Universitat Autònoma de Barcelona

DEPARTAMENT DE MATEMÀTIQUES

Doctorat en Matemàtiques

Curs Acadèmic 2013-2014

TESI DOCTORAL:

Hamiltonian linear type centers and nilpotent centers of linear plus cubic polynomial vector fields

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Chapter 1

Introduction

In the qualitative theory of real planar polynomial differential systems two of the main problems are the determination of limit cycles and the center-focus problem, i.e. to distinguish when a singular point is either a focus or a center. In this work we provide normal forms for Hamiltonian systems with cubic homogeneous nonlinearities which have a center at the origin, and classify these systems with respect to the topological equivalence of their global phase portraits on the Poincaré disk. This classification will further allow to start the study of how many limit cycles can bifurcate from the periodic orbits of the Hamiltonian centers with only linear and cubic terms when they are perturbed inside the class of all cubic polynomial differential systems. Before going any further we shall talk about some preliminary concepts and definitions that we will use throughout this work. For more details see [14].

1.1 Preliminary definitions

Let A be an open set in $\mathbb{R}^2$. We define a *vector field of class* C^r as a C^2 map $X : A \to \mathbb{R}^2$ where $X(x, y)$ represents the tip of the vector whose tail is at the point $(x, y) \in A$. The *orbits* of the vector field X are the solutions $\varphi(t) = (x(t), y(t))$ of the differential equation

$$(\dot{x}, \dot{y}) = X(x, y), \tag{1.1}$$

where the dot denotes the sderivative with respect to time t. Therefore when we say "vector field X" and "differential system (1.1)" we mean the same thing. Here x and y are called the *dependent variables*, and t is called the *independent variable*. An orbit is called a *periodic orbit* if there exists a $c > 0$ such that $\varphi(t) = \varphi(t + c)$ for every t. A *limit cycle* is a periodic orbit which has a neighborhood that does not contain any additional periodic orbit.

The *flow* of a vector field is defined as usual, see for instance page 3 of [14]. The union of orbits of the vector field X constitute its *phase portrait*.

A *bifurcation diagram* illustrates how the phase portrait of a vector field depends on its parameters.

A point (x, y) is called a *singular point* (or an *equilibrium point*) if $X(x, y) = 0$. If a singular point has a neighborhood that does not contain any other singular point, then that singular point is called an *isolated singular point*.

We define the *linear part* of X at a point as the Jacobian matrix of X at that point. We say that a singular point is *non–elementary* if both of the eigenvalues of the linear part of the vector field at that point are zero, and *elementary* otherwise. If both of the eigenvalues of the linear part of the vector field at an elementary singular point are real, then the singular point is called *hyperbolic*. A non–elementary singular point is called *degenerate* if the linear part is identically zero, otherwise it is called *nilpotent*.

The notion of center goes back to Poincaré, see [24]. He defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled of periodic orbits with the exception of the singular point. If an analytic system has a center, it is known that after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called a *linear type center*;

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y), \tag{1.2}$$

called a *nilpotent center*;

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

called a *degenerate center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

A *saddle* ,a *node*, a *focus* and a *cusp* are defined in the usual way, for more details see for instance [14] pages 7 and 110. A *separatrix of a saddle* is an orbit that tends to that saddle either as $t \to \infty$, or $t \to -\infty$. Clearly a saddle has four separatrices.

We now talk a little about the Poincaré compactification. Let $\mathbb{S}^2$ be the set of points $(s_1, s_2, s_3) \in \mathbb{R}^3$ such that $s_1^2 + s_2^2 + s_3^2 = 1$. We will call this set *the Poincaré sphere*. Given a polynomial vector field

$$X(x, y) = (\dot{x}, \dot{y}) = (P(x, y), Q(x, y)) \tag{1.3}$$

in $\mathbb{R}^2$ of *degree d* (where d is the maximum of the degrees of the polynomials P and Q) it can be extended analytically to the Poincaré sphere by projecting each point $x \in \mathbb{R}^2$ identified by $(x_1, x_2, 1) \in \mathbb{R}^3$ onto the Poincaré sphere using the straight line through x and the origin of $\mathbb{R}^3$. In this way we obtain two copies of X: one on the *northern hemisphere* $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ and another on the *southern hemisphere* $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 < 0\}$. The *equator* $\mathbb{S}^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. The *local charts* needed for doing the calculations on the Poincaré sphere are

$$U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}, \quad V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, with the corresponding *local maps*

$$\varphi_i(s) : U_i \to \mathbb{R}^2, \quad \psi_i(s) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i(s) = -\psi_i(s) = (s_m/s_i, s_n/s_i)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$. The expression for the corresponding vector field on $\mathbb{S}^2$ in the local chart U_1 is given by

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right); \tag{1.4}$$

the expression for U_2 is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right); \tag{1.5}$$

and the expression for U_3 is just

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v), \tag{1.6}$$

where d is the degree of the vector field X. The expressions for the charts V_i are those for the charts U_i multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence, to study the vector field X, it is enough to study its Poincaré compactification restricted to the northern hemisphere plus $\mathbb{S}^1$, which we denote by $\mathbb{D}$ call *the Poincaré disk*. To draw the phase portraits we will consider the projection $\pi(s_1, s_2, s_3) = (s_1, s_2)$ of the Poincaré disk onto the unit disk centered at the origin.

Finite singular points of X are the singular points of its compactification which are in $\mathbb{D}^2 \setminus \mathbb{S}^1$, and they can be studied using U_3. *Infinite singular points* of X are the singular points of the corresponding vector field on the Poincaré disk $\mathbb{D}$ lying on $\mathbb{S}^1$. Clearly a point $s \in \mathbb{S}^1$ is an infinite singular point if and only if so is $-s \in \mathbb{S}^1$, and the local behavior of one is the same as the other multiplied by $(-1)^{d-1}$. Hence to study the infinite singular points it suffices to look only at $U_1|_{v=0}$ and at the origin of U_2.

We say that two vector fields on the Poincaré disk $\mathbb{D}$ are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D} \to \mathbb{D}$ which sends orbits to orbits preserving or reversing the direction of the flow.

A polynomial differential system (1.3) is called *Hamiltonian* if there exists a nonconstant polynomial $H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\dot{x} = H_y, \quad \dot{y} = -H_x,$$

where H_x denotes the partial derivative of H with respect to x. H is called the *Hamiltonian polynomial*. For a Hamiltonian vector field on the Poincaré disk the *separatrices* are (i) the separatrices of finite and infinite saddles, (ii) the finite and infinite singular points, and (iii) all the orbits at infinity. Let Σ be the set of all separatrices, Σ is a closed set in $\mathbb{D}$. The open components of $\mathbb{D} \setminus \Sigma$ are called *canonical regions*. The union of Σ with an orbit from each canonical region is called a *separatrix configuration*. . The next theorem of Neumann [23] gives a characterization of two topologically equivalent vector fields in the Poincaré disk.

Theorem 1 (Neumann's Theorem)**.** *Two continuous flows in $\mathbb{D}$ with isolated singular points are topologically equivalent if and only if their separatrix configurations are equivalent.*

This theorem implies that once a separatrix configuration of a vector field in the Poincaré disk is determined, the global phase portrait of that vector field is obtained up to topological equivalence.

Finally we mention without getting into too much detail an important result that classifies the finite singular points of Hamiltonian planar polynomial differential systems. For a detailed definition of the (topological) *index* of a singular point see for instance Chapter 6 of [14], but for our intents and purposes the following theorem known as the *Poincaré Formula* provides enough information for the subject. Similarly a *parabolic sector*, a *hyperbolic sector* and an *elliptic sector* are defined in the standard way, for details see page 18 of [14]. A vector field is said to have the *finite sectorial decomposition property* at a singular point p if either p is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors.

Theorem 2 (Poincaré Formula)**.** *Let q be an isolated singular point having the finite sectorial decomposition property. Let $e, h,$ and p denote the number of elliptic, hyperbolic, and parabolic sectors of q, respectively. Then the index of q is $(e - h)/2 + 1$.*

For details on Theorem 2 see page 179 of [14].

Proposition 3. *Finite singular points of Hamiltonian planar polynomial vector fields are either centers, or have a finite union of an even number of hyperbolic sectors.*

Proof. It is known that an analytic planar differential system has the finite sectorial decomposition property, for details see [14]. Moreover, if the system is Hamiltonian, its flow preserves area, see [1]. So a singular point of a Hamiltonian system cannot be a focus, or have elliptic or parabolic sectors. Finally, since the index of a singular point formed by hyperbolic sectors is $1 - h/2$, with h being the number of its hyperbolic sectors, it follows that h is even. For more details about the index, see [14]. □

1.2 Background and our main results

An algorithm for the characterization of linear type centers was provided by Poincaré [25] and Lyapunov [20], see also Chazy [6] and Moussu [22]. For an algorithm for the characterization of the nilpotent centers and some class of degenerate centers see the works of Chavarriga *et al.* [5], Giacomini *et al.* [16], Cima and Llibre [8], and Giné and Llibre [17].

The classification of centers for real planar polynomial differential systems started with the classification of centers for quadratic systems, and these results go back mainly to Dulac [13], Kapteyn [18, 19] and Bautin [2]. In [28] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. The bifurcation diagrams of these systems were done by Schlomiuk [26] and Żołądek [31]. There are many partial results for the centers of planar polynomial differential systems of degree larger than two. For instance the linear type centers for cubic systems of the form linear plus homogeneous nonlinearities were characterized by Malkin [21], and Vulpe and Sibirski [29]. We must mention that in this work we do not use their characterization, instead we introduce a different set of normal forms. Some interesting results on some subclasses of cubic systems are those of Rousseau and Schlomiuk [27], and the ones of Żołądek [32, 33]. For polynomial differential systems of the form linear plus homogeneous nonlinearities of degree greater than three the centers at the origin are not characterized, but there are partial results for degrees four and five for the linear type centers, see for instance Chavarriga and Giné [3, 4].

In this work we provide the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. It is shown in [8] that the degenerate centers of such vector fields are topologically equivalent to 1.18 of Figure 1.1. Hence this work completes the classification of the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a center at the origin.

We note that the problem of finding the global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center at the origin has been considered also in [15]. There the authors provide a general algorithm using polar coordinates for finding the phase portraits of Hamiltonian linear type centers with arbitrary n–th order homogeneous nonlinearities. In addition, as an application of this algorithm they provide the global phase portraits of the Hamiltonian linear type centers having cubic nonlinearities. However, while it differs from our work in terms of the tools used, there are also some differences in the results obtained. They find the same global phase portraits as us, except for the locations of the cusp points in their phase portraits 21 and 22 (which, in our case, correspond to the phase portraits 1.12 and 1.9 of Figure 1.1, respectively). According to our results, if we perturb linearly the phase portraits 21 and 22 in the same class we should obtain the phase portraits 17 and 20, respectively (which, in our case, correspond to the phase portraits 1.11 and 1.8 of Figure 1.1, respectively). However this clearly is not possible. We remark that for the Hamiltonian nilpotent centers of the form linear plus cubic homogeneous terms there are no previous results.

We now state our main results. We first provide normal forms and the global phase portraits in the Poincaré disk for all the Hamiltonian linear type center or nilpotent centers of linear plus cubic homogeneous planar polynomial vector fields. These results are summarized in Theorems 4 and 5, respectively.

Theorem 4. *Any Hamiltonian linear type planar polynomial vector field with linear plus cubic homogeneous terms has a linear type center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

(I) $\dot{x} = ax + by,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3$

(II) $\dot{x} = ax + by - x^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3x^2y,$

(III) $\dot{x} = ax + by - 3x^2y + y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2,$

(IV) $\dot{x} = ax + by - 3x^2y - y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2,$

(V) $\dot{x} = ax + by - 3\mu x^2y + y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,$

(VI) $\dot{x} = ax + by - 3\mu x^2y - y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,$

where $a, b, \beta, \mu \in \mathbb{R}$ *with* $b \neq 0$ *and* $\beta > 0$. *Moreover, the global phase portraits of these six families of systems are topologically equivalent to the following of Figure 1.1:*

(a) 1.1 *or* 1.2 *for systems* (*I*);

(b) 1.3 *for systems* (II)*;*

(c) 1.4*,* 1.5 *or* 1.6 *for systems* (III)*;*

(d) 1.1*,* 1.2*,* 1.7*,* 1.8 *or* 1.9 *for systems* (IV)*;*

(e) 1.3*,* 1.10*,* 1.11 *or* 1.12 *for systems* (V)*;*

(f) 1.13–1.23 *for systems* (VI)*.*

We will prove Theorem 4 in Chapter 2.

Theorem 5. *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

(VII) $\dot{x} = ax + by$, $\dot{y} = -\dfrac{a^2}{b}x - ay + x^3$, *with* $b < 0$,

(VIII) $\dot{x} = ax + by - x^3$, $\dot{y} = -\dfrac{a^2}{b}x - ay + 3x^2y$, *with* $a > 0$.

(IX) $\dot{x} = ax + by - 3x^2y + y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $ab \neq 0$, *and* $a^2/b - 6b > 0$,

(X) $\dot{x} = ax + by - 3x^2y - y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $a \neq 0$, *and* $b < 0$,

(XI) $\dot{x} = ax + by - 3\mu x^2y + y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 - b^4 - 6a^2b^2\mu)/b > 0$,

(XII) $\dot{x} = ax + by - 3\mu x^2y - y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 + b^4 + 6a^2b^2\mu)/b < 0$,

where $a, b, c, \mu \in \mathbb{R}$*. Moreover the global phase portraits of these six families of systems are topologically equivalent to the following of Figure 1.1:*

(a) 1.1 *for systems* (VII) *and* (X)*;*

(b) 1.3 *for systems* $(VIII)$*;*

(c) 1.4*,* 1.5 *or* 1.6 *for systems* (IX)*;*

(d) 1.3*,* 1.10*,* 1.11 *or* 1.12 *for systems* (XI)*;*

(e) 1.13*,* 1.14*,* 1.15 *or* 1.18 *for systems* (XII)*.*

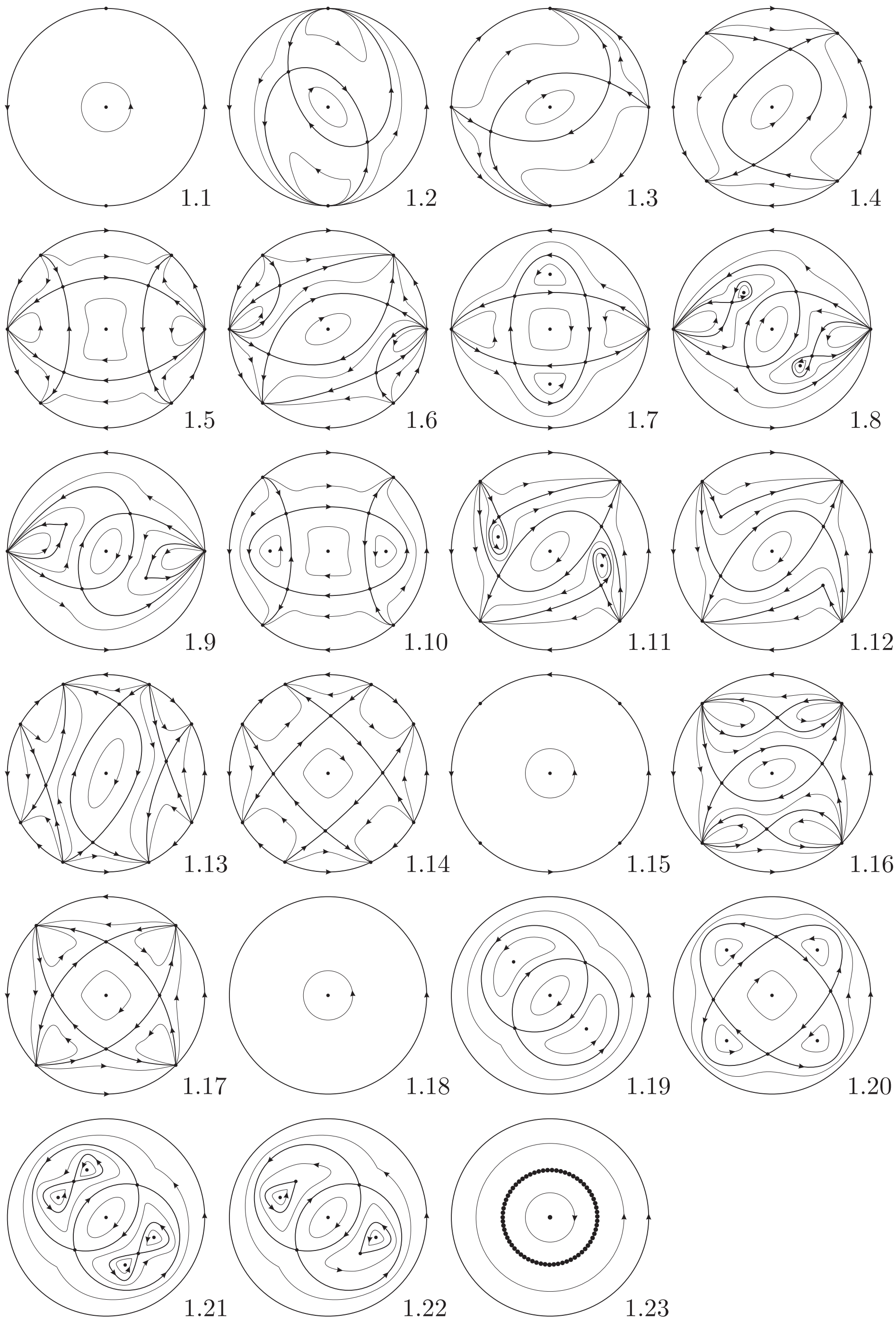

Figure 1.1: Global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type or nilpotent center at the origin. The separatrices are in bold.

We will prove Theorem 5 in Chapter 3.

Second we provide the bifurcation diagrams of the families of vector fields (I)–(XII) of Theorems 4 and 5. The bifurcation diagrams for the centers of Theorems 7 and 8 in the particular case when they are reversible were also given in [15].

We note that the parameters of these families can be further simplified, however, at this point such simplifications do not contribute to the proofs. On the other hand, to obtain better and simpler bifurcation diagrams we shall make use of those simplifications. Hence we make the following remark before stating our next results.

Remark 6. *Using the change of variables $(u, v) = (x/\sqrt{\beta}, y/\sqrt{\beta})$, the time rescale $d\tau = \beta dt$, and redefining parameters $\bar{a} = a/\beta$ and $\bar{b} = b/\beta$, we can assume $\beta = 1$ in the families of systems (I)–(VI). We also note that in the families (III)–(VI) the cases with $a < 0$ are obtained from those with $a > 0$ simply by making the change $(t, x) \mapsto (-t, -x)$. Therefore we will assume $a \geq 0$ for these systems.*

A system in class (XI) with $a = c = 0$ can be transformed to a system inside the same class with $a = b = 0$ and $c \neq 0$ doing the change $(x, y) \mapsto (y, x)$, $c \mapsto b$ and $\mu \mapsto -\mu$. Hence when $c = 0$ we can assume $a \neq 0$. Similarly we can assume $a \neq 0$ in systems (XII) whenever $c = 0$ (in this case the change of variables is $(x, y) \mapsto (-y, x)$).

When $a \neq 0$, via the rescaling of the variables $(x, y, t) \mapsto (x/\sqrt{|a|}, y/\sqrt{|a|}, |a|t)$ and the parameter $b \mapsto b/|a|$ we can assume $a = 1$ in the families of systems $(IX) - (XII)$.

Using Remark 6 we present our results on the bifurcation diagrams in the following two theorems.

Theorem 7. *The global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms having a linear type center at the origin are topologically equivalent to the following ones of Figure 1.1 using the notation of Theorem 4.*

(a) For systems (I) the phase portrait is

- 1.1 *when $b < 0$;*
- 1.2 *when $b > 0$.*

(b) For systems (II) the unique phase portrait is 1.3.

(c) For systems (III) the phase portrait is

- 1.4 *when $b < 0$;*
- 1.5 *when $b > 0$ and $a = 0$;*

1.6 *when* $b > 0$ *and* $a > 0$.

The corresponding bifurcation diagram is shown in Figure 1.2.

(d) *For systems* (IV) *the phase portrait is*

1.1 *when* $b < 0$;

1.2 *when* $b > 0$, $D = 0$ *and* $a = 0$, *or when* $b > 0$ *and* $D > 0$;

1.7 *when* $b > 0$, $D < 0$ *and* $a = 0$;

1.8 *when* $b > 0$, $D < 0$ *and* $a > 0$;

1.9 *when* $b > 0$, $D = 0$ *and* $a > 0$.

See (4.4) *for the definition of* D. *The corresponding bifurcation diagram is shown in Figure 1.3.*

(e) *For systems* (V) *we can assume* $b > 0$, *and the phase portrait is*

1.3 *when* $\mu \leq 0$, *or when* $\mu > 0$ *and* $D_4 < 0$, *or when* $\mu > 0$, $D_4 = 0$ *and* $a = 0$;

1.10 *when* $\mu > 0$, $D_4 > 0$ *and* $a = 0$;

1.11 *when* $\mu > 0$, $D_4 > 0$ *and* $a > 0$;

1.12 *when* $\mu > 0$, $D_4 = 0$ *and* $a > 0$.

See (4.12) *for the definition of* D_4. *The corresponding bifurcation diagram for the case* $\mu > 0$ *is shown in Figure 1.4.*

(f) *For systems* (VI) *we can assume* $b > 0$ *whenever* $\mu < -1/3$, *and the phase portrait is*

1.13 *when* $\mu < -1/3$ *and* $b \neq \sqrt{1+a^2}$;

1.14 *when* $\mu < -1/3$ *and* $b = \sqrt{1+a^2}$;

1.15 *when* $\mu = -1/3$ *and* $b < 0$;

1.16 *when* $\mu = -1/3$, $b > 0$ *and* $b \neq \sqrt{1+a^2}$;

1.17 *when* $\mu = -1/3$ *and* $b = \sqrt{1+a^2}$;

1.18 *when* $\mu > -1/3$ *and* $b < 0$;

1.19 *when* $\mu > -1/3$, $b > 0$, $D_4 < 0$, *or when* $\mu > -1/3$, $b > 0$, $D_4 = D_3 = 0$ *and either* $a \neq 0$ *or* $\mu \neq 1/3$ *or* $b \neq 1$;

1.20 *when* $1/3 - 2a/(3\sqrt{1+a^2}) > \mu > -1/3$, $D_4 > 0$ *and* $b = \sqrt{1+a^2}$, *or when* $\mu > 1/3$, $b > 0$, $D_4 > 0$ *and* $a = 0$;

1.21 *when* $\mu > -1/3$, $b > 0$, $D_4 > 0$ *and* $b \neq \sqrt{1+a^2}$, *or when* $\mu > 1/3 + 2a/(3\sqrt{1+a^2})$, $b = \sqrt{1+a^2}$, $D_4 > 0$ *and* $a \neq 0$;

1.22 *when* $\mu > -1/3$, $b > 0$, $D_4 = 0$ *and* $D_3 \neq 0$;

1.23 *when* $a = 0$*,* $\mu = 1/3$ *and* $b = 1$.

See (4.26) *and* (4.39) *for the definitions of* D_4 *and* D_3*, respectively. The corresponding bifurcation diagrams are shown in Figures 5–9.*

We will prove Theorem 7 in Chapter 4.

Theorem 8. *The global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms having a nilpotent center at the origin are topologically equivalent to the following ones of Figure 1.1 using the notation of Theorem 5.*

(a) For systems (VII) *and* (X) *the unique phase portrait is* 1.1.

(b) For systems $(VIII)$ *the unique phase portrait is* 1.3.

(c) For systems (IX) *the phase portrait is*

1.4 *when* $b < 0$*;*
1.5 *when* $b = 0$*;*
1.6 *when* $b > 0$.

(d) For systems (XI) *we can assume* $b \geq 0$*, and the phase portrait is*

1.3 *when* $b = 0$ *and* $\mu \leq 0$*, or when* $b > 0$ *and* $D < 0$*;*
1.10 *when* $b = 0$ *and* $\mu > 0$*;*
1.11 *when* $b > 0$ *and* $D > 0$*;*
1.12 *when* $b > 0$ *and* $D = 0$.

Here $D = -b^2 - 6b^2\mu + 4(1 - b^4)\mu^3 + 3b^2\mu^4$*, and the corresponding bifurcation diagrams are shown in Figure 1.10.*

(e) For systems (XII) *the phase portrait is*

1.13 *when* $\mu > -1/3$ *and* $b \neq 0, 1$*;*
1.14 *when* $\mu < -1/3$ *and* $b = 0, 1$*;*
1.15 *when* $\mu = -1/3$*;*
1.18 *when* $\mu > -1/3$.

The corresponding bifurcation diagrams are shown in Figure 1.11.

We will prove Theorem 8 in Chapter 5.

We remark that all the equations controlling the bifurcations of the global phase portraits described in Theorems 7 and 8 are algebraic curves. We must mention that essentially Chapters 2 and 3 are published in the journals J. Differential Equations and Advances in Mathematics, repectively (see

[9] and [10]). The Chapters 4 and 5 are submitted to publication, see [11] and [12].

As we mentioned at the beginning of this chapter, the normal forms, the phase portraits and the bifurcation diagrams provided in Theorems 4, 5, 7 and 8 will lead to new studies in the number of limit cycles that bifurcate from the periodic orbits of the families of differential systems (I)–(XII) when they are perturbed inside the class of all cubic polynomial differential systems. This last study was made for the quadratic polynomial differential systems, see the paper [7] and the references quoted therein.

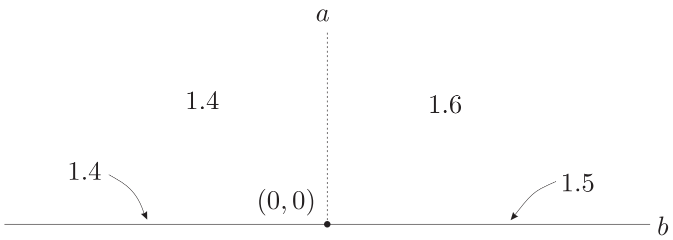

Figure 1.2: Bifurcation diagram for systems (III).

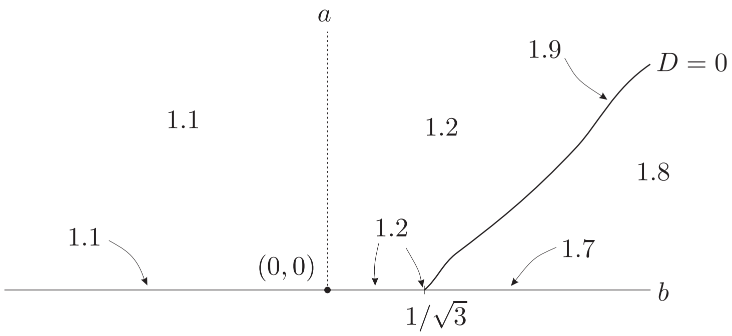

Figure 1.3: Bifurcation diagram for systems (IV).

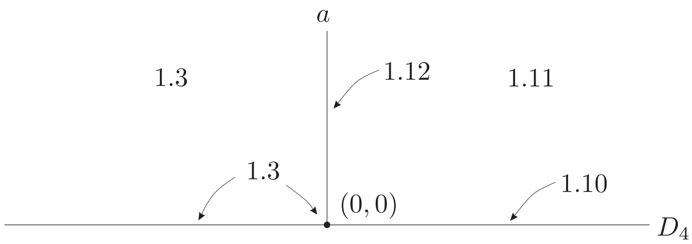

Figure 1.4: Bifurcation diagram for systems (V) with $\mu > 0$.

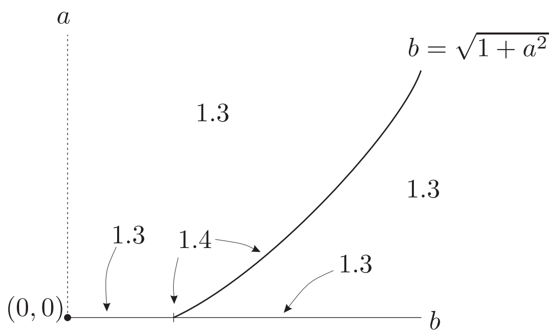

Figure 1.5: Bifurcation diagram for systems (VI) with $\mu < -1/3$ and $b > 0$.

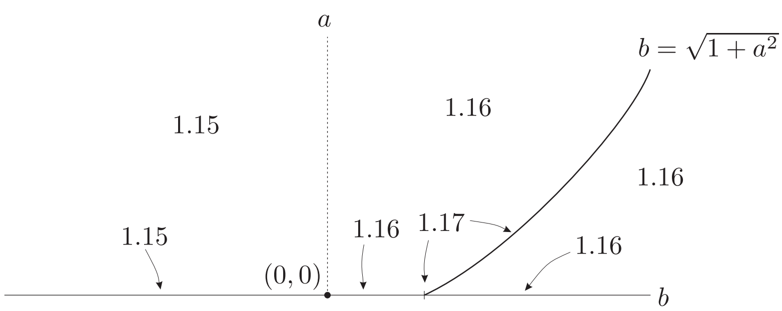

Figure 1.6: Bifurcation diagram for systems (VI) with $\mu = -1/3$.

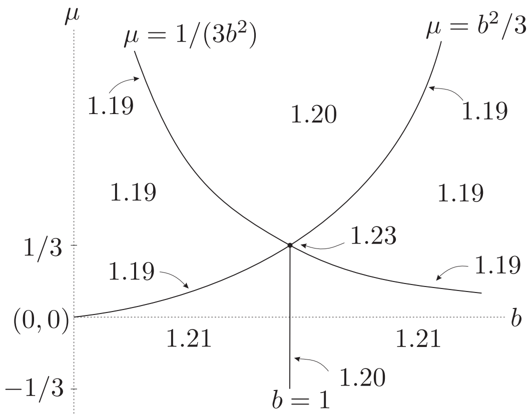

Figure 1.7: Bifurcation diagram for systems (VI) with $\mu > -1/3$ and $a = 0$.

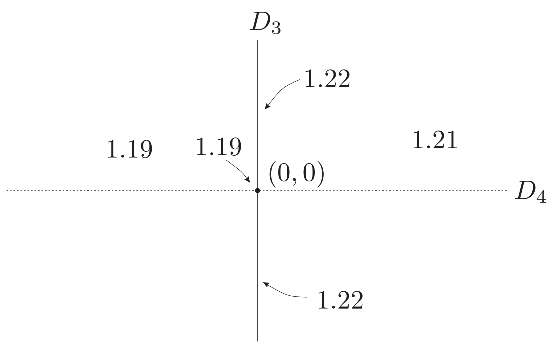

Figure 1.8: Bifurcation diagram for systems (VI) with $\mu > -1/3$, $a > 0$ and $b \neq \sqrt{1+a^2}$.

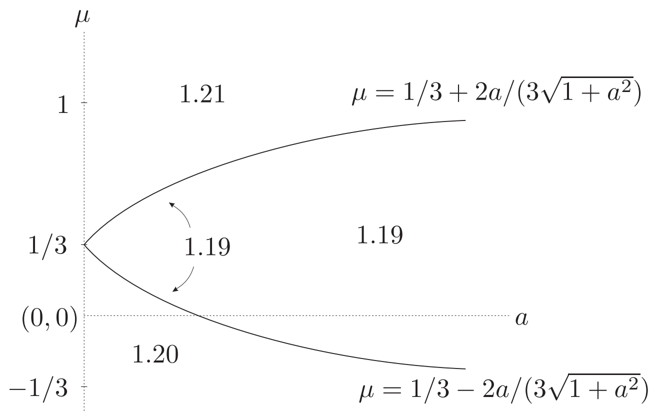

Figure 1.9: Bifurcation diagram for systems (VI) with $\mu > -1/3$, $a > 0$, $b = \sqrt{1+a^2}$ and $D_4 > 0$.

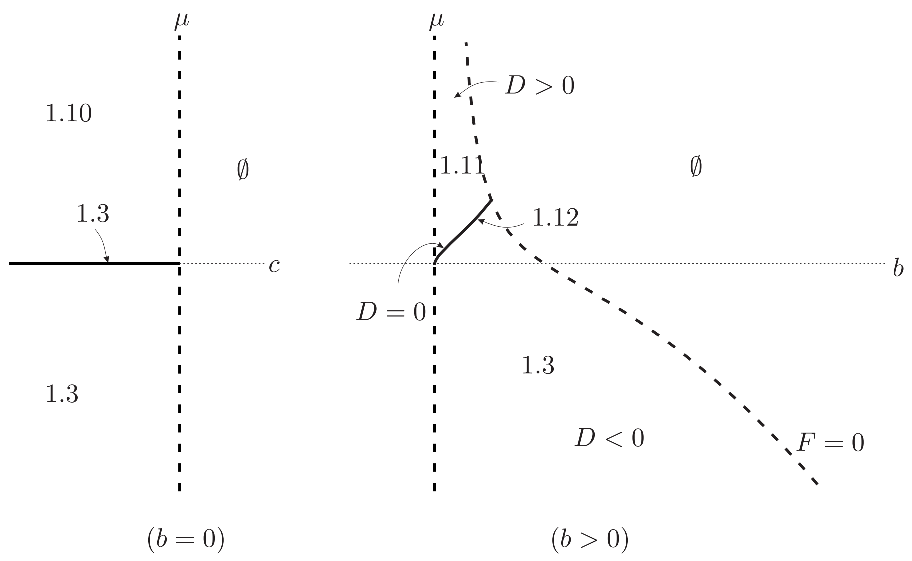

Figure 1.10: The bifurcation diagrams for systems (XI) when $b=0$ and when $b>0$. Note that when $b>0$ we have $c=0$. In the figure $F=1-b^4-6b^2\mu$.

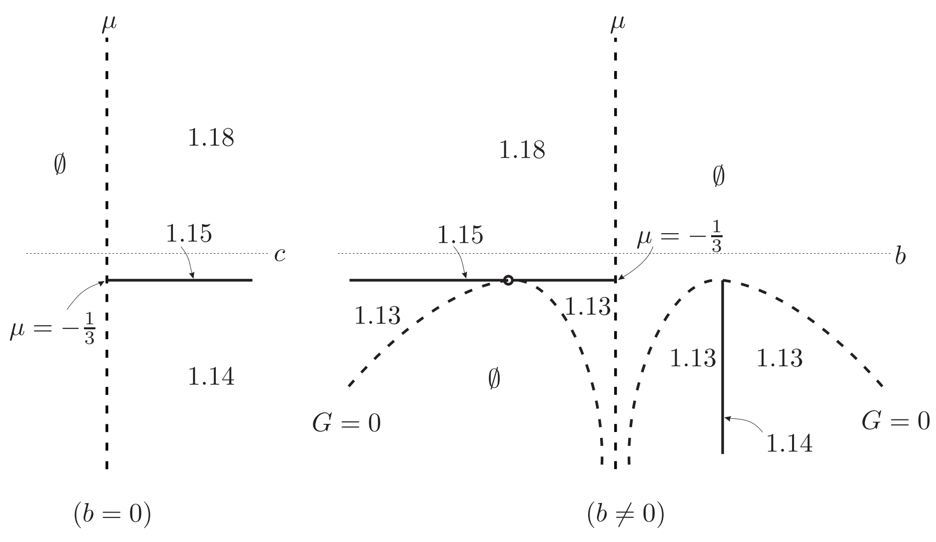

Figure 1.11: The bifurcation diagram for systems (XII) when $b=0$ and when $b\neq 0$. Note that when $b\neq 0$ we have $c=0$. In the figure $G=1+b^4+6b^2\mu$.