Dynamical Classification of a Family of Birational Maps of $\mathbb{C}^2$ via Algebraic Entropy*

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Abstract

This work dynamically classifies a 9−parametric family of birational maps $f : \mathbb{C}^2 \to \mathbb{C}^2$. From the sequence of the degrees d_n of the iterates of f, we find the dynamical degree $\delta(f)$ of f. We identify when d_n grows periodically, linearly, quadratically or exponentially. The considered family includes the birational maps studied by Bedford and Kim in [4] as one of its subfamilies.

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1 Introduction

In this work we consider the family of fractional maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the form:

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right), \tag{1}$$

where the parameters are complex numbers.

This family of maps can be extended to the projective plane $P\mathbb{C}^2$ by considering the embedding $(x_1, x_2) \in \mathbb{C}^2 \mapsto [1 : x_1 : x_2] \in P\mathbb{C}^2$ into projective space. The induced map

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$F : P\mathbb{C}^2 \to P\mathbb{C}^2$ has three components $F_i[x_0 : x_1 : x_2]$, $i = 1, 2, 3$ which are homogeneous polynomials of degree two. For general values of the parameters the three components don't have a common factor: we say that these maps have degree two. Similarly we can define the degree of $F^n = F \circ \cdots \circ F$ for each $n \in \mathbb{N}$. It can be seen that if $f(x_1, x_2)$ is a birational map, then the sequence of its degrees satisfies a homogeneous linear recurrence with constant coefficients (see [18] for instance or Section 3). This is governed by the characteristic polynomial $\mathcal{X}(x)$of a certain matrix associated to F. The other information we get from $\mathcal{X}(x)$ is the *dynamical degree* $\delta(F)$ which is it's largest real root, and is defined as

$$\delta(F) := \lim_{n\to\infty} (\deg(F^n))^{\frac{1}{n}}, \tag{2}$$

see [3, 4, 5, 6, 17, 18]. The logarithm of this quantity has been called the *algebraic entropy.*

The application of algebraic entropy in the field of dynamical systems has been growing in recent years, see for instance [3, 4, 5, 6, 7, 8, 11, 17, 18]. On the other hand, the study of the dynamics generated by birational mappings in the plane is also a current issue, see for instance [1, 3, 4, 5, 17] also [14, 15, 16, 20, 21, 22, 23, 25, 24, 28].

It is known (see [26]) that the algebraic entropy is an upper bound of the *topological entropy*, which in turn is a dynamic measure of the complexity of the mapping.

The algebraic entropy for the maps (1) highly depends on the choice of parameters. For this reason our study includes all the possible values of the parameters of f to determine the growth rate d_n and $\delta(F)$. Therefore the results that we get can be seen as a dynamical classification of family (1). Furthermore, they generalize the results obtained in [4], which the authors consider the subfamily of (1)with $\alpha_0 = 0$, $\alpha_1 = 0$ and $\alpha_2 = 1$.

Birational mappings $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ have an indeterminacy set $\mathcal{I}(F)$ of points where F is ill-defined as a continuous map. This set is given by:

$$\mathcal{I}(F) = \{[x_0 : x_1 : x_2] \in P\mathbb{C}^2 : F_1[x_0 : x_1 : x_2] = 0, F_2[x_0 : x_1 : x_2] = 0, F_3[x_0 : x_1 : x_2] = 0]\}.$$

On the other hand, if we consider one irreducible component V of the determinant of the Jacobian of F, it is known (see Proposition 3.3 in [17]) that $F(V)$ reduces to a point in $\mathcal{I}(F^{-1})$. The set of these curves which are sent to a single point is called the *exceptional locus* of F and it is denoted by $\mathcal{E}(F)$.

It is known that the dynamical degree depends on the orbits of the indeterminacy points of the inverse of F under the action of F, see [18, 19, 27]. Indeed, the key point is whether the iterates of such points coincide with any of the indeterminacy points of F.

Generically our map F has three indeterminacy points. The exceptional locus is formed by three straight lines, each two of them intersecting on a single indeterminate point of F. We call them *non degenerate mappings.* But there is a subfamily such that the exceptional locus is formed by only two straight lines. We call these mappings *degenerate mappings* and they are studied in the paper [10]. Hence we are not going to consider them here.

To find $\delta(F)$ and d_n and to study its behaviour, we use a Theorem of Bedford and Kim (see [3]) to find the characteristic polynomial which provides d_n.

The results obtained in this paper are the starting point for studying the dynamic properties of the elements of family (1). We can expect certain types of behaviors, for instance, if we want to know the mappings which are globally periodic, we have to look at the ones whose sequence of degrees is periodic. To find the mappings that are integrable (i. e., mappings that preserve the level curves of some rational function) or mappings that preserve some fibrations, we have to look for the mappings whose sequence of degrees d_n grows linearly or quadratically in n, see [18]. We can encounter chaos whenever d_n grows exponentially. As a continuation of this work, in the following articles, "Zero entropy for some birational maps of $\mathbb{C}^2$", see [9], and "Finding invariant fibrations for some birational maps of $\mathbb{C}^2$", see [10], we give all the maps of type (1) which have zero entropy in the particular cases $\gamma_1 = 0$ in the first one and for the degenerate cases in the second one, giving explicitly the invariants when they exist.

The details of prerequisites and results of this work can be found in [12].

The article is organized as follows: The main results are announced in Section 2. The preliminary results which include the basic settings of the work, some background of birational maps and Picard group and the structure of the orbits'lists are introduced in Section 3. In Section 4 we give the proof of the results. Finally in Section 5 we present the rest of the zero entropy mappings which are not included in the two mentioned papers [9] and [10].

2 Main results

The results that we find are presented in the following theorems $1, 2, 3, 4$ and 5. In all of them we consider that the coefficients of the map f are such that f is a birational map and F has degree two (see Lemma 5).

Consider the family of fractional maps $f : \mathbb{C}^2 \to \mathbb{C}^2$:

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y} \right),$$

where the parameters are complex numbers. We call

$$F[x_0 : x_1 : x_2] = [F_1[x_0 : x_1 : x_2] : F_2[x_0 : x_1 : x_2] : F_3[x_0 : x_1 : x_2]], \tag{3}$$

the extension of $f(x,y)$ to the projective plane, where

$$F_1[x_0:x_1:x_2] = x_0(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2),$$
$$F_2[x_0:x_1:x_2] = (\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2)(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2),$$
$$F_3[x_0:x_1:x_2] = x_0(\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2).$$

The indeterminacy set of $F[x_0:x_1:x_2]$ is $\mathcal{I}(F) = \{O_0, O_1, O_2\}$, with

$$O_0 = [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01}], \quad O_1 = [0 : \alpha_2 : -\alpha_1], \quad O_2 = [0 : \gamma_2 : -\gamma_1], \tag{4}$$

where $(\beta\gamma)_{ij} := \beta_i\gamma_j - \beta_j\gamma_i$ for $i,j \in \{0,1,2\}$.

By calling $f^{-1}(x,y)$ the inverse of $f(x,y)$ and by $F^{-1}[x_0:x_1:x_2]$ its extension on $P\mathbb{C}^2$, also a indeterminacy set $\mathcal{I}(F^{-1})$ exists: $\mathcal{I}(F^{-1}) = \{A_1, A_2, A_3\}$, with

$$A_0 = [0:1:0] \quad , \quad A_1 = [0:0:1],$$

$$A_2 = [(\beta\gamma)_{12}\,(\alpha\gamma)_{12} : (\alpha_0\,(\beta\gamma)_{12} - \alpha_1\,(\beta\gamma)_{02} + \alpha_2\,(\beta\gamma)_{01})\,(\alpha\gamma)_{12} : (\alpha\beta)_{12}\,(\beta\gamma)_{12}]. \tag{5}$$

We denote by $\delta^* = \frac{1+\sqrt{5}}{2}$ the golden mean, which is the largest root of the polynomial $x^2 - x - 1$.

Theorem 1. *Let $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ be a birational degree two non degenerate map of type (1) and suppose that $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are all non zero. Then either,*

(i) If it exists $p \in \mathbb{N}$ such that $F^p(A_2) = O_0$, then the characteristic polynomial associated with F is

$$\mathcal{W}_p = x^{p+2} - 2x^{p+1} + x - 1,$$

$\delta(F)$ is given by the largest root of the polynomial $\mathcal{W}_p$ and d_n grows quadratically.

(ii) If no such p exists then $\delta(F) = 2$ and d_n grows quadratically.

Notice that Theorem 1 says us that family (1) generically has dynamical degree equal 2.

Theorem 2. *Let $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ be a birational degree two non degenerate map of type (1) and suppose that $\gamma_1 = 0$. Then $\gamma_2, \alpha_1, \beta_1$ are non zero and the following hold:*

1. *Assume that $\alpha_2 = 0$ and let $\tilde{F}$ be the extension of F after blowing-up the points A_0, A_1. If $\tilde{F}^p(A_2) = O_0$ for some $p \in \mathbb{N}$ then the characteristic polynomial associated with F is given by*

$$\mathcal{X}_p = (x^{p+1} + 1)(x-1)^2(x+1),$$

and the sequence of degrees d_n of F is periodic with period $2p+2$. If no such p exists then the characteristic polynomial associated with F is

$$\mathcal{X} = (x-1)^2\,(x+1),$$

and the sequence of degrees d_n grows linearly.

2. *Assume that $\alpha_2 \neq 0$ and let $\tilde{F}$ be the induced map after blowing up the point A_0. Then the following hold:*

   - *If $\tilde{F}^p(A_2) = O_0$ for some $p \in \mathbb{N}$ and $\tilde{F}^{2k}(A_1) \neq O_1$ for all $k \in \mathbb{N}$ then the characteristic polynomial associated with F is given by*

     $$\mathcal{X}_p = x^{p+1}(x^2 - x - 1) + x^2,$$

     and

     – *for $p = 0$, $p = 1$ the sequence of degrees d_n is bounded,*

     – *for $p = 2$ the sequence of degrees d_n grows linearly,*

     – *for $p > 2$ the sequence of degrees d_n grows exponentially.*

   - *Assume that $\tilde{F}^{2k}(A_1) = O_1$ for some $k \in \mathbb{N}$. Let $\tilde{F}_1$ be the induced map after we blow-up the points $A_0, A_1, \tilde{F}(A_1), \ldots, \tilde{F}^{2k}(A_1) = O_1$. If $\tilde{F}_1^p(A_2) \neq O_0$ for all $p \in \mathbb{N}$, then the characteristic polynomial associated with F is given by*

     $$\mathcal{X}_k = x^{2k+1}(x^2 - x - 1) + 1,$$

     and the sequence of degrees grows exponentially. Furthermore $\delta(F) \to \delta^*$ as $k \to \infty$.

   - *If $\tilde{F}^{2k}(A_1) = O_1$ and $\tilde{F}_1^p(A_2) = O_0$ for some $p, k \in \mathbb{N}$ then the characteristic polynomial associated with F is given by*

     $$\mathcal{X}_{(k,p)} = x^{p+1}(x^{2k+3} - x^{2k+2} - x^{2k+1} + 1) + x^{2k+3} - x^2 - x + 1,$$

     and

     – *for $p > \frac{2\,(1+k)}{k}$ the sequence of degrees d_n grows exponentially for all p, $k \in \mathbb{N}$;*

     – *for $(p,k) \in \{(3,\,2),\,(4,\,1)\}$ the sequence of degrees d_n is periodic or grows quadratically;*

     – *for $(p,k) \in \{(0,\,k),\,(1,\,k),\,(2,\,k),\,(3,\,1)\}$ the sequence of degrees d_n is periodic.*

- *Assume that* $\tilde{F}^{2k}(A_1) \neq O_1$ *and* $\tilde{F}^{p}(A_2) \neq O_0$ *for all* $k,\, p \in \mathbb{N}$. *Then the characteristic polynomial associated with* F *is given by*

$$\mathcal{X}(x) = x^2 - x - 1,$$

and the sequence of degrees grows exponentially with $\delta(F) = \delta^*$.

Theorem 3. *Let* $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ *be a birational degree two non degenerate map of type (1) and suppose that* $\gamma_2 = 0$. *Then* $\gamma_1, \alpha_2, \beta_2$ *are non zero and the following hold:*

1. *Let* F *be the map for* $\alpha_1 = 0$ *and let* $\tilde{F}$ *be the induced map after blowing up the points* $A_0\,, A_1$. *If* $\tilde{F}^p(A_2) = O_0$ *for some* $p \in \mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Y}_p = x^{p+1}(x^3 - x - 1) + (x^3 + x^2 - 1),$$

and

- *for* $p \in \{0,\, 1,\, 2,\, 3,\, 4,\, 5\}$ *the sequence of degrees* d_n *is periodic of period* $6, 5, 8, 12, 18$ *and* 30 *respectively;*
- *for* $p = 6$ *the sequence of degrees* d_n *it grows quadratically or it is periodic of period* 30*;*
- *for* $p > 6$ *the sequence of degrees* d_n *grows exponentially.*

If no such p *exists then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Y} = x^3 - x - 1,$$

and the sequence of degrees grows exponentially with $\delta(F) = \delta_*$.

2. *Let* F *be the map for* $\alpha_1 \neq 0$ *and let* $\tilde{F}$ *be the induced map after blowing up the point* A_1. *Then the following hold:*

- *If* $\tilde{F}^p(A_2) = O_0$ *for some* $p \in \mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Y}_p = x^{p+1}(x^2 - x - 1) + x^2 - 1,$$

and the sequence of degrees has exponential growth rate. Furthermore $\delta(F) \to \delta^*$ *as* $p \to \infty$.

- *If* $\tilde{F}^q(A_2) = O_1$ *for some* $q \in \mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Y}_q = x^{q+1}(x^2 - x - 1) + x^2,$$

and for $q \geq 2$*:*

– *The sequence of the degrees grows linearly when* $q = 2$.

– *The sequence of the degrees grows exponentially when* $q > 2$.

For $q \in \{0, 1\}$ *there are no such mappings.*

- *Assume that* $\tilde{F}^p(A_2) \neq O_0$ *and* $\tilde{F}^q(A_2) \neq O_1$ *for all* q, $p \in \mathbb{N}$. *Then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Y}(x) = x^2 - x - 1,$$

and the sequence of degrees grows exponentially with $\delta(F) = \delta^*$.

Theorem 4. *Let* $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ *be a birational degree two non degenerate map of type (1) and suppose that* $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ *and* $\alpha_1\alpha_2 = 0$. *Then:*

1. *Assume that* $\alpha_1 = 0$ *and let* $\tilde{F}$ *be the induced map after blowing up the point* A_0. *Then the following hold:*

- *If* $\tilde{F}^p(A_1) = O_0$ *for some* $p \in \mathbb{N}$ *and* $\tilde{F}^q(A_2) \neq O_0$ *for all* $q \in \mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Z}_p = x^{p+1}(x^2 - x - 1) + x^2,$$

and for $p \geq 2$*:*

– *The sequence of the degrees grows linearly for* $p = 2$.

– *The sequence of the degrees grows exponentially for* $p > 2$.

For $p \in \{0, 1\}$ *there are no such mappings.*

- *If* $\tilde{F}^q(A_2) = O_0$ *for some* $q \in \mathbb{N}$ *and* $\tilde{F}^p(A_1) \neq O_0$ *for all* $p \in \mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Z}_q = x^{q+1}(x^2 - x - 1) + x^2 - 1,$$

and the sequence of degrees has exponential growth rate. Furthermore $\delta(F) \to \delta^*$ *as* $q \to \infty$.

- *If* $\tilde{F}^p(A_1) = O_0$ *and* $\tilde{F}^q(A_2) = O_0$ *for some* p, $q \in \mathbb{N}$, *then* $p \neq q$ *and*

– *for* $p > q$ *the characteristic polynomial associated with* F *is* $\mathcal{Z}_q$.

– *for* $p < q$ *the characteristic polynomial associated with* F *is* $\mathcal{Z}_p$.

- *If* $\tilde{F}^p(A_1) \neq O_0$ *and* $\tilde{F}^q(A_2) \neq O_0$ *for all* p, $q \in \mathbb{N}$ *then the characteristic polynomial associated with* F *is given by*

$$\mathcal{Z}(x) = x^2 - x - 1,$$

and the sequence of degrees grows exponentially with $\delta(F) = \delta^*$.

2. *Assume $\alpha_2 = 0$ and let $\tilde{F}$ be the induced map after blowing up the point A_1. If there exists some $p \in \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$, then the characteristic polynomial associated with F is given by*

$$\mathcal{Z}_p = (x^{p+1} + 1)(x-1)^2,$$

and for all $p \in \mathbb{N}$ the sequence of degrees d_n grows linearly. If no such p exists then the characteristic polynomial associated with F is given by $\mathcal{Z} = (x-1)^2$, and d_n grows linearly.

3 Preliminary results

3.1 Settings

Consider

$$f(x,y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right).$$

The exceptional locus of $F[x_0 : x_1 : x_2]$ is $\mathcal{E}(F) = \{S_0, S_1, S_2\}$, where

$$S_0 = \{x_0 = 0\}, \quad S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\},$$

$$S_2 = \{(\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01})\, x_0 + \alpha_1(\beta\gamma)_{12} x_1 + \alpha_2(\beta\gamma)_{12} x_2 = 0\},$$

and the exceptional locus of $F^{-1}[x_0 : x_1 : x_2]$ is $\mathcal{E}(F^{-1}) = \{T_0, T_1, T_2\}$, where

$$\begin{aligned} T_0 &= \{(\gamma_0(\alpha\beta)_{12} - \gamma_1(\alpha\beta)_{02} + \gamma_2(\alpha\beta)_{01})\, x_0 - (\beta\gamma)_{12} x_1 = 0\}, \\ T_1 &= \{(\alpha\beta)_{12} x_0 - (\alpha\gamma)_{12} x_2 = 0\}, \qquad T_2 = \{x_0 = 0\}. \end{aligned}$$

It is easy to see that F maps each S_i to A_i where the $A_i's$ are defined in (5) and that the inverse of F maps T_i to O_i for $i \in \{0,1,2\}$, see (4). To specify this behaviour we write $F : S_i \twoheadrightarrow A_i$ (also $F^{-1} : T_i \twoheadrightarrow O_i$).

We are interested in the mappings (1) when they are birational maps which are not degree one maps. Next lemma informs about the set of parameters which are available in this study. Also the degenerate case and the non degenerate cases are distinguished. We recognize the non degenerate case when F has three distinct exceptional curves. When f has two exceptional curves of such type, then we are in degenerate case.

Recall that a birational map is a map $f : \mathbb{C}^2 \to \mathbb{C}^2$ with rational components such that there exists an algebraic curve V and another rational map g such that $f \circ g = g \circ f = id$ in $\mathbb{C}^2 \setminus V$.

Lemma 5. *Consider the mappings*

$$f(x_1,x_2)=\left(\alpha_0+\alpha_1x_1+\alpha_2x_2,\frac{\beta_0+\beta_1x_1+\beta_2x_2}{\gamma_0+\gamma_1x_1+\gamma_2x_2}\right),\ (\gamma_1,\gamma_2)\neq(0,0)\neq\ (\alpha_1,\alpha_2).$$

Then:

(a) *The mapping f is birational if and only if the vectors $(\beta_0,\beta_1,\beta_2)$, $(\gamma_0,\gamma_1,\gamma_2)$ are linearly independent and $((\alpha\beta)_{12},(\alpha\gamma)_{12})\neq(0,0)$, $((\alpha\gamma)_{12},(\beta\gamma)_{12})\neq(0,0)$, and either $((\alpha\beta)_{12},(\beta\gamma)_{12})\neq(0,0)$ or $(\beta_1,\ \beta_2)=(0,0)$.*

(b) *The mapping f is degenerate if and only if $(\beta\gamma)_{12}=0$ or $(\alpha\gamma)_{12}=0$.*

Proof. The conditions in *(a)* are necessary for f to be invertible as if the vectors $(\beta_0,\beta_1,\beta_2)$, $(\gamma_0,\gamma_1,\gamma_2)$ are linearly dependent then the second component of f is a constant, also if $((\alpha\beta)_{12},\ (\alpha\gamma)_{12})=(0,0)$ or $((\alpha\gamma)_{12},(\beta\gamma)_{12})=(0,0)$ then f only depends on $\alpha_1\,x_1+\alpha_2\,x_2$ or on $\gamma_1x_1+\gamma_2x_2$. If $((\alpha\beta)_{12},(\beta\gamma)_{12})=(0,0)$ and $(\beta_1,\ \beta_2)\neq(0,0)$ then f only depends on $\beta_1x_1+\beta_2x_2$.

Now assume that conditions (a) are satisfied. Then the inverse of f which formally is

$$f^{-1}(x,y)=\left(\frac{-(\alpha\beta)_{02}+\beta_2x+(\alpha\gamma)_{02}y-\gamma_2xy}{(\alpha\beta)_{12}-(\alpha\gamma)_{12}y},\frac{(\alpha\beta)_{01}-\beta_1x+(\alpha\gamma)_{10}y+\gamma_1xy}{(\alpha\beta)_{12}-(\alpha\gamma)_{12}y}\right),$$

is well defined. Furthermore the numerators of the determinants of the Jacobian of f and f^{-1} are

$$\alpha_1(\beta\gamma)_{02}-\alpha_2(\beta\gamma)_{01}+\alpha_1(\beta\gamma)_{12}x+\alpha_2(\beta\gamma)_{12}y \tag{6}$$

and

$$\alpha_0(\beta\gamma)_{12}-\alpha_1(\beta\gamma)_{02}+\alpha_2(\beta\gamma)_{01}-(\beta\gamma)_{12}y, \tag{7}$$

respectively. It is easily seen that conditions (a) imply that both (6) and (7) are not identically zero. Hence, $f\circ f^{-1}=f^{-1}\circ f=id$ in $\mathbb{C}^2\setminus V$, where V is the algebraic curve determined by the common zeros of (6) and (7).

To see *(b)* we know that since S_i maps to A_i, this implies that the points A_0,A_1,A_2 are not all distinct. Since $A_0\neq A_1$ we have two possibilities: $A_0=A_2$ or $A_1=A_2$. Condition $A_0=A_2$ writes as $(\beta\gamma)_{12}\,(\alpha\gamma)_{12}=0$ and $(\alpha\beta)_{12}(\beta\gamma)_{12}=0$. From *(a)*, the vector $((\alpha\beta)_{12},(\alpha\gamma)_{12})\neq(0,0)$. Hence $(\beta\gamma)_{12}$ must be zero. In a similar way it is seen that $A_1=A_2$ if and only if $(\alpha\gamma)_{12}=0$. ∎

3.2 Birational mappings and Picard group

Given the birational map f let $F[x_0:x_1:x_2]$ be the extension of $f(x_1,x_2)$ at $P\mathbb{C}^2$ and consider $\mathcal{I}(F)$ and $\mathcal{E}(F)$. To get rid of indeterminacies we do a series of blowups. More precisely, if $F^k(A_i) = O_j$ we perform the blowingup at the points $A_i, F(A_i), \ldots, F^k(A_i) = O_j$.

Given a point $p \in \mathbb{C}^2$, let (X, π), be the blowing-up of $\mathbb{C}^2$ at the point p. Then,

$$\pi^{-1}p = \pi^{-1}(0,0) = \{((0,0), [u:v])\} := E_p \simeq P\mathbb{C}^1$$

and if $q = (x,y) \neq (0,0)$, then

$$\pi^{-1}q = \pi^{-1}(x,y) = ((x,y), [x:y]) \in X.$$

Given the point $((0,0),[u:v]) \in E_p$ (resp. $((x,y),[x:y])$) we are going to represent it by $[u:v]_{E_p}$ (resp. by $(x,y) \in \mathbb{C}^2$ or by $[1:x:y] \in P\mathbb{C}^2$ if it is convenient). After every blow up we get a new expanded space X and the induced map $\tilde{F} : X \to X$. Hence in this work we deal with complex manifolds X obtained after performing a finite sequence of blow-ups. Indeterminacy sets and exceptional locus can also be defined if we consider meromorphic functions defined on complex manifolds. If X is a complex manifold we are going to consider the Picard group of X, denoted by $\mathcal{P}ic(X)$. Then $\mathcal{P}ic(P\mathbb{C}^2)$ is generated by the class of L, where L is a generic line in $P\mathbb{C}^2$. As usual, given a curve C on $\mathbb{C}^2$, the *strict transform* of C is the adherence of $\pi^{-1}(C \setminus \{p\})$, in the Zariski topology, and we denote it by $\hat{C}$. If the base points of the blow-ups are $\{p_1, p_2, \ldots, p_k\} \subset P\mathbb{C}^2$ and $E_i := \pi^{-1}\{p_i\}$ then it is known that $\mathcal{P}ic(X)$ is generated by $\{\hat{L}, E_1, E_2, \ldots, E_k\}$, where L is a generic line in $P\mathbb{C}^2$ (see [3, 4]). Furthermore $\pi : X \longrightarrow P\mathbb{C}^2$ induces a morphism of groups $\pi^* : \mathcal{P}ic(P\mathbb{C}^2) \longrightarrow \mathcal{P}ic(X)$, with the property that for any complex curve $C \subset P\mathbb{C}^2$,

$$\pi^*(C) = \hat{C} + \sum m_i \, E_i, \tag{8}$$

where m_i is the algebraic multiplicity of C at p_i.

On the other hand, if F is a birational map defined on $P\mathbb{C}^2$, then there is a natural extension of F on X, which we denote by $\tilde{F}$. And $\tilde{F}$ induces a morphism of groups, $\tilde{F}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$ just by taking classes of preimages. The interesting thing here is that

$$\tilde{F}^*(\hat{L}) \; = \; d\,\hat{L} \; + \; \sum_{i=1}^{k} c_i \, E_i \quad , \quad c_i \in \mathbb{Z}$$

where d is the degree of F. By iterating F, we get the corresponding formula by changing F by F^n and d by d_n. In order to deduce the behavior of the sequence d_n it is convenient to deal with maps $\tilde{F}$ such that

$$(\tilde{F}^n)^* = (\tilde{F}^*)^n. \tag{9}$$

Maps $\tilde{F}$ satisfying condition (9) are called *Algebraically Stable maps* (AS for short), (see [18]).

In order to get AS maps we will use the following useful result showed by Fornaess and Sibony in [19] (see also Theorem 1.14) of [18]:

$$\text{The map } \tilde{F} \text{ is AS if and only if for every exceptional curve } C \text{ and all } n \geq 0\,,\ \tilde{F}^n(C) \notin \mathcal{I}(\tilde{F}). \tag{10}$$

It is known (see Theorem 0.1 of [18]) that one can always arrange for a birational map to be AS considering an extension of f. If it is the case and we call $\mathcal{X}(x) = x^k + \sum_{i=0}^{k-1} c_i\, x^i$ the characteristic polynomial of $A := (\tilde{F}^*)$, then since $\mathcal{X}(A) = 0$ and d_i is the $(1,1)$ term of A^i we get that

$$d_k = -(c_0 + c_1 d_1 + c_2 d_2 + \cdots + c_{k-1} d_{k-1}),$$

i. e., the sequence d_n satisfies a homogeneous linear recurrence with constant coefficients. The dynamical degree is then the largest real root of $\mathcal{X}(x)$.

The following result is useful in our work. It is a direct consequence of Theorem 0.2 of [18]. Given a birational map F of $P\mathbb{C}^2$, let $\tilde{F}$ be its regularized map so that the induced map $\tilde{F}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$ satisfies $(\tilde{F}^n)^* = (\tilde{F}^*)^n$. Then

Theorem 6. *(See [18]) Let $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ be a birational map, $\tilde{F}$ be its regularized map and let $d_n = deg(F^n)$. Then up to bimeromorphic conjugacy, exactly one of the following holds:*

- *The sequence d_n grows quadratically, $\tilde{F}$ is an automorphism and f preserves an elliptic fibration.*
- *The sequence d_n grows linearly and f preserves a rational fibration. In this case $\tilde{F}$ cannot be conjugated to an automorphism.*
- *The sequence d_n is bounded, $\tilde{F}$ is an automorphism and f preserves two generically transverse rational fibrations.*
- *The sequence d_n grows exponentially.*

In the first three cases $\delta(F) = 1$ while in the last one $\delta(F) > 1$. Furthermore in the first and second, the invariant fibrations are unique.

3.3 Lists of orbits.

We derive our results in the non-degenerate case by using Theorem 7 below, established and proved in [3]. The proof of that is based in the same tools explained in the above paragraph. In order to determine the matrix of the extended map in the Picard group, it is necessary to distinguish between different behaviors of the iterates of the map on the indeterminacy points of its inverse.

The theorem is written for a general family G of quadratic maps of the form $G = L \circ J$. As we will see the maps of family (1), when the triangle is non-degenerate, are linearly conjugated to such a maps. Here L is an invertible linear map and J is the involution in $P\mathbb{C}^2$ as follows:

$$J[x_0 : x_1 : x_2] = [x_1x_2 : x_0x_2 : x_0x_1].$$

We find that the involution J has an indeterminacy locus $\mathcal{I} = \{\epsilon_0, \epsilon_1, \epsilon_2\}$ and a set of exceptional curves $\mathcal{E} = \{\Sigma_0, \Sigma_1, \Sigma_2\}$, where $\Sigma_i = \{x_i = 0\}$ for $i = 0, 1, 2$, and $\epsilon_i = \Sigma_j \bigcap \Sigma_k$ with $\{i, j, k\} = \{0, 1, 2\}$ and $i \neq j \neq k$, $i \neq k$. Let $\mathcal{I}(G^{-1}) := \{a_0, a_1, a_2\}$, the elements of this set are determined by $a_i := G(\Sigma_i - \mathcal{I}(J)) = L\,\epsilon_i$ for $i = 0, 1, 2$; see [3].

To follow the orbits of the points of $\mathcal{I}(G^{-1})$ we need to understand the following definitions and construction of lists of orbits in order to apply the result of Theorem 7.

We assemble the orbit of a point $p \in P\mathbb{C}^2$ under the map G as follows. For a point $p \in \mathcal{E}(G) \cup \mathcal{I}(G)$ we say that the orbit $\mathcal{O}(p) = \{p\}$. Now consider that there exits a $p \in P\mathbb{C}^2$ such that its $n^{th}-$ iterate belongs to $\mathcal{E}(G) \cup \mathcal{I}(G)$ for some n, whereas all the other $n-1$ iterates of p under G are never in $\mathcal{E}(G) \cup \mathcal{I}(G)$. This is to say that for some n the orbit of p reaches an exceptional curve of G or an indeterminacy point of G. We thus define the orbit of p as $\mathcal{O}(p) = \{p, G(p), ..., G^n(p)\}$ and we call it a *singular orbit*. If for some $p \in P\mathbb{C}^2$ in turns out that p and all of its iterates under G are never in $\mathcal{E}(G) \cup \mathcal{I}(G)$ for all n, we set as $\mathcal{O}(p) = \{p, G(p), G^2(p)...\}$ and $\mathcal{O}(p)$ is *non singular orbit*. We now make another characterization of these orbits. Consider that a singular orbit reaches an indeterminacy point of G, this is to say that $G^n(p) \in \mathcal{I}(G)$ but its not in $\mathcal{E}(G)$. We call such orbits as *singular elementary orbits* and we refer them as SE-orbits. To apply Theorem 7 we need to organize our SE orbits into lists in the following way.

Two orbits $\mathcal{O}_1 = \{a_1, ..., \epsilon_{j_1}\}$ and $\mathcal{O}_2 = \{a_2, ..., \epsilon_{j_2}\}$ are in the same list if either $j_1 = 2$ or $j_2 = 1$, that is, if the ending index of one orbit is the same as the beginning index of the other. We have the following possibilities:

- *Case 1: One SE-orbit,* $\mathcal{O}_i = \{a_i, ..., \epsilon_{\tau(i)}\}$. Then we have the list $\mathcal{L} = \{\mathcal{O}_i = \{a_i, ..., \epsilon_{\tau(i)}\}\}$. If $\tau(i) = i$ we say that $\mathcal{L}$ is a closed list. Otherwise it is an open list.

- *Case 2: Two SE-orbits,* $\mathcal{O}_i = \{a_i, ..., \epsilon_{\tau(i)}\}\}$ and $\mathcal{O}_j = \{a_i, ..., \epsilon_{\tau(j)}\}\}$. In this case we can have either two closed lists,

$$\mathcal{L}_1 = \{\mathcal{O}_i = \{a_i, ..., \epsilon_i\}\} \quad \text{and} \quad \mathcal{L}_2 = \{\mathcal{O}_j = \{a_j, ..., \epsilon_j\}\} \quad \text{with} \quad i \neq j$$

or one open and one closed list

$$\mathcal{L}_1 = \{\mathcal{O}_i = \{a_i, ..., \epsilon_i\}\} \quad \text{and}$$
$$\mathcal{L}_2 = \{\mathcal{O}_j = \{a_j, ..., \epsilon_k\}\} \quad \text{with} \quad i \neq j\,,\, j \neq k\,,\, k \neq i$$

or a single list

$$\mathcal{L} = \{\mathcal{O}_i = \{a_i, ..., \epsilon_j\}, \mathcal{O}_j = \{a_j, ..., \epsilon_{\tau(j)}\}\} \quad \text{with} \quad i \neq j$$

which is closed if $\tau(j) = i$ and an open list otherwise.

Notice that we cannot have two open lists because there are at most three SE-orbits.

- *Case 3: Three SE orbits*: In this case we can have either *three closed lists*

$$\mathcal{L}_1 = \{\mathcal{O}_0 = \{a_0, ..., \epsilon_0\}\} \quad \text{and} \quad \mathcal{L}_2 = \{\mathcal{O}_1 = \{a_1, ..., \epsilon_1\}\} \quad \text{and}$$
$$\mathcal{L}_3 = \{\mathcal{O}_2 = \{a_2, ..., \epsilon_2\}\},$$

or *two closed lists*

$$\mathcal{L}_1 = \{\mathcal{O}_i = \{a_i, ..., \epsilon_j\}, \mathcal{O}_j = \{a_j, ..., \epsilon_i\}\} \text{ and}$$
$$\mathcal{L}_2 = \{\mathcal{O}_k = \{a_k, ..., \epsilon_k\}\} \quad \text{with} \quad i \neq k \neq j \quad \text{and} \quad i \neq j$$

or *one closed list*

$$\mathcal{L} = \{\mathcal{O}_0 = \{a_0, ..., \epsilon_1\}, \mathcal{O}_1 = \{a_1, ..., \epsilon_2\}, \mathcal{O}_2 = \{a_2, ..., \epsilon_0\}\}.$$

We now define two polynomials $\mathcal{T}_{\mathcal{L}}$ and $\mathcal{S}_{\mathcal{L}}$ which we will use to state theorem 7. Let n_i denote the sum of the number of elements of an orbit $\mathcal{O}_i$ and let $\mathcal{N}_{\mathcal{L}} = n_u + ... + n_{u+\mu}$ denote the sum of the numbers of elements of each list $|\mathcal{L}|$. If $\mathcal{L}$ is closed then $\mathcal{T}_{\mathcal{L}} = x^{\mathcal{N}_{\mathcal{L}}} - 1$ and if $\mathcal{L}$ is open then $\mathcal{T}_{\mathcal{L}} = x^{\mathcal{N}_{\mathcal{L}}}$. Now we define $\mathcal{S}_{\mathcal{L}}$ for different lists as follows:

$$\mathcal{S}_{\mathcal{L}}(x) = \begin{cases} 1 & \text{if } |\mathcal{L}| = \{n_1\}, \\ x^{n_1} + x^{n_2} + 2 & \text{if } \mathcal{L} \text{ is closed and } |\mathcal{L}| = \{n_1, n_2\}, \\ x^{n_1} + x^{n_2} + 1 & \text{if } \mathcal{L} \text{ is open and } |\mathcal{L}| = \{n_1, n_2\}, \\ \sum_{i=1}^{3} \left[x^{\mathcal{N}_{\mathcal{L}} - n_i} + x^{n_i}\right] + 3 & \text{if } \mathcal{L} \text{ is closed and } |\mathcal{L}| = \{n_1, n_2, n_3\}, \\ \sum_{i=1}^{3} x^{\mathcal{N}_{\mathcal{L}} - n_i} + \sum_{i \neq 2} x^{n_i} + 1 & \text{if } \mathcal{L} \text{ is open and } |\mathcal{L}| = \{n_1, n_2, n_3\}. \end{cases}$$

Theorem 7. *([4]) If $G = L \circ J$, then the dynamical degree $\delta(G)$ is the largest real zero of the polynomial*

$$\mathcal{X}(x) = (x-2) \prod_{\mathcal{L} \in \mathcal{L}^c \cup \mathcal{L}^o} \mathcal{T}_{\mathcal{L}}(x) + (x-1) \sum_{\mathcal{L} \in \mathcal{L}^c \cup \mathcal{L}^o} S_L(x) \prod_{\mathcal{L}' \neq \mathcal{L}} \mathcal{T}_{\mathcal{L}'}(x).$$

Here $\mathcal{L}$ runs over all the orbit lists.

4 Proof of the results

F is a non-degenerate when the sets $\mathcal{E}(F)$, $\mathcal{E}(F^{-1})$ have the three elements, each two of them intersecting on distinct points of $\mathcal{I}(F)$, $\mathcal{I}(F^{-1})$ presented in section 2.1. In this section we consider this to do the following study.

We consider the involution $J[x_0 : x_1 : x_2]$ introduced in section 2.3, and two invertible linear maps M_1 and M_2 of $P\mathbb{C}^2$ such that M_1 sends each Σ_i to S_i and $M_2(A_i) = \epsilon_i$ for $i = 0, 1, 2$. Then the mapping $M_2 \circ F \circ M_1$ is quadratic and sends each Σ_i to ϵ_i. Therefore $M_2 \circ F \circ M_1$ must be of the form $[\lambda_0\, x_1\, x_2 : \lambda_1 x_0 x_2 : \lambda_2\, x_0\, x_1]$ with $\lambda_i \neq 0$ for each $i = 0, 1, 2$, that is $M_2 \circ F \circ M_1 = D \circ J$ where D=diag$(\lambda_0, \lambda_1, \lambda_2)$. Calling $L = M_1^{-1} \circ M_2^{-1} \circ D$ we get that $F \circ M_1 = M_2^{-1} \circ D \circ J = M_1 \circ L \circ J$, that is the mappings F and $G := L \circ J$ are linearly conjugated. Calling $a_i := G(\Sigma_i - \mathcal{I}(J)) = L\epsilon_i$ for $i = 0, 1, 2$ we are going to identify each $a_i \in \mathcal{I}(G^{-1})$ with $A_i \in \mathcal{I}(F^{-1})$.

From now on we are going to assume that $f(x, y)$ is birational (see conditions (a) in Lemma 5), that $F[x_0 : x_1 : x_2]$ has degree two and that it is not not degenerated (i. e., $(\alpha\gamma)_{12} \neq 0 \neq (\beta\gamma)_{12}$). The exceptional set of F and F^{-1} can be seen in the following figure:

The mapping F is bijective from $P\mathbb{C}^2 \setminus \{S_0, S_1, S_2\}$ to $P\mathbb{C}^2 \setminus \{T_0, T_1, T_2\}$. The only points in T_i which have preimage by F are A_j, A_k with $i \notin \{j, k\}$, $j \neq k$ which have as preimages S_j and S_k respectively.

To prove the results we use the following strategy. First we perform the necessary blow-up's in order to have an extension of F, that is $\tilde{F}$ and it is AS. Then we construct the lists of the orbits of points A_i and we apply Theorem 7. We now give the proofs of Theorems 1 to 4. They are as follows.

1. **Proof of Theorem 1**

   Proof. The conditions on the parameters imply that $F(A_0) = A_0$ with $A_0 \notin \mathcal{I}(F)$ and $F(A_1) = A_0$. Since $A_0, A_1 \in S_0 \in \mathcal{E}(F)$, thus we find that their orbits are $\mathcal{O}_0 = \{A_0\}$ and $\mathcal{O}_1 = \{A_1\}$ which are singular but not elementary. Now it remains to analyze the behavior of iterates of A_2. We claim that $\nexists p \in \mathbb{N} : F^p(A_2) = O_1$ and $\nexists p \in \mathbb{N} : F^p(A_2) = O_2$. It is so because if $F^p(A_2) = O_1$ then, since $O_1 \in S_0 = T_2$ it would imply $O_1 = A_0$ or $O_1 = A_1$, that is $\alpha_1 = 0$ or $\alpha_2 = 0$. Similarly, $F^p(A_2) = O_2$ implies $\gamma_1 = 0$ or $\gamma_2 = 0$. Therefore the only possibility is that any iterate of A_2 reaches O_0. Thus we assume the following cases:

   (a) Assume that $F^p(A_2) \neq O_0$ for all $p \in \mathbb{N}$. Then the map F is itself AS. Hence, $\delta(f) = 2$.

(b) Now assume that $F^p(A_2) = O_0$ for some $p \in \mathbb{N}$. Thus we have a SE orbit of A_2 which is: $\mathcal{O}_2 = \{A_2, F(A_2), ..., F^p(A_2) = O_0\}$. In this case we have only one list $\mathcal{L}_o$ which is open. That is:

$$\mathcal{L}_o = \{\mathcal{O}_2 = \{A_2, F(A_2), ..., F^p(A_2) = O_0\}\}.$$

To find the characteristic polynomial we use Theorem 7. We find that $\mathcal{N}_{\mathcal{L}_o} = p$, $\mathcal{T}_{\mathcal{L}_o} = x^p$ and $\mathcal{S}_{\mathcal{L}_o} = 1$. Then the $\delta(F)$ is the largest root of the characteristic polynomial $\mathcal{Y}_p(x) := x^{p+2} - 2x^{p+1} + x - 1$.

Observe that for all the values of $p \in \mathbb{N}$ the above polynomial has always the largest root $\lambda > 1$. This is because $\mathcal{Y}_p(1) = -1 < 0$ and $\mathcal{Y}_p(2) = 1 > 0$, therefore there always exists a root $\lambda > 1$ such that $\mathcal{Y}_p(\lambda) = 0$. Hence d_n has exponential growth rate.

∎

2. **Proof of Theorem 2**

Assume that $\gamma_1 = 0$. From Lemma 5 we know that α_1, β_1 and γ_2 are non zero. We distinguish two cases, depending on α_2.

- Consider the case when $\alpha_2 = 0$. Observe that $S_0 \twoheadrightarrow A_0 = O_2$ and $S_1 \twoheadrightarrow A_1 = O_1$. Hence we blow up the points A_0, A_1 to get the exceptional fibres E_0, E_1. Let X be the new space and let $\tilde{F} : X \to X$ be the extended map on X. In order to know $\tilde{F}$ we see $[u : v]_{E_0} \in S_0$ (resp. $[u : v]_{E_1} \in S_0$) as $\lim_{t\to 0}[tu : 1 : tv]$ (resp. $\lim_{t\to 0}[tu : tv : 1]$), we evaluate $F[tu : 1 : tv]$ (resp. $F[tu : tv : 1]$) and take limits again. We get:

$$\tilde{F}[0 : x_1 : x_2] = [x_2 : x_1 + \beta x_2]_{E_0}\,,\ \tilde{F}[u : v]_{E_0} = [0 : \alpha_1 v : u] \in T_2 = S_0$$

and

$$\begin{aligned}\tilde{F}[x_0 : x_1 : -\tfrac{\gamma_0}{\gamma_2}x_0] &= [x_0 : \alpha_0 x_0 + \alpha_1 x_1]_{E_1}\\ \tilde{F}[u : v]_{E_1} &= [\gamma_2 u : \gamma_2(\alpha_0 u + \alpha_1 v) : \beta_2 u] \in T_1\end{aligned}$$

Then the map $\tilde{F}$ sends the curve $S_0 \to E_0 \to S_0$ and $S_1 \to E_1 \to T_1$. We observe that no new point of indeterminacy is created therefore $\mathcal{I}(\tilde{F}) = \{O_0\}$ and $\mathcal{E}(\tilde{F}) = \{S_2\}$. Assume that there exists $p \in \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$. Then we blow up $A_2, \tilde{F}(A_2), \tilde{F}^2(A_2), \ldots, \tilde{F}^p(A_2) = O_0$ getting the exceptional fibres which we call $E_2, E_3, \ldots, E_{p+2}$. Set $\tilde{F}_1 : X_1 \to X_1$ the extended map. Performing the blow up at O_0, since T_0 is sent to O_0 via F^{-1}, we have that $\tilde{F}_1^{-1} : T_0 \to E_{p+2}$.

Then $S_2 \to E_2 \to E_3 \to \cdots \to E_{p+1} \to E_{p+2} \to T_0$. Hence $\tilde{F}_1 : X_1 \to X_1$ is an AS map and also an automorphism. Now we have two closed lists as follows

$$\mathcal{L}_{c_1} = \{\mathcal{O}_0 = \{A_0 = O_2\}, \quad \mathcal{O}_2 = \{A_2,\, \tilde{F}(A_2)\,, \ldots, \tilde{F}^p(A_2) = O_0\}\},$$

$$\mathcal{L}_{c_2} = \{\mathcal{O}_1 = \{A_1 = O_1\}\}.$$

Then by using Theorem 7 we find that the characteristic polynomial associated to F is $\mathcal{X} = (x^{p+1}+1)(x-1)^2(x+1)$. If p is even then $x^{p+1}+1$ has the factor $x+1$ and $\mathcal{X} = (x-1)^2\,(x+1)^2\,(x^p - x^{p-1} + \cdots - x + 1)$. Hence the sequence of degrees is $d_n = c_0 + c_1\,n + c_2\,(-1)^n + c_3\,n\,(-1)^n + c_4\,\lambda_1^n + c_5\,\lambda_2^n + \ldots + c_{p+3}\,\lambda_p^n$, where c_i are constants and $\lambda_1,\, \lambda_2, \ldots, \lambda_p$ are the roots of polynomial $x^p - x^{p-1} + \cdots - x + 1$. By looking at d_n we see that f does not grow quadratically or exponentially. As our map $\tilde{F}_1$ is an automorphism then by using the results from Diller and Favre in [18] we see that also cannot have linear growth. Therefore we must have $c_1 = c_3 = 0$. Hence the sequence of degrees must be periodic. This implies that $d_{2p+2+n} = d_n$ i.e. the sequence of degrees is periodic with period $2p+2$. If p is odd then d_n is also periodic of period $2p+2$.

If $\tilde{F}^p(A_2) \neq O_0$ for all $p \in \mathbb{N}$, then we have two lists which are open and closed as follows:

$$\mathcal{L}_o = \{\mathcal{O}_0 = \{A_0 = O_2\}\} \quad , \quad \mathcal{L}_c = \{\mathcal{O}_1 = \{A_1 = O_1\}\}.$$

Then $\delta(F)$ is determined by the polynomial $(x-1)^2(x+1)$, and $\delta(f) = 1$. The sequence of degrees is $d_n = \frac{5}{4} + \frac{1}{2}\,n - \frac{1}{4}\,(-1)^n$.

- Now consider that $\alpha_2 \neq 0$. The parameters $\alpha_1, \beta_1, \gamma_2$ are all non zero. Observe that $S_0 \twoheadrightarrow A_0 = O_2$. The orbit of A_0 is SE. By blowing up A_0 we get the exceptional fibre E_0 and the new space X. The induced map $\tilde{F} : X \to X$ sends the curve $S_0 \to E_0 \to S_0$. Observe that now $\mathcal{I}(\tilde{F}) = \{O_0, O_1\}$ and $\mathcal{E}(\tilde{F}) = \{S_1, S_2\}$. We see that $A_1 \neq O_1$ and the exceptional curve $S_1 \twoheadrightarrow A_1 \in S_0$. We observe that the collision of orbits discussed in preliminaries is happening here. The orbit of A_1 under $\tilde{F}$ is as follows:

$$S_1 \twoheadrightarrow A_1 \to [\gamma_2 : \beta_2]_{E_0} \to [0 : \alpha_1(\gamma_0 + \beta_2) : \beta_1] \in S_0 \to \cdots$$

After some iterates we can write the expression of $\tilde{F}^{2k}(A_1)$ for all $k > 0 \in \mathbb{N}$ as $\tilde{F}^{2k}(A_1) = [0 : \alpha_1(\gamma_0 + \beta_2)(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) : \beta_1] \in S_0$. Observe that for some value of $k \in \mathbb{N}$ it is possible that $\tilde{F}^{2k}(A_1) = O_1$. This happens when the following *condition* k is satisfied for some k.

$$\alpha_1^2(\gamma_0 + \beta_2)(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{k-1}) + \alpha_2\beta_1 = 0. \tag{11}$$

For such $k \in \mathbb{N}$ the orbit of A_1 is SE. By blowing up the points of this orbit we get the new space X_1 and the induced map $\tilde{F}_1$. Then under the action of $\tilde{F}_1$ we have

$$S_1 \to G_0 \to G_1 \to G_2 \to \cdots \to G_{2k-1} \to G_{2k} \to T_1.$$

Then $\mathcal{I}(\tilde{F}_1) = \{O_0\}$ and $\mathcal{E}(\tilde{F}_1) = \{S_2\}$.

Now if the orbit of A_1 is SE and if $\tilde{F}_1^p(A_2) = O_0$ that is the orbit of A_2 is also SE for some $p \in \mathbb{N}$ then we have three SE orbits. If *condition* k is not satisfied then with the extended map $\tilde{F}$ we have $\mathcal{I}(\tilde{F}) = \{O_0, O_1\}$. Therefore we have two options: $\tilde{F}^p(A_2) = O_0$ or $\tilde{F}^p(A_2) = O_1$.

We claim that for all $p \in \mathbb{N}$, $\tilde{F}^p(A_2) \neq O_1$. Assume that $\tilde{F}^p(A_2) = O_1$ and assume that $F^j(A_2) \notin S_0$ for $j = 1, 2, \ldots, p-1$. $\tilde{F}^p(A_2) = F^p(A_2) = O_1$. Since $O_1 \in S_0$ and $A_2 \notin S_0$ if $F^p(A_2) = O_1$ then p would be greater than zero and since $S_0 = T_2$, it would imply that $O_1 = A_1$ or $O_1 = A_2$, which is not the case (recall that the only points in T_2 which have a preimage are A_1 and A_2).

Contrarily, if it exists some $l \in \mathbb{N}, l < p$ such that $F^j(A_2) \notin S_0$ for $j = 1, 2, \ldots, l-1$ but $F^l(A_2) \in S_0 \setminus \{O_1\}$ then $F^l(A_2)$ must be equal to A_1 or A_2 that is, $F^l(A_2) = A_1$ or $F^l(A_2) = A_2$. The second case is not possible as A_2 is a fixed point. In the first case $\tilde{F}^p(A_2) = \tilde{F}^{p-l}(F^l(A_2)) = \tilde{F}^{p-l}(A_1) = O_1$ which implies that $p = l + 2r$ and $\tilde{F}^{2r}(A_1) = O_1$. Hence the orbit of A_1 must be SE and that condition k must be satisfied for $k = r$ which is a contradiction. It implies that the only available possibility for $\mathcal{O}_2$ to be SE is to have that for some p, $\tilde{F}^p(A_2) = O_0$. After the blow up process we get

$$S_2 \to E_1 \to E_2 \to \cdots \to E_p \to E_{p+1} \to T_0.$$

The extended map $\tilde{F}_2$ is an automorphism when we have three SE orbits.

The above discussion gives us three different cases.

- One SE orbit: This happens when $A_0 = O_2$ with the conditions that $\tilde{F}^{2k}(A_1) \neq O_1$ and $\tilde{F}^p(A_2) \neq O_0$ for all $k, p \in \mathbb{N}$. Therefore we have only one list $\mathcal{L}_o$ which is open that is $\mathcal{L}_o = \{\mathcal{O}_0 = \{A_0 = O_2\}\}$. By using theorem 7 we find that $\delta(F) = \delta^* = \frac{\sqrt{5}+1}{2}$. which is given by the greatest root of the polynomial $X(x) = x^2 - x - 1$. Therefore it has exponential growth.
- Two SE orbits (a): It is the case when $A_0 = O_2$, $\tilde{F}^p(A_2) = O_0$ and $\tilde{F}^{2k}(A_1) \neq O_1$ for all $k \in \mathbb{N}$. By organizing the orbits into lists we have one closed list $\mathcal{L}_c = \{\mathcal{O}_0 = \{A_0 = O_2\}, \quad \mathcal{O}_2 = \{A_2, \tilde{F}(A_2), ..., \tilde{F}^p(A_2) = O_0\}\}$.

By utilizing theorem 7 we find that the characteristic polynomial associated to F is $\mathcal{X}_p = x^{p+1}(x^2 - x - 1) + x^2$. For $p = 0$ and $p = 1$ the sequence of degrees satisfies $d_{n+3} = d_n$ and $d_{n+4} = d_{n+3}$ respectively which corresponds towards boundedness of f.

For $p = 2$ we get the polynomial $\mathcal{X}_2 = x^2(x+1)(x-1)^2$. Looking at the first degrees we get that the sequence of degrees is $d_n = -1 + 2\,n$.

For $p > 2$, we observe that $\mathcal{X}_p(1) = 0$, $\mathcal{X}'_p(1) = 2-p < 0$ and $\lim_{x\to+\infty} \mathcal{X}_p(x) = +\infty$. Hence $\mathcal{X}_p$ always has a root $\lambda > 1$ and the result follows.

- Two SE orbits (b): When we have $A_0 = O_2$, $\tilde{F_1}^{2k}(A_1) = O_1$ and $\tilde{F_1}^{p}(A_2) \neq O_0$ for all $p \in \mathbb{N}$ then there is one open and one closed list and $\mathcal{X}_k = x^{2k+1}(x^2 - x - 1) + 1$. We observe that for all the values of $k \in \mathbb{N}$, $k \geq 1$ the polynomial $\mathcal{X}_k$ has always a root $\lambda > 1$. Therefore f has exponential growth.
- Three SE orbits: In this case we have $A_0 = O_2$, $\tilde{F}^{2k}(A_1) = O_1$, $\tilde{F}^{p}(A_2) = O_0$, for a certain $p, k \in \mathbb{N}$. We have two closed lists as follows:

$$\mathcal{L}_c = \{\mathcal{O}_0 = \{A_0 = O_2\}, \quad \mathcal{O}_2 = \{A_2,\, \tilde{F}(A_2)\,, ...,\, \tilde{F}^p(A_2) = O_0\}\},$$

$$\mathcal{L}_c = \{\mathcal{O}_1 = \{A_1,\, \tilde{F}(A_1)_{E_0}\,, ...,\, \tilde{F}^{2k}(A_1)_{S_0} = O_1\}\}.$$

From theorem 7 we can write $\mathcal{X}_{(k,p)} = x^{p+1}(x^{2k+3} - x^{2k+2} - x^{2k+1} + 1) + x^{2k+3} - x^2 - x + 1$. The map $\tilde{F}_2$ is an automorphism for all the values (k, p). According to Diller and Favre in [18] the growth of degrees of iterates of an automorphism could be bounded, quadratic or exponential but it cannot be linear as in such a case the map is never an automorphism. For this we observe the behavior of $\mathcal{X}_{(k,p)}$ around $x = 1$. we consider it's Taylor expansion near $x = 1$:

$$\mathcal{X}_{(k,p)}(x) = 2(2 - kp + 2k)(x-1)^2 + O(|x-1|^3).$$

Thus $\mathcal{X}_{(k,p)}$ vanishes at $x = 1$ $x = 1$ and has a maximum on it $p > \frac{2(1+k)}{k}$. Since $\lim_{x\to+\infty}\mathcal{X}_{(k,p)}(x) = +\infty$, always exists a root greater than one. If $p \leq \frac{2(1+k)}{k}$, $k \geq 1$ then the pairs (k, p) are in the set: $A_{(k,p)} = \{((k \geq 1), 0), ((k \geq 1), 1), ((k \geq 1), 2), (1, 3), (2, 3), (1, 4)\}$.

For $(k, p) = (k, 0)$, when k is even the sequence of degrees is

$$d_n = c_0 + c_1 n + c_2\,(-1)^n + c_3\,(-1)^n\,n + c_4\,\lambda_1^n + c_5\,\lambda_2^n + ... + c_{2k+3}\,\lambda_{2k}^n,$$

where c_i are constants and λ's are the roots of polynomial $x^{2k+2} = 1$ different from ± 1. If k is odd then

$$d_n = l_0 + l_1 n + l_2\,(-1)^n + l_3\,\mu_1^n + l_4\,\mu_2^n + \cdots + l_{2k+3}\,\mu_{2k+1}^n,$$

where l_i are constants and μ's are the roots of polynomial $(x^{k+1}-1)(x^k+x^{k-1}+\cdots+x+1)$. Since $\tilde{F}_2$ is an automorphism for all (k,p), using [18] we have $c_1=0=c_3$ and also $l_1=0$. This implies that $d_{2k+2+n}=d_n$, i. e., the sequence of degrees is periodic with period $2k+2$. The argument for the proof of other values of $(k,p)\in A_{(k,p)}$ follows accordingly.

3. **Proof of Theorem 3**

From hypothesis and from Lemma 5 we know that $\alpha_2\gamma_1\neq 0$ and $\beta_2\gamma_1\neq 0$ therefore α_2,β_2 and γ_1 cannot be zero. There exist two different cases to study depending on α_1.

- Consider that $\alpha_1=0$. Then $A_0=O_1$ and $A_1=O_2$. We get new space X by blowing up A_0 and A_1. E_0, E_1 are the exceptional fibres on these points respectively. The extended map $\tilde{F}:X\to X$ sends $S_0\to E_0\to T_1$ and $S_1\to E_1\to T_2$. Therefore the orbits of A_0 and A_1 are SE. No new indeterminacy points have appeared therefore $\mathcal{I}(\tilde{F})=\{O_0\}$ and $\mathcal{E}(\tilde{F})=\{S_2\}$. If $\tilde{F}^p(A_2)=O_0$ for some $p\in\mathbb{N}$ then the orbit of A_2 is SE. Let $\tilde{F}_2:X_1\to X_1$ be the extended map on new space X_1 we get after blowing up the points of orbit of A_2. Then $\tilde{F}_2$ sends $S_2\to E_2\to E_3\to\cdots\to E_{p+2}\to T_0$. Then $\tilde{F}_2$ is an AS map and is an automorphism.

  We see that we have one closed list and by utilizing Theorem 7 the characteristic polynomial associated to F is $\mathcal{Y}_p=x^{p+1}(x^3-x-1)+(x^3+x^2-1)$. For $p=0$, the sequence of degrees $d_n=c_1+c_2\,(-1)^n+c_3(\lambda_1)^n+c_4(\lambda_2)^n$, where λ_1, λ_2 are the two roots of $x^2+x+1=0$. Hence d_n satisfies $d_{n+6}=d_n$, i.e., it is periodic of period 6. For $p\leq 5$ the argument for the proof is similar with periods 5, 8, 12, 18 and 30 accordingly. When $p=6$, the sequence of degrees $d_n=c_1+c_2\,n+c_3\,n^2+c_4\,(-1)^n+c_5\,(\lambda_1)^n+c_6(\lambda_2)^n+c_7(\lambda_3)^n+c_8(\lambda_4)^n+c_9(\lambda_5)^n+c_{10}(\lambda_6)^n$. As $\tilde{F}_2$ is an automorphism, from Theorem 6, the sequence of degrees does not grow linearly. Then either, $c_3\neq 0$ and d_n grows quadratically or $c_2=0=c_3$ and d_n is periodic of period 30.

  For $p>6$ there always exists a root $\lambda>1$ of $\mathcal{Y}_p$. Hence the sequence of degrees grows exponentially for such values of p.

  Now suppose that no p exists such that $\tilde{F}^p(A_2)=O_0$. In this case $\delta(F)$ is given by the greatest real root of the polynomial $\mathcal{Y}(x)=x^3-x-1$.

- Now consider that $\alpha_1\neq 0$. Observe that in general $S_0\twoheadrightarrow A_0\neq O_i$ for any $i\in\{0,1,2\}$ and $F(A_0)=A_0$. Thus $\mathcal{O}_0=\{A_0\}$ is not a SE orbit. Now $S_1\twoheadrightarrow A_1=O_2$. We blow up the point $A_1=O_2$. Therefore the orbit of A_1 is SE. Let

X be the new space after blowing up A_1 and let E_1 be the exceptional fibre at this point. The induced map $\tilde{F} : X \to X$ sends the curve $S_1 \to E_1 \to T_2 = S_0$. Then $\mathcal{I}(\tilde{F}) = \{O_0\,, O_1\}$ and $E(\tilde{F}) = \{S_0\,, S_2\}$.

The curve $S_2 \twoheadrightarrow A_2$ and $O_1 \in S_0 = T_2$. Note that $F^p(A_2) \neq O_1$. As the only points on T_2 which have preimages are A_0 and A_1. Then if the orbit of A_2 reaches O_1 at some iterate of F then O_1 should be equal to either A_0 or A_1. As $\alpha_1 \neq 0$ hence $A_0 \neq O_1 \neq A_1$. This implies that $F^p(A_2) \neq O_1$ for all p but it is possible that $\tilde{F}^p(A_2) = O_1$. Then in general there are two possibilities: $\tilde{F}^p(A_2) = O_0$ for some $p \in \mathbb{N}$ or $\tilde{F}^q(A_2) = O_1$ for some $q \in \mathbb{N}$. In both cases the orbit of A_2 is SE.

Now if there exists some $p \in \mathbb{N}$ such that $\tilde{F}^p(A_2) = O_0$, then to get X_1 we blow-up all the points of the orbit of A_2. The extended map $\tilde{F}_1 : X_1 \to X_1$ sends $S_2 \to E_2 \to E_3 \to \cdots \to E_{p+2} \to T_0$. This shows that $\tilde{F}_1$ is an AS map.

Now we have one open list and the characteristic polynomial associated to F is $\mathcal{Y}_p = x^{p+1}(x^2 - x - 1) + x^2 - 1$. Observe that for all the values of $p \in \mathbb{N}$ the polynomial $\mathcal{Y}_p$ always has the largest root $\lambda > 1$. Hence d_n grows exponentially and $\delta(F)$ approaches to the value $\delta^* = \frac{1+\sqrt{5}}{2}$ as $p \to \infty$.

Also if there exists some $q \in \mathbb{N}$ such that $\tilde{F}^q(A_2) = O_1$, then $\tilde{F}_1$ is an AS map. We have one closed list and the characteristic polynomial associated to F is $\mathcal{Y}_q = x^{q+1}(x^2 - x - 1) + x^2$. Note that there are no mappings for $q \in \{0\,, 1\}$. As $A_2[1] \neq O_1[1]$ therefore $q = 0$ is not possible. For $q = 1$ we have two possibilities. First when $A_2 \notin S_1$, then the condition $\tilde{F}(A_2) = F(A_2) = O_1$. But the orbit of A_2 can never reach O_1 because $O_1 \in T_2$ and $O_1 \neq A_0$. Now if $A_2 \in S_1$ then we have the condition $\tilde{F}(A_2) = O_1$. In this case $\tilde{F}(A_2) \in E_1$ but it is clear that $O_1 \notin E_1$ therefore $q = 1$ is not possible.

For $q = 2$ we get the polynomial $x^2(x+1)(x-1)^2$ and the sequence of degrees is $d_n = c_2\,(-1)^n + c_3 + c_4\, n$. Looking at the first degrees we get $d_n = -1 + 2n$. For $q > 2$, we observe that $\mathcal{Y}_q$ always has a root $\lambda > 1$ and the result follows.

4. Proof of Theorem 4

Considering the hypothesis we know that $\gamma_1 \neq 0 \neq \gamma_2$ therefore from lemma 5 the parameters $\{\alpha_0\,, \beta_0,\, \beta_1,\, \beta_2,\, \gamma_0\}$ and one of $\{\alpha_1\,, \alpha_2\}$ at the same moment can be zero. Hence two different cases, $\alpha_1 = 0$ and $\alpha_2 = 0$ are considered as follows:

- Consider that $\alpha_1 = 0$ and $\alpha_2 \neq 0$.

  Observe that $S_0 \twoheadrightarrow A_0 = O_1$. Let X be the new space we get after blowing up the point A_0 and let E_0 be the exceptional fibre at this point. The induced map $\tilde{F} : X \to X$ sends the curves $S_0 \to E_0 \to T_1$. Hence $\mathcal{I}(\tilde{F}) = \{O_0\,, O_2\}$

and $\mathcal{E}(\tilde{F}) = \{S_1\,, S_2\}$. Note that A_1 is not indeterminate for F and $A_1 \in S_0$ therefore orbit of A_1 collides with A_0. But we observe that $\tilde{F}^q(A_1) \neq O_2$ for all q as $O_2 \in S_0 = T_2$. However it is possible that $\tilde{F}^p(A_1) = O_0$ for some $p \in \mathbb{N}$. If there exists such p then we blow up the points of the orbit of A_1 to get the exceptional fibres E_i's. Let X_1 be the new space. Then the extended map $\tilde{F}_1$ sends $S_1 \to E_1 \to E_2 \to \cdots \to E_{p+1} \to T_0$. Now $\mathcal{I}(\tilde{F}_1) = \{O_2\}$ and $\mathcal{E}(\tilde{F}_1) = \{S_2\}$. The exceptional curve $S_2 \twoheadrightarrow A_2$. But the orbit of A_2 can never reach O_2 as $O_2 \in T_2$. Thus the orbit of A_2 is not SE. Hence the map $\tilde{F}_1$ is AS.

In this case we have one closed list by using Theorem 7 the characteristic polynomial associated to F is $\mathcal{Z}_p = x^{p+1}(x^2 - x - 1) + x^2$. The proof for all values of p is similar to the last part of theorem 3.

Now assume that $\tilde{F}^p(A_1) \neq O_0$ for all $p \in \mathbb{N}$ such that the orbit of A_1 is not SE then for some q it is possible that $\tilde{F}^q(A_2) = O_0$. In this case after the blow up of the orbit of A_2, in the extended space X_1 the induced map $\tilde{F}_1$ acts as $S_2 \to E_1 \to E_2 \to \cdots \to E_{q+1} \to T_0$. Thus the orbit of A_2 is SE and the map $\tilde{F}_1$ is AS.

We have one open list and the characteristic polynomial associated to F is $\mathcal{Z}_q = x^{q+1}(x^2-x-1)+x^2-1$. We observe that for all the values of $q \in \mathbb{N}$ the polynomial $\mathcal{Z}_q$ always the largest root $\lambda > 1$ and d_n grows exponentially.

We now consider the case when for some p, $q \in \mathbb{N}$ we have $\tilde{F}^p(A_1) = O_0$ and $\tilde{F}^q(A_2) = O_0$.

We claim that p must be different from q. Assume that $\tilde{F}^k(A_1) \neq A_2$ and $\tilde{F}^j(A_2) \neq A_1$ for any $0 < k < p$ and $0 < j < q$. Because otherwise these points can have multiple preimages. Then $p = q$ gives the condition that $\tilde{F}^p(A_1) = \tilde{F}^p(A_2) = O_0$ implies that $A_1 = A_2$, as F is bijective except for some particular points. But this gives a contradiction as all A_i's must be different in this case. Now if there exists some k or j such that $\tilde{F}^k(A_1) = A_2$ or $\tilde{F}^j(A_2) = A_1$ then there is collision of orbits. This gives that either, $k > 0$ or $j > 0$ which shows that $p \neq q$.

Now consider that $q > p$. Then the orbit of A_2 must collides with the orbit of A_1. Because if not then this claims that $\tilde{F}^k(A_2) \neq A_1$ for all $0 < k < q$. As $q > p$ then for some $j > 0$ we can write $q = p + j$. This gives $\tilde{F}^q(A_2) = \tilde{F}^{j+p}(A_2) = O_0 = \tilde{F}^p(A_1)$. As O_0 has unique preimage and there is no collision this implies that no orbit enters any T_1 or T_2. Therefore the points $\tilde{F}^{j+p}(A_2)$ and $\tilde{F}^p(A_1)$ have unique preimages. Then for $\tilde{F}^{j+p}(A_2) = O_0 = \tilde{F}^p(A_1)$ we can find the preimages by iterating p times with $\tilde{F}^{-1}$. This gives us $\tilde{F}^{-p}(\tilde{F}^{j+p}(A_2)) = \tilde{F}^{-p}(\tilde{F}^p(A_1))$

which implies that $\tilde{F}^j(A_2) = A_1$ for some $0 < j < q$, which gives contradiction to our claim. This implies that in the case when $q > p$ or $q < p$ we always have collision of orbits of A_2 with A_1 or A_1 with A_2.

Now for $q > p$ we must have $\tilde{F}^k(A_2) = A_1$ for some $0 < k < q$. Then we see that:

$$S_2 \twoheadrightarrow A_2 \to \tilde{F}(A_2) \to \cdots \to \tilde{F}^{q-p}(A_2) = A_1 \to \tilde{F}(A_1) \to \cdots \to \tilde{F}^p(A_1) = O_0.$$

This implies that the orbit of A_2 is no more SE. Hence two SE orbits are the orbits of A_0 and A_1. This shows that the characteristic polynomial is Z_p in this case. Similarly, in the second case the characteristic polynomial is Z_q.

Finally, if $\tilde{F}^p(A_1) \neq O_0$ and $\tilde{F}^p(A_2) \neq O_0$ for any $p \in \mathbb{N}$ then we have one SE orbit and the characteristic polynomial is given by $\mathcal{Z}(x) = x^2 - x - 1$. Then the dynamical degree $\delta(f) = \delta^*$.

- Now consider $\alpha_2 = 0$, from Lemma 5 we have $\alpha_1\,, \gamma_1\,, \gamma_2$ non zero. Observe that A_0 is not an indeterminate for F and is a fixed point of F. Hence the orbit of A_0 is not SE.

  Now $S_1 \twoheadrightarrow A_1 = O_1$. After blowing up the point A_1 to get the exceptional fibre E_1, the induced map $\tilde{F} : X \to X$ for new space X sends the curve $S_1 \to E_1 \to T_1$. Now $\mathcal{I}(\tilde{F}) = \{O_0\,, O_2\}$ and $\mathcal{E}(\tilde{F}) = \{S_0\,, S_2\}$. The exceptional curve $S_2 \twoheadrightarrow A_2$. Note that $\tilde{F}^p(A_2) \neq O_2$ for all $p \in \mathbb{N}$ as $O_2 \in T_2$. Then for some $p \in \mathbb{N}$ it is possible that $\tilde{F}^p(A_2) = O_0$. In this case the orbit of A_2 is SE. Let X_1 be the expanded space we get after blowing up the orbit of A_2 the observe that now $\tilde{F}_1 : X_1 \to X_1$ is an AS map but is not an automorphism as S_0 still collapses.

  In this case we have one closed list and one open and the characteristic polynomial associated to F is $\mathcal{Z}_p = (x^{p+1} + 1)(x-1)^2$.

  Now if no such p exists so that $\tilde{F}^p(A_2) = O_0$ then we have one open list $\delta(F)$ is given by the largest root of the polynomial $\mathcal{Z}(x) = (x-1)^2$, which is one.

5 Zero entropy cases

From the results of Theorems 1, 2, 3 and 4 we can present the maps with zero algebraic entropy.

Looking at Theorem 1 we see that the maps which satisfy their hypothesis have not zero entropy. From Theorem 2 we find very interesting maps with zero entropy. This case is studied in detail in the paper [9], giving all the prescribed invariant fibrations and also recognizing which of such a maps are periodic and/or integrable.

Among the mappings satisfying the hypothesis of Theorem 3 we have the ones with $\alpha_1 = 0$. After an affine change of coordinates this maps can be written as

$$f(x,y) = \left(y, \frac{\beta_0 + y}{\gamma_0 + x} \right).$$

These are the maps which we deal when we want to study a linear fractional recurrence of order two, and they are analized in paper [4]. When $\alpha_1 \neq 0$ the only case with dynamical degree equals one is when $\tilde{F}^2(A_2) = O_1$, where $\tilde{F}$ is the mapping induced by F after blowing up the point A_1. To find the maps with this condition in principle we have two possibilities, with or without collision of orbits. When $A_2 \notin S_1$ then the condition $\tilde{F}^2(A_2) = F^2(A_2) = O_1$ never is satisfied. It is because $O_1 \in S_0 = T_2$ and the only points on T_2 which have preimage by F are A_0 and A_1 and we see that $A_0 \neq O_1 \neq A_1$. If $A_2 \in S_1$, that is if $\beta_0 = \alpha_0$, then

$$\tilde{F}^2(A_2) = \tilde{F}(\tilde{F}([1:0:-\alpha_1])) = \tilde{F}([1:\alpha_0 - \alpha_1]_{E_1}) = [0:\alpha_0 - \alpha_1 : 1].$$

Hence, condition $\tilde{F}^2(A_2) = O_1 = [0:1:-\alpha_1]$ is satisfied for $\alpha_0 = \frac{\alpha_1^2 - 1}{\alpha_1} = \beta_0$ and we get the uniparametric family of mappings

$$f(x,y) = \left(\omega + \alpha_1 x + y, \frac{\omega + y}{x} \right) \;,\; \omega = \frac{\alpha_1^2 - 1}{\alpha_1} \;,\; \alpha_1 \neq 0.$$

Since the corresponding sequence of degrees grows linearly, f preserves a unique rational fibration. In fact,

$$V(x,y) = \frac{(1 + \alpha_1 x)(\alpha_1 + \alpha_1 x + y)}{x}$$

satisfies $V(f(x,y)) = \alpha_1 V(x,y)$. When $\alpha_1^n = 1$ for some $n \in \mathbb{N}$ then defining $W(x,y) = V(x,y)^n$ we see that $W(f(x,y)) = W(x,y)$, that is f is integrable. We observe that we also know that these maps never are periodic maps as the degrees grow linearly.

In a similar way, we find

$$f(x,y) = \left(\alpha_2 y, \frac{\beta_2 y}{-\alpha_2 \beta_2 + x + y} \right)$$

which satisfies the hypothesis of Theorem 4 with $\alpha_1 = 0$ and has the unique invariant fibration

$$V(x,y) = \frac{(\beta_2 - y)(\alpha_2 \beta_2 - x)}{y}$$

with the property $V(f(x,y)) = -\alpha_2 V(x,y)$. As before when $(-\alpha_2)^n = 1$, then $W(x,y) = V(x,y)^n$ is a first integral of $f(x,y)$.

Finally, if $f(x,y)$ satisfies the hypothesis of Theorem 4 with $\alpha_1 \neq 0$ and has zero entropy, after an affine change of coordinates can be written as

$$f(x,y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0 + y}{x + y} \right).$$

Clearly $V(x,y)=x$ is an invariant fibration satisfying $V(f(x,y))=\alpha_0+\alpha_1 V(x,y)$. When $\alpha_1^n=1\neq\alpha_1$ then calling $h(x):=\alpha_0+\alpha_1 x$ we have that $W(x,y)=x\,h(x)\,h^2(x)\,\ldots\,h^{n-1}(x)$ is a first integral of $f(x,y)$. Also when $\alpha_1=1$ and $\alpha_0=0$ the maps are integrable.

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