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Integrability and algebraic entropy of k-periodic non-autonomous Lyness recurrences

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ABSTRACT

This work deals with non-autonomous Lyness type recurrences of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $\{a_n\}_n$ is a k-periodic sequence of complex numbers with minimal period k. We treat such non-autonomous recurrences via the autonomous dynamical system generated by the birational mapping $F_{a_k} \circ F_{a_{k-1}} \circ \cdots \circ F_{a_1}$ where F_a is defined by $F_a(x, y) = (y, \frac{a+y}{x})$. For the cases $k \in \{1, 2, 3, 6\}$ the corresponding mappings have a rational first integral. By calculating the dynamical degree we show that for $k = 4$ and for $k = 5$ generically the dynamical system is no longer rationally integrable. We also prove that the only values of k for which the corresponding dynamical system is rationally integrable for all the values of the involved parameters, are $k \in \{1, 2, 3, 6\}$.

1. Introduction and main results

Consider the non-autonomous Lyness difference equations of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \tag{1}$$

where $\{a_n\}_n$ is a k-periodic sequence of real numbers. Such recurrences have been studied in [5,7,10,14,15] and recently in [6].

In discrete integrability community there is a rising interest in the study of integrability of the systems of difference equations with periodic coefficients. The integrability of some maps coming from periodic difference equations have been studied in [6,10,12,13,16]. However the investigation in the area of non-integrability of periodic difference equations is rare. The work in this article makes an evident contribution in the area of non-integrable periodic difference equations by providing large number of classes of non-integrable difference equation with periodic coefficients.

From a mathematical biology point of view, it is interesting to study these non-autonomous difference equations because they can be used to model different biological systems which possess the change in time. These biological systems can possess certain behavior due to some changes in their environment. We consider those changes which are cyclic. That is to say that the parameters of the biological system are periodic.

In our study we consider the more general case when the parameters and the variables belong to the complex space.

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