ON A CRITERIUM OF GLOBAL ATTRACTION FOR ORDINARY DIFFERENTIAL EQUATIONS

B. COLL,[*] A. GASULL[†] AND R. PROHENS[‡]

[*] [‡] *Dept. de Matemàtiques i Informàtica*
Universitat de les Illes Balears
07071, Palma de Mallorca. Spain
E-mail: tomeu.coll@uib.es, rafel.prohens@uib.es

[†] *Dept. de Matemàtiques*
Edifici C, Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona. Spain
E-mail: gasull@mat.uab.es

LaSalle, in his book of 1976 (see Ref. 8), proposes to study several conditions which might imply global attraction of a fix point for a discrete dynamical system $x_{n+1} = F(x_n)$, defined in $\mathbb{R}^m$. Assuming that the fix point is the origin, one of his proposals appears after writing $F(x) = A(x)x$, and then imposing conditions on the eigenvalues of the matrices $A(x)$, for all $x \in \mathbb{R}^m$. Cima *et al.* (see Ref. 3) have given an adaptation to ordinary differential equations of these conditions. In that paper the authors study the effect of imposing both conditions, either in the case of discrete dynamical systems or for ordinary differential equations. They also observe that such a decomposition of $F(x)$ is in general not unique. In this note we consider the extension of LaSalle's Condition to ordinary differential equations, when the choice of $A(x)$ is somehow canonical. Concretely, the choice of $A(x)$ that we consider in this work is $A_c(x) = \int_0^1 DF(sx)\,ds$, where the integration of the matrix $DF(x)$ is made term by term. Unfortunately, our conclusion is that the condition obtained for ordinary differential equations, when the decomposition of $F(x)$ is given by $F(x) = A_c(x)x$, is just useful to give global attraction in dimension one.

1. Introduction and main results

In 1976, J. P. LaSalle raised and answered[8] several questions concerning the global asymptotic stability of equilibrium points of autonomous discrete dynamical systems. Concretely, he gives conditions which imply that the origin of a discrete dynamical system defined on $\mathbb{R}$ is a global attractor. He also proposes to study the natural extension of these conditions to $\mathbb{R}^m$. Once the fix point is placed at the origin, one of the LaSalle's ideas is to write the map F, which defines the dynamical system $x^{(n+1)} = F(x^{(n)})$, as $F(x) = A(x)x$ where $A(x)$ is a real $m \times m$ matrix and to impose a natural assumption on $A(x)$: *For all $x \in \mathbb{R}^m$, the eigenvalues of $A(x)$ have modulus smaller that 1.* Cima *et al.* propose[3] the following extension of LaSalle's Condition, to ordinary differential equations $\dot{x} = F(x)$ defined in $\mathbb{R}^m$ and having the origin as a critical point: *For all $x \in \mathbb{R}^m$, the eigenvalues of $A(x)$ have negative real part.* Observe that both conditions come from what happens when $F(x)$ is linear, i. e. $F(x) = Ax$.

Notice that in both cases, the matrix $A(x)$ satisfying $F(x) = A(x)x$, needs not to be unique and hence the fact that LaSalle's Condition implies global attraction might depend on the choice of $A(x)$. In fact, the authors prove[3] that for ordinary differential equations, the condition stated above does not imply global attraction for dimension $m \geq 2$. The counterexample given in that paper is

$$\begin{aligned} \dot{x}_1 &= (1 - x_1)x_2, \\ \dot{x}_2 &= -(1 - x_2)^2 x_1 + (x_2 - 2)x_2, \end{aligned} \tag{1}$$

in $\mathbb{R}^2$ and extended to $\mathbb{R}^m$ by adding the equations $\dot{x}_i = -x_i$ for $i = 3, \ldots, m$. Notice that we can choose $A(x)$ as

$$A(x) = \begin{pmatrix} -x_2 & 1 \\ -(1 - x_2)^2 & x_2 - 2 \end{pmatrix}.$$

This matrix has for all values of x the double eigenvalue -1 and on the other hand the straight line $x_1 = 1$ is an invariant line. Hence the origin is not a global attractor. It seems to us that LaSalle's Condition is too ambiguous in the description of the matrix A. We think that the decomposition of $A(x)$ should be more canonical. In fact, notice that for instance, also the following choices of A could be considered:

$$A_{f,g}(x) = \begin{pmatrix} -x_2 + f(x_1, x_2)x_2 & 1 - f(x_1, x_2)x_1 \\ -(1 - x_2)^2 + g(x_1, x_2)x_2 & x_2 - 2 - g(x_1, x_2)x_1 \end{pmatrix},$$

for any couple of smooth function f and g, and there is no special reason to choose $f = g = 0$.

For this reason in this note we propose to study the following problem:

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