\deg(G) = \deg(C)$.

Once we have proved the following proposition we will be in position to give a characterization of the polynomial isochronous global centers.

PROPOSITION 3.17. *Assume that G and C are polynomials and that the origin is a global center. Then $T(h) \to 0$ as $h \to +\infty$ except in case that $G(x) = gx^2$ with $g > 0$ and $C(x) = c$ with $c > 0$.*

Proof. Let m and n denote respectively the degree of G and C. Since the origin is a global center we note that $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$ and, by (a) in Lemma 3.16, that $m > n$. On the other hand an easy computation shows that

$$\sqrt{C(x)}\, g'(x) = \frac{\operatorname{sgn}(x)}{4} \frac{G'(x)\, C(x) - G(x)\, C'(x)}{\sqrt{G(x)}\, C(x)}.$$

Using that $m \neq n$, the above expression shows that the function $x \mapsto \sqrt{C(x)}\, g'(x)$ has order $(m/2) - 1$ at infinity. Therefore

$$\lim_{|x| \to +\infty} \sqrt{C(x)}\, g'(x) = +\infty$$

if $m > 2$. Now, by applying Corollary 3.5 we can assert that if $m > 2$ then

$$\lim_{h \to +\infty} T(h) = 0.$$

Hence it only remains to consider when $m \leqslant 2$. However, since G and C are polynomials of even degree by (a) and (b) in Lemma 3.1, the fact that

$m>n$ implies $m=2$ and $n=0$. Finally, using (a) and (b) in Lemma 3.1 again we conclude that $G(x)=gx^2$ with $g>0$ and $C(x)=c$ with $c>0$. ∎

Next result provides a purely algebraic characterization for a polynomial isochronous global center in terms of its coefficients.

THEOREM 3.18. *Assume that C and G are polynomials. Then the origin is an isochronous global center of period ω if and only if $C(x)=c$ with $c>0$ and $G(x)=((2\pi/\omega)\,x)^2$.*

Proof. If the origin is an isochronous global center then, making use of Proposition 3.17, $G(x)=gx^2$ with $g>0$ and $C(x)=c$ with $c>0$. Now, if its period is ω then (c) in Proposition 3.4 shows that $g=(2\pi/\omega)^2$.

Conversely, if $C(x)=c$ with $c>0$ and $G(x)=((2\pi/\omega)\,x)^2$ then the analytic change of coordinates given by

$$u=\frac{2\pi}{\omega}\frac{x}{\sqrt{2c}}, \qquad v=\frac{B(x)+2cy}{\sqrt{2c}}$$

brings system (6) to the form

$$\begin{cases} \dot{u}=-\dfrac{2\pi}{\omega}\,v, \\[2mm] \dot{v}=\dfrac{2\pi}{\omega}\,u. \end{cases}$$

Therefore the origin is an isochronous global center of period ω for the system (6). ∎

PROPOSITION 3.19. *If G and C are polynomials and the origin is an isochronous nonglobal center then* $\deg(G)=\deg(C)\geqslant 4$.

Proof. Since the origin is an isochronous center it follows from applying Corollary 3.7 that $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is not bounded. Then applying Lemma 3.16 we have that $\deg(G)=\deg(C)$ because the center is nonglobal. Let us suppose that $x_{\mathrm{S}}=+\infty$ (the other case is similar) and say that $\deg(C)=n$.

If for all $x\in(x_{\mathrm{I}}, x_{\mathrm{S}})$ we define the auxiliary function

$$F(x)=\frac{\sqrt{G(x)}\;C(x)}{|G'(x)\,C(x)-G(x)\,C'(x)|},$$

then it follows from derivating with respect to x the isochronicity condition given in Theorem 3.8 that it holds

$$F(g^{-1}(x))+F(g^{-1}(-x))=\frac{\omega}{2\pi} \qquad \text{for all} \quad x\in(-\sqrt{h_0}, \sqrt{h_0}). \tag{33}$$

Here we used the definition of g and that for all $x\in(x_{\mathrm{I}}, +\infty)\backslash\{0\}$ it holds

$$\operatorname{sgn}(G'(x)\, C(x)-G(x)\, C'(x))=\operatorname{sgn}(x)$$

by (c) in Lemma 3.1. Notice that $F(g^{-1}(x))>0$ for all $x\in(-\sqrt{h_0}, \sqrt{h_0})$ due to (a) in Lemma 3.1. Moreover if we denote the degree of $G'C-GC'$ by r then it follows that

$$r\leqslant 2(n-1) \tag{34}$$

and that the function F has order $\frac{3}{2}n-r$ at infinity. Now, a necessary condition in order that (33) holds is that

$$\tfrac{3}{2}n-r\leqslant 0, \tag{35}$$

otherwise the left hand of expression (33) would tend to infinity as $x\nearrow\sqrt{h_0}$. This is so because $x_{\mathrm{S}}=+\infty$ and

$$\lim_{x\nearrow\sqrt{h_0}} g^{-1}(x)=x_{\mathrm{S}}$$

by (a) in Lemma 3.3. This proves the result because (34) and (35) imply that $n\geqslant 4$. ∎

Remark 3.20. It is clear that the combination of Theorem 3.18 and Proposition 3.19 determines all the isochronous centers with $\deg(C)\leqslant 3$.

Next result will give some of the geometric properties of the period annulus of a nonglobal isochronous center.

COROLLARY 3.21. *Assume that A, B and C are polynomials and that the origin is a nonglobal center. If $\partial\mathscr{P}$ contains the infinite critical point given by $x=0$ then $T(h)\to+\infty$ when $h\nearrow h_0$.*

Proof. Note that in this situation there exist a bounded sequence $(x_n)_{n\in N}$ and an unbounded sequence $(y_n)_{n\in N}$ such that $(x_n, y_n)\in\partial P$ for all $n\in N$. Now the result follows from Proposition 3.6. ∎

We note finally that if A, B and C are polynomials and the origin is an isochronous nonglobal center then the period annulus in the Poincaré disc is qualitatively one of the given in Fig. 2.

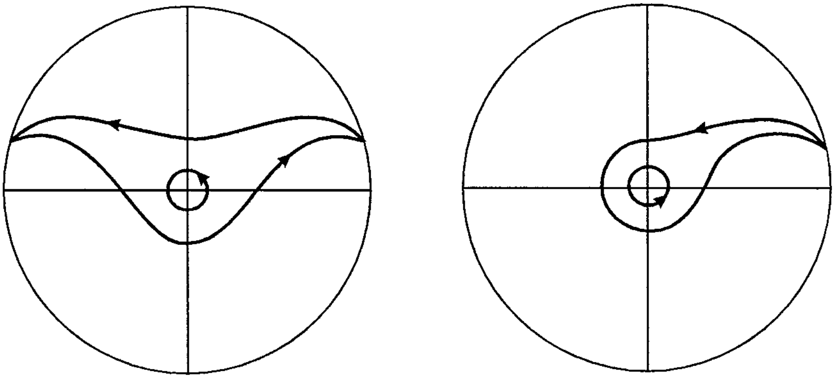

FIG. 2. Possible period annulus for an isochronous nonglobal center.

Notice that by Corollary 3.21 none of the infinite critical points in $\partial\mathscr{P}$ is given by the direction $x=0$. That there are at most two follows from using that $H(x, y)=h$ has at most two solutions for any fixed y and h. Finally (c) in Lemma 3.3 shows that when there are two infinite critical points in $\partial\mathscr{P}$, if one is given by the direction $\theta_0 \in (-\pi/2, \pi/2)$ then the other one is given either by the direction $\pi-\theta_0$ (as the example in Fig. 2) or by the direction $\pi+\theta_0$.

Polynomial Isochronous Nonglobal Centers

We now take advantage of Proposition 3.13 to give a method to construct polynomial Hamiltonian systems which have a nontrivial and nonglobal isochronous center.

Assume that G and C are even polynomials satistying the required condition in Proposition 3.13 and that there exist two polynomials A and B with $G=4AC-B^2$ such that the origin is a nondegenerate center of system (6). Notice then that the origin is an isochronous center.

First we shall study which must be the form of G and C in order that the isochronous center is nonglobal. Since $(x_{\mathrm{I}}, x_{\mathrm{S}})$ is unbounded by Corollary 3.7, the fact that G and C are even implies that $(x_{\mathrm{I}}, x_{\mathrm{S}})=\mathbb{R}$. Then, since we want the center to be nonglobal, using Lemma 3.16 it follows that $\deg(G)=\deg(C)$. On the other hand, (a) and (b) in Lemma 3.1 show respectively that C does not have any real zero and that $x=0$ is the unique real zero of G.

Using that $\operatorname{sgn}(G'(x)\,C(x)-G(x)\,C'(x))=\operatorname{sgn}(x)$ for all $x\neq 0$ by (c) in Lemma 3.1, one can easily verify that the required condition in Proposition 3.13 is equivalent to

$$\left(\frac{G}{C}\right)'(x)=\operatorname{sgn}(x)\,\frac{4\pi}{\omega}\,\frac{\sqrt{G(x)}}{C(x)} \qquad \text{for all} \quad x\in\mathbb{R}. \tag{36}$$

Notice that $G(x)=P(x)^2$ for some polynomial P because the left hand of expression (36) is a rational function. Since $G(0)=G'(0)=0$ and $G''(0)>0$ by (b) in Lemma 3.1, it follows that $P(0)=0$ and $P'(0)\neq 0$. Let us assume that $P'(0)>0$ (the other case is similar). Then $\operatorname{sgn}(P(x))=\operatorname{sgn}(x)$ for all $x\neq 0$ and this shows that (36) is equivalent to

$$2P'(x)\,C(x)-P(x)\,C'(x)=\frac{4\pi}{\omega}\,C(x) \qquad \text{for all} \quad x\in\mathbb{R}.$$

Using the above expression it is easy to show that if $C(z_0)=0$ for some $z_0\in\mathbb{C}$ then $P(z_0)=0$, and that the multiplicity of P at any of its zeros is exactly one.

All these facts lead us to consider G and C with the form:

$$G(x)=x^2\prod_{i=1}^{n}(x^2+b_i)^2 \qquad \text{and} \qquad C(x)=c\prod_{i=1}^{n}(x^2+b_i)^{k_i+2}, \tag{37}$$

where $c>0$, $b_i>0$ for $i=1,2,...,n$, $k_i\in\mathbb{N}\cup\{-1,-2\}$ for $i=1,2,...,n$ and $\sum_{i=1}^{n}k_i=1$. Then it is obvious that G and C are even polynomials with the same degree. Of course the general form of each factor should be $a_ix^2+c_ix+b_i$ with $c_i^2-4a_ib_i<0$ but this would make the computation more complicated.

Until now we have found some necessary conditions in order that the origin is an isochronous nonglobal center and these conditions lead us to take G and C of the form given in (37). Next, assuming that G and C have this concrete form, we shall find an equivalent expression to (36).

Taking into account the definition of G and C given in (37) it follows that

$$\frac{G(x)}{C(x)}=\frac{x^2}{c}\prod_{i=1}^{n}\frac{1}{(x^2+b_i)^{k_i}}.$$

Then a computation shows that

$$\begin{aligned}\left(\frac{G}{C}\right)'(x)&=\frac{2x}{c}\prod_{i=1}^{n}\frac{1}{(x^2+b_i)^{k_i}}+\frac{x^2}{c}\left(\prod_{i=1}^{n}\frac{1}{(x^2+b_i)^{k_i}}\right)'\\&=\frac{2x}{c}\prod_{i=1}^{n}\frac{1}{(x^2+b_i)^{k_i}}\left(1-x^2\sum_{i=1}^{n}\frac{k_i}{x^2+b_i}\right).\end{aligned}$$

On the other hand, using again the definition of G and C, we have that

$$\operatorname{sgn}(x)\,\frac{\sqrt{G(x)}}{C(x)}=\frac{x}{c}\prod_{i=1}^{n}\frac{1}{(x^2+b_i)^{k_i+1}}.$$

Now it follows from (36) that the condition required in Proposition 3.13 is satisfied when

$$\frac{2\pi}{\omega}\prod_{i=1}^{n}\frac{1}{x^2+b_i}=1-x^2\sum_{i=1}^{n}\frac{k_i}{x^2+b_i} \qquad \text{for all} \quad x\in\mathbb{R}. \tag{38}$$

We will discuss the simplest cases. That is, when $n=1$ or $n=2$. In the first case it follows from (38) that $b_1=2\pi/\omega$ and $k_1=1$. Thus,

$$G(x)=x^2\left(x^2+\frac{2\pi}{\omega}\right)^2 \qquad \text{and} \qquad C(x)=c\left(x^2+\frac{2\pi}{\omega}\right)^3.$$

Now we must find two polynomials A and B with $G=4AC-B^2$ such that the origin is a nondegenerate center of system (6). Clearly we must search them of the form $A(x)=x^2\widetilde{A}(x)$ and $B(x)=x\widetilde{B}(x)$. Using that A, B, C, and G have repeated factors it easily follows that taking

$$A(x)=\frac{\omega}{8c\pi}x^2 \qquad \text{and} \qquad B(x)=\sqrt{\frac{\omega}{2\pi}}\,x^2\left(x^2+\frac{2\pi}{\omega}\right)$$

it holds $G=4AC-B^2$. One can check that taking A, B, and C as above then the origin is a nondegenerate center of system (6). Now, since G and C are even polynomials satisfying the condition required in Proposition 3.13 we conclude that the origin is an isochronous center of period ω. Moreover, by Lemma 3.16, the period annulus is not the whole plane since $\deg(G)=\deg(C)$. In brief, taking $c=\frac{1}{2}$ and $\omega=2\pi$ for simplicity, we have shown:

EXAMPLE 3.22. The planar Hamiltonian system given by

$$H(x,y)=\frac{x^2}{2}+x^2(x^2+1)\,y+\frac{1}{2}(x^2+1)^3\,y^2$$

has an isochronous nonglobal center of period 2π at the origin.

One can check that the system given in Example 3.22 can be linearized by means of

$$u=\frac{x}{\sqrt{x^2+1}} \qquad \text{and} \qquad v=\frac{x^2+(x^2+1)^2\,y}{\sqrt{x^2+1}},$$

and that the level curve $H=\frac{1}{2}$ is unbounded.

Next we consider the case $n=2$. In this case the relation (38) has two sets of solutions,

$$\left\{k_1=2, k_2=-1, b_1=\sqrt{\frac{\pi}{\omega}}, b_2=2\sqrt{\frac{\pi}{\omega}}\right\}$$

and

$$\left\{k_1=3, k_2=-2, b_1=2\sqrt{\frac{3\pi}{\omega}}, b_2=\sqrt{\frac{3\pi}{\omega}}\right\}.$$

For instance we will study the first one. In this case,

$$G(x)=x^2\left(x^2+\sqrt{\frac{\pi}{\omega}}\right)^2\left(x^2+2\sqrt{\frac{\pi}{\omega}}\right)^2 \quad \text{and}$$

$$C(x)=c\left(x^2+\sqrt{\frac{\pi}{\omega}}\right)^4\left(x^2+2\sqrt{\frac{\pi}{\omega}}\right).$$

As before, we must find two polynomials A and B with $G=4AC-B^2$ such that the origin is a nondegenerate center. The couple of polynomials of lowest degree are

$$A(x)=\frac{\sqrt{\omega^3}}{16c\sqrt{\pi^3}}x^2\left(x^2+2\sqrt{\frac{\pi}{\omega}}\right)\left(x^2+4\sqrt{\frac{\pi}{\omega}}\right)$$

and

$$B(x)=\frac{\sqrt[4]{\omega^3}}{2\sqrt[4]{\pi^3}}x^2\left(x^2+\sqrt{\frac{\pi}{\omega}}\right)\left(x_2+2\sqrt{\frac{\pi}{\omega}}\right)\left(x_2+3\sqrt{\frac{\pi}{\omega}}\right).$$

Again, one can easily verify that A, B, and C as above give a nondegenerate center at the origin. By the same arguments as before, and taking $\omega=\pi$ and $c=\frac{1}{4}$ for simplicity, we have shown:

EXAMPLE 3.23. The planar Hamiltonian system given by

$$H(x, y)=\frac{x^2}{4}(x^2+2)(x^2+4)$$
$$+\frac{x^2}{2}(x^2+1)(x^2+2)(x^2+3)\,y+\frac{1}{4}(x^2+1)^4(x^2+2)\,y^2$$

has an isochronous nonglobal center of period π at the origin.

A linearization of the system given in Example 3.23 is

$$u = \frac{x\sqrt{2(x^2+2)}}{x^2+1} \qquad \text{and} \qquad v = \sqrt{\frac{x^2+2}{2}}\,\frac{x^2(x^2+3)+y(x^2+1)^3}{x^2+1},$$

and on the other hand it is easy to verify that the level curve $H=1$ is unbounded.

Remark 3.24. These examples are nontrivial isochronous centers. If they were trivial ones then they would be counterexamples to the Jacobian conjecture since in both cases the period annulus is not the whole plane.

In fact if there is a Jacobian pair (P, Q) satisfying

$$H(x, y) = A(x) + B(x)\, y + C(x)\, y^2 = \frac{P(x, y)^2 + Q(x, y)^2}{2} \tag{39}$$

then it follows that C is constant and hence the associated Hamiltonian system has an isochronous global center at the origin by Proposition 3.19.

This can be shown with the following argument. If there exists a Jacobian pair (P, Q) such that (39) holds then $P(x, y) = p_1(x)\, y + p_2(x)$ and $Q(x, y) = q_1(x)\, y + q_2(x)$ where p_1, p_2, q_1, and q_2 are polynomials satisfying

$$\begin{vmatrix} p_1'(x)\, y + p_2'(x) & p_1(x) \\ q_1'(x)\, y + q_2'(x) & q_1(x) \end{vmatrix} = k \qquad \text{for all} \quad (x, y) \in \mathbb{R}^2,$$

for some constant k. One can easily verify that the required condition above is satisfied if and only if there exist some constants λ and $r \neq 0$ such that

$$p_1(x) = \frac{\lambda k}{r}, \qquad p_2(x) = \lambda q_2(x) + rx, \qquad \text{and} \qquad q_1(x) = \frac{k}{r}.$$

Since $C(x) = p_1(x)^2 + q_1(x)^2$ this shows that C must be constant.

We conclude this section noting that the classification of the polynomial isochronous centers of this family is still incomplete. They are characterized only when $\deg(C) \leqslant 3$. Another open problem is the existence of polynomial Hamiltonian systems with an isochronous nonglobal center at the origin not satisfying that $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$. One can verify that $(x_{\mathrm{I}}, x_{\mathrm{S}}) = \mathbb{R}$ in Example 3.22 and Example 3.23. The system given in Example 3.12 has x_{I} finite but it is not polynomial.

4. CUBIC HAMILTONIAN SYSTEMS

In this section we determine all the cubic Hamiltonian systems that have an isochronus center at the origin.

We shall first give a necessary condition in order that the origin is an isochronous center for the general Hamiltonian system given by

$$H(x, y) = \sum_{i=2}^{n+1} H_i(x, y), \tag{40}$$

where $n \geqslant 2$ and $H_i(x, y)$ is an homogeneous polynomial of degree i for $i = 2, 3, ..., n+1$. Denoting $H_i(r\cos\theta, r\sin\theta) = g_i(\theta)\, r^i$ for $i = 2, 3, ..., n+1$, we prove:

Proposition 4.1. *If the origin is an isochronous center for the system given by* (40) *then* $n \geqslant 3$ *and there exists a direction* $\theta_0 \in [\,0, 2\pi)$ *such that* $g_{n+1}(\theta_0) = g'_{n+1}(\theta_0) = g_n(\theta_0) = 0$.

Proof. First of all notice that the origin must be a nondegenerate center because it is isochronous (see [3] for instance). This implies that

$$g_2(\theta) \neq 0 \qquad \text{for all} \quad \theta \in [0, 2\pi). \tag{41}$$

Moreover the period annulus is unbounded, otherwise there would be a critical point in its boundary.

We consider first the case in which the period annulus is the whole plane. In this case $H(x, y) \neq 0$ for all $(x, y) \neq (0, 0)$, otherwise it is easy to see that there would be a critical point different from the origin. For each $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ we define

$$F(r, \theta) = g_2(\theta) + g_3(\theta)\, r + \cdots + g_{n+1}(\theta)\, r^{n-1}.$$

Then, since for all $r > 0$ and $\theta \in [0, 2\pi)$ it holds $H(r\cos\theta, r\sin\theta) = r^2 F(r, \theta)$, we can assert that

$$F(r, \theta) \neq 0 \qquad \text{for all} \quad r > 0 \quad \text{and} \quad \theta \in [0, 2\pi). \tag{42}$$

This shows that n is odd because if not $g_{n+1}(\theta)\, g_{n+1}(\theta + \pi) < 0$, and this would imply that $F(r, \theta)\, F(r, \theta + \pi) < 0$ for $r > 0$ large enough.

On the other hand, in [3] it is proved that there exists $\theta_0 \in [\,0, 2\pi)$ such that $g_{n+1}(\theta_0) = g'_{n+1}(\theta_0) = 0$. We will prove that $g_n(\theta_0) = 0$ must be also satisfied. If it is false then $F(r, \theta_0)$ is a polynomial of degree $n-2$ in r and, since n is odd, this implies that there exists $r_0 \in \mathbb{R}$ such that $F(r_0, \theta_0) = 0$. Notice that $r_0 \neq 0$ because $F(0, \theta_0) = g_2(\theta_0) \neq 0$ due to (41). Then the fact

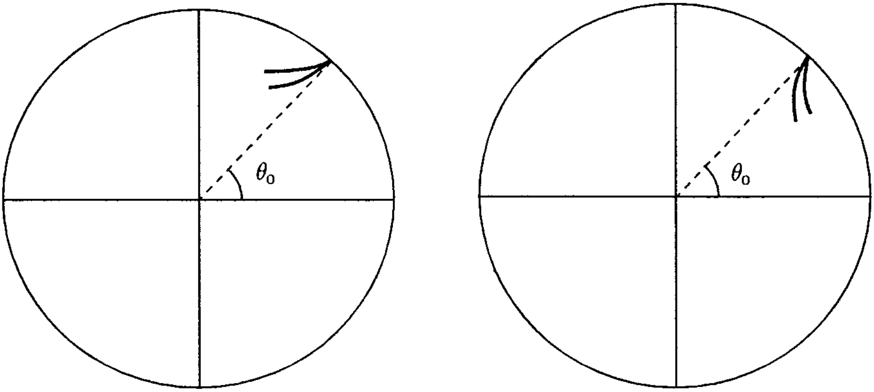

FIG. 3. Possible location of the hyperbolic sector.

that $F(r_0, \theta_0) = F(-r_0, \theta_0 + \pi) = 0$ contradicts (42). This shows that $g_n(\theta_0) = 0$, and that $n \geqslant 3$ follows from the fact that it must be odd.

Finally we consider the case in which the period annulus is not the whole plane. Then, since it must be unbounded, there exists an infinite critical point in the Poincaré's compactification with a hyperbolic sector having both sepatrices lying in the finite part (see [4]). If we say that the direction of this infinite critical point is given by $\theta_0 \in [0, 2\pi)$ then $g_{n+1}(\theta_0) = 0$. Moreover $g'_{n+1}(\theta_0) = 0$, if not it is easy to verify that this infinite critical point has only parabolic sectors (see [4] for instance). Let ℓ denote the straight line passing through the origin and having the direction given by θ_0. A result in [4] shows that for any compact set K, $\ell \cap (\mathbb{R}^2 \setminus K)$ is not contained in the hyperbolic sector. Therefore the unique possible situations are shown in Fig. 3.

Say that the boundary of the period annulus is given by the level curve $H = h_0$ and for each fixed $\theta \in [0, 2\pi)$ define

$$H_\theta(r) = H(r \cos\theta, r \sin\theta) = g_2(\theta)\, r^2 + g_3(\theta)\, r^3 + \cdots + g_{n+1}(\theta)\, r^{n+1}.$$

Then $H_\theta(r)$ is a polynomial in r of degree at most $n+1$ for all $\theta \in [0, 2\pi)$. We claim that if $g_n(\theta_0) \neq 0$ then there exist $\varepsilon > 0$ and $M > 0$ such that for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ the equation $H_\theta(z) - h_0 = 0$ has at most one solution $z_\theta \in \mathbb{C}$ with $|z_\theta| > M$. It is clear that this will be in contradiction with Fig. 3. So, assume that $g_n(\theta_0) \neq 0$ and take a Jordan curve Γ in $\mathbb{C}$ containing the n zeros of the polynomial $H_{\theta_0}(z) - h_0$ in its interior. We choose $\varepsilon > 0$ such that

$$\sup_{z \in \Gamma} |H_\theta(z) - H_{\theta_0}(z)| < \inf_{z \in \Gamma} |H_{\theta_0}(z) - h_0|$$

for all $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$. Then it is clear that for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ it holds

$$|H_\theta(z) - h_0 - (H_{\theta_0}(z) - h_0)| < |H_\theta(z) - h_0| + |H_{\theta_0}(z) - h_0| \qquad \text{for all} \quad z \in \Gamma.$$

In this situation, applying the Rouche's Theorem, we can assert that the number of zeros of the polynomial $H_{\theta_0}(z) - h_0$ inside $\operatorname{Int} \Gamma$ is equal to the number of zeros of the polynomial $H_\theta(z) - h_0$ inside $\operatorname{Int} \Gamma$ for any fixed $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$. Clearly the first number is n by construction while the second one is at most $n+1$ for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$. If we take $M = \sup\{|z| : z \in \Gamma\}$ then it is clear that this shows that for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ there exists at most one root z_θ of the equation $H_\theta(z) - h_0 = 0$ with $|z_\theta| > M$. This proves the claim and hence that $g_n(\theta_0) \neq 0$ yields a contradiction. Therefore $g_n(\theta_0) = 0$, and that $n \geqslant 3$ follows from (41). ∎

Remark 4.2. An alternative proof for $g_n(\theta_0) = 0$ in the nonglobal case can be done by means of an accurated analysis of all the possible phase portraits of an infinite critical point with a nilpotent linear part.

We note that in [9] is given a classification of the isochronous cubic Hamiltonian systems which are Darboux linearizable. We are now in position to complete the classification.

THEOREM E. *A cubic Hamiltonian system has an isochronous center at the origin if and only if after a linear change of coordinates can be written as*

$$H(x, y) = (k_1 x)^2 + (k_2 y + P(x))^2,$$

where k_1 and k_2 are different from zero and $P(x) = k_3 x + k_4 x^2$.

Proof. Assume that the origin is an isochronous center. Then by making use of Proposition 4.1 with $n = 3$ it follows that there exists $\theta_0 \in [0, 2\pi)$ such that $g_3(\theta_0) = g_4(\theta_0) = g_4'(\theta_0) = 0$. Without loss of generality we can assume that $\theta_0 = 0$ (if not we make a rotation of axis) and hence that the Hamiltonian can be written as

$$H(x, y) = a_1 x^2 + a_2 xy + a_3 y^2 + (b_1 x^2 + b_2 xy + b_3 y^2)\, x + (c_1 x^2 + c_2 xy + c_3 y^2)\, x^2.$$

Now, a reordering of the terms shows that $H(x, y) = A(x) + B(x)\, y + C(x)\, y^2$ with $A(x) = (a_1 + b_1 x + c_1 x^2)\, x^2$, $B(x) = (a_2 + b_2 x + c_2 x^2)\, x$ and

$C(x)=a_3+b_3x+c_3x^2$. Since $\deg(C)\leqslant 2$ it follows from Proposition 3.19 that the origin must be a global center. In this case Theorem 3.18 asserts that $C(x)=c$ and $4A(x)\,C(x)-B(x)^2=gx^2$ for some positive constants c and g. Then a computation shows that

$$H(x, y)=\left(\sqrt{\frac{g}{c}}\frac{x}{2}\right)^2+\left(\frac{B(x)}{2\sqrt{c}}+\sqrt{c}\,y\right)^2.$$

This proves the necessity because that H has at most degree 4 in x implies $\deg(B)\leqslant 2$.

It has now to be shown the sufficiency. So assume that the Hamiltonian is

$$H(x, y)=(k_1x)^2+(k_2y+k_3x+k_4x^2)^2, \tag{43}$$

with k_1 and k_2 different from zero. In this case it is readily seen that the change of coordinates $(u, v)=(k_1x, k_2y+k_3x+k_4x^2)$ brings the Hamiltonian system associated to (43) to

$$\begin{cases} \dot{u}=-2k_1k_2v, \\ \dot{v}=2k_1k_2u. \end{cases}$$

Thus, the Hamiltonian system associated to (43) is linearizable. This ends the proof of the theorem. ∎

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