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Local analytic first integrals of planar analytic differential systems

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ABSTRACT

We study the existence of local analytic first integrals of a class of analytic differential systems in the plane, obtained from the Chua's system studied in L.O. Chua (1992, 1995), N.V. Kuznetsov et al. (2011), G.A. Leonov et al. (2012) [6,7,11,13]. The method used can be applied to other analytic differential systems.

1. Introduction and statement of results

The nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system defined on the plane $\mathbb{R}^2$ the existence of a first integral determines completely its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a differential system in $\mathbb{R}^2$, how to recognize if this system has a first integral*?

The easiest planar differential systems having a first integral are the Hamiltonian ones. The integrable planar differential systems which are not Hamiltonian are, in general, very difficult to detect. Many different methods have been used for studying the existence of first integrals for non-Hamiltonian differential systems based on: Noether symmetries [4], the Darbouxian theory of integrability [8], the Lie symmetries [16], the Painlevé analysis [3], the use of Lax pairs [12], the direct method [9] and [10], the linear compatibility analysis method [18], the Carlemann embedding procedure [5] and [2], the quasimonomial formalism [3], the Ziglin's method [19], the Morales–Ramis theory [15], etc.

The main objective of this Letter is to show how to study the existence or non-existence of analytic first integrals of planar analytic differential systems, when the standard theorems providing sufficient conditions for the non-existence do not work.

We consider analytic differential systems

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y), \tag{1}$$

defined in an open subset U of $\mathbb{R}^2$. We say that a non-constant analytic function $H : U \to \mathbb{R}^2$ is an *analytic first integral* of system (1) in U if H is constant on the solution curves of system (1), or equivalently

$$f(x, y)H_x + g(x, y)H_y = 0,$$

in U. Of course, H_x denotes the derivative of H with respect to x.

There exist well-known results providing sufficient conditions for the non-existence of local analytic first integrals of system (1), as for instance the following two theorems.

Theorem 1. *(See Poincaré [17].) Assume that the eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ at some singular point p of the analytic differential system (1) do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0,$$

for any positive integers k_1 and k_2. Then system (1) has no analytic first integrals defined in a neighborhood of p.

Theorem 2. *(See Li et al. [14].) Assume that the eigenvalues λ_1 and λ_2 at some singular point p of the analytic differential system (1) satisfy that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then system (1) has no analytic first integrals in a neighborhood of p if p is isolated in the set of all singular points of system (1).*

The problem for studying the non-existence of local analytic first integrals of system (1) in a neighborhood of a singular point appears when the sufficient conditions of Theorems 1 and 2 cannot be applied. In this work we deal with such a case. More precisely, we will study the local analytic integrability of the analytic differential system

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