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CENTER-FOCUS AND ISOCHRONOUS CENTER PROBLEMS FOR DISCONTINUOUS DIFFERENTIAL EQUATIONS

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Abstract. The study of the center focus problem and the isochronicity problem for differential equations with a line of discontinuities is usually done by computing the whole return map as the composition of the two maps associated to the two smooth differential equations. This leads to large formulas which usually are treated with algebraic manipulators. In this paper we approach to this problem from a more theoretical point of view. The results that we obtain relate the order of degeneracy of the critical point of the discontinuous differential equations with the order of degeneracy of the two smooth component differential equations. Finally we apply them to some families of examples.

1. **Introduction and Main Results.** Planar systems defined by a vector field with a line of discontinuities appear frequently in the applications, see for instance the classical book [4], the web page [27] or Examples 4.4 and 4.5 in Section 4. These systems, indistinctly called discontinuous or non-smooth, are also studied just from a theoretical point of view in several papers, see for instance [20, 21, 22]. In this work we are interested in the center and the isochronicity problem for such systems. Assuming the generic case, these kind of systems can be written in $(z, \overline{z})$-complex coordinates, $z = x + iy = \mathrm{Re}(z) + i\mathrm{Im}(z) \in \mathbb{C}$, as

$$\dot{z} = \begin{cases} F_1(z, \overline{z}) & \text{if} \quad \mathrm{Im}(z) \geq 0, \\ F_2(z, \overline{z}) & \text{if} \quad \mathrm{Im}(z) \leq 0, \end{cases} \tag{1}$$

where $\dot{z} = \frac{dz}{dt}$, $t \in \mathbb{R}$ and $F_1(z, \overline{z})$, $F_2(z, \overline{z})$ are complex analytic functions in a neighbourhood of $z = 0$ whose series expansion starts with linear terms of the form $(a_1 + i)z$ and $(a_2 + i)z$ respectively, where a_1, $a_2 \in \mathbb{R}$. Observe that on the line $\mathrm{Im}(z) = 0$ there are two vector fields defined. Following [12] we have that for system (1) the flow near the origin is well defined and that there is no sliding motion.

We note that, in fact, equation (1) allow us to define two more differential equations in a neighbourhood of $z = 0$. They are given by the smooth vector fields

$$\dot{z} = F_j(z, \overline{z}), \tag{1.j}$$

for $j = 1, 2$.

Consider the (r, θ)-polar coordinates, $r^2 = z\overline{z}$ and $\theta = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)})$. In these coordinates, let $r(\theta; \rho) = \sum_{i \geq 1} u_i(\theta)\rho^i$ be the solution of equation (1) with initial condition $r(0; \rho) = \rho$. It is clear that for ρ small enough the (complete) return

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