ON THE BIFURCATION OF LIMIT CYCLES DUE TO POLYNOMIAL PERTURBATIONS OF HAMILTONIAN CENTERS

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Abstract. We study the number of limit cycles bifurcating from the period annulus of a real planar polynomial Hamiltonian ordinary differential system with a center at the origin when it is perturbed in the class of polynomial vector fields of a given degree.

1. Introduction and statement of the main results

In the qualitative theory of real planar polynomial differential systems one of the main problems is the determination of limit cycles of a given vector field. The notion of limit cycle goes back to Poincaré, see [12]. He defined a limit cycle for a vector field in the plane as a periodic orbit of the differential system isolated in the set of all periodic orbits. The first works in determining the number of limit cycles of a given vector field can be traced back to Liénard [9] and Andronov [1]. After these works, the detection of the number of limit cycles of a polynomial differential system, intrinsically related with the so-called 16th Hilbert problem [7], has been extensively studied in the mathematical community, see for instance the books [3, 14] and the papers [5, 6, 10, 11].

One of the main tools of producing limit cycles is perturbing a system having a center. The notion of center goes back to Poincaré, see [12], who defined a center for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point. If a system has a center then when we perturb it we may have a limit cycle that bifurcates in the perturbed system from some of the periodic orbits forming a center. This tool is one of the most effective ways of producing limit cycles but it requires the knowledge of the first integral of the unperturbed system (the one having a center). It is well-known that the determination of first integrals is also a very hard problem. This is why in this paper we will focus on an unperturbed planar differential system from which we know a first integral of it.

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More precisely, in this paper we consider the planar polynomial Hamiltonian system

$$\dot{x} = -F_y, \quad \dot{y} = F_x, \tag{1}$$

with Hamiltonian $F = F(x, y)$ of the form

$$F(x, y) = \sum_{i=1}^{n} c_i H(x, y)^i,$$

where $H(x, y) = (x^2 + y^2)/2$ and the first $c_i \neq 0$ is positive. We assume $c_n \neq 0$ for convenience so that the degree of system (1) is $2n-1$. Note that H is a first integral of system (1), and since H has a local minimum at the origin of system (1) has a center at the origin.

In order to simplify the notation we will write system (1) as

$$\dot{x} = -y \cdot G(x, y), \quad \dot{y} = x \cdot G(x, y), \tag{2}$$

where

$$G(x, y) = \sum_{i=1}^{n} i c_i \left(\frac{x^2 + y^2}{2} \right)^{i-1} := \Gamma(H(x, y)). \tag{3}$$

We note that the circles $x^2 + y^2 =$ constant on which $G(x, y) \neq 0$ are periodic orbits, and that the ones on which $G(x, y) = 0$ are filled of singular points of the differential system (2), and consequently of the differential system (1). Let $\eta > 0$ be the smallest real number such that the circle $x^2 + y^2 = \eta^2$ is filled of singular points if it exists, otherwise $\eta = +\infty$. We also note that all the singular points of the differential system (2) except the origin are on circles $x^2 + y^2 =$ constant where $G(x, y) = 0$. So the period annulus of the center at the origin (i.e. the connected set formed by the union of all the periodic orbits surrounding the origin and having the origin in its inner boundary) is the annulus $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \eta^2\}$.

First we will study the number of limit cycles that appear when system (1) (or (2)) is perturbed in the class of all polynomial differential systems in the form

$$\dot{x} = -yG(x, y) + \varepsilon A(x, y), \quad \dot{y} = xG(x, y) + \varepsilon B(x, y), \tag{4}$$

where A and B are arbitrary real polynomials such that

$$m = \max\{\deg(A), \deg(B)\}.$$

Second, we will study the number of limit cycles that appear when system (1) (or (2)) is perturbed in the class of all polynomial differential systems in the form

$$\dot{x} = -yG(x,y) + \sum_{i=1}^{\infty} \varepsilon^i A_i(x,y),$$
$$\dot{y} = xG(x,y) + \sum_{i=1}^{\infty} \varepsilon^i B_i(x,y), \tag{5}$$

where A_i and B_i are arbitrary real polynomials such that

$$m = \max\{\deg(A_i), \deg(B_i)\} = m \text{ for all } i = 1, 2, \ldots.$$

Let $\eta_0 = \eta$ if $\eta < +\infty$, and let $\eta_0 < +\infty$ if $\eta = +\infty$. Then we can parameterize the set of periodic orbits surrounding the origin and intersecting the interval $(0, \eta_0)$ by h such that $h = x^2/2 = H(x, 0)$ with $x \in (0, \eta_0)$. We define the Poincaré map $P_\varepsilon(h)$ for system (4) on $(0, x_0)$, and the *displacement map* $\Delta(h, \varepsilon) = P_\varepsilon(h) - h$ which has a power series representation

$$\Delta(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots$$

that converges for sufficiently small ε. The functions $M_i(h)$, defined for $h \geq 0$, are called the *i–th Melnikov function*, and each positive simple zero of the first non–vanishing Melnikov function corresponds to a limit cycle of system (4).

In order to study the limit cycles that bifurcate from an unpertubed system when we perturb it, the vast majority of the papers study the simple zeros of $M_1(h)$, assuming that it is the first non-vanishing Melnikov function. There are much fewer papers studying the simple zeros of $M_2(h)$ assuming that it is the first non-vanishing Melnikov function, and there very few papers which study the simple zeros of $M_2(h)$ assuming that it is the first non-vanishing Melnikov function. In this paper we will study the simple zeroes of all the Melnikov functions M_k for an arbitrary k, assuming that it is the first non-vanishing Melnikov function.

As far as we know there are only two papers that provide a similar result working with Melnikov functions at any order and perturbing the linear center $\dot{x} = -y$, $\dot{y} = -x$. The first one goes back to Iliev [8] were he was the first one in doing so. Due to the fact that this is an extremely hard problem involving very difficult computations, he started with the linear center, i.e., system (2) with $G = 1$. Following the ideas of Iliev, in [2] the authors study the number of zeros of the Melnikov function at any order for system (2) in the case in which $G = (x^2 + y^2)^{m-1}$. Our system (2) generalizes the systems studied in [8] and [2] because the unperturbed part is taken to be more general and we extend their results to this more general situation.

Our first main result is the following.

Theorem 1. *The first non–vanishing Melnikov function $M_k(h)$ for system* (4) *has at most $[k(m-1)/2]+(k-1)(n-1)$ positive zeros counting their multiplicities.*

Here $[p]$ denotes the integer part of the real number p.

The second main result of the paper is the following.

Theorem 2. *The first non–vanishing Melnikov function $M_k(h)$ for system* (5) *has at most*

(*i*) $[(m-1)/2]+2(k-1)(n-1)$ *positive zeros counting their multiplicities if* $m \leq 2n-1$,

(*ii*) $[k(m-1)/2]+(k-1)(n-1)$ *positive zeros counting their multiplicities if* $m \geq 2n-1$.

When $G=1$ the bounds obtained in Theorem 2 coincide with the upper bounds obtained in [8]. When $n=m-1$ and $c_i=0$ for $i=1,\ldots,m-1$, system (2) coincide with the one studied in [2]. However, the upper bounds provided by Theorem 2 are larger than those obtained in [2] due to the fact that our G is full. Since a particular case of our system is the system studied in [2] where the authors obtain a better bound and prove that the bounds are not reached, and since the bounds in Theorem 1 are the same as those in statement (*ii*) of Theorem 2, we believe that the bounds in Theorem 2 are not going to be reached.

The paper is divided as follows. In section 2 we introduce three lemmas that will be used in the proofs of Theorem 1 and 2. The proof of Theorem 1 is given in section 3 and the proof of Theorem 2 is given in section 4.

2. Preliminary results

We first present a lemma, proved in [8], which will be a key factor in calculating the Melnikov functions.

Lemma 3. *Any polynomial one–form τ of degree s can be expressed as*

$$\tau = dQ(x,y) + q(x,y)\,dH + \alpha(H)y\,dx \tag{6}$$

where $Q(x,y)$, $q(x,y)$ and $\alpha(h)$ are polynomials of degree $s+1$, $s-1$ and $[(s-1)/2]$, respectively.

Corollary 4. *For a polynomial one–form τ of degree s we have $\int_{H=h}\tau = -\alpha(h)2\pi h$ where $\alpha(h)$ is a polynomial of degree $[(s-1)/2]$.*

Proof. The proof follows directly from Lemma 3. □

Corollary 5. *Let τ be a polynomial one–form of degree s. Then the one–form τ/G^l can be expressed as*

$$\frac{\tau}{G^l} = dS + r\,dH + \frac{\alpha(H)}{G^l}y\,dx,$$

where

$$S=\frac{Q}{G^l},\quad r=\frac{qG+lQJ}{G^{l+1}},\quad J=\sum_{i=2}^{n} i(i-1)H^{i-2},$$

such that $Q(x,y)$, $q(x,y)$ *and* $\alpha(h)$ *are polynomials of degree* $s+1$, $s-1$ *and* $[(s-1)/2]$, *respectively.*

Proof. Choose the polynomials Q, q and α as in Lemma 3. Then the proof follows by substitution. □

We rewrite system (4) as

$$dH-\varepsilon\frac{\omega}{G}=0,$$

where $\omega=A(x,y)dy-B(x,y)dx$ is a polynomial one–form of degree m. We will calculate the Melnikov functions for system (4) using the following well–known result due to Françoise [4] and Roussarie [13].

Lemma 6. *For system* (4) *we have*

$$M_1(h)=\int_{H=h}\Omega_1,$$

where $\Omega_1=\omega/G$. *In addition, if for some* $k\geq 2$ *we have* $M_1(h)=\ldots=M_{k-1}(h)\equiv 0$, *then*

$$M_k(h)=\int_{H=h}\Omega_k,$$

where $\Omega_k=r_{k-1}\omega/G$, *and* r_{k-1} *is determined successively by* $\Omega_i=dS_i+r_i\,dH$ *for* $i=1,\ldots,k-1$.

We rewrite system (5) as

$$dH-\varepsilon\frac{\omega_1}{G}-\varepsilon^2\frac{\omega_2}{G}-\ldots=0,$$

where $\omega_i=A_i(x,y)dy-B_i(x,y)dx$ is a polynomial one–form of degree m for each $i=1,2,\ldots$. We will calculate the Melnikov functions for system (4) using the following well–known result due to Françoise [4] and Roussarie [13].

Lemma 7. *For system* (5) *we have*

$$M_1(h)=\int_{H=h}\Omega_1,$$

where $\Omega_1=\omega/G$. *In addition, if for some* $k\geq 2$ *we have* $M_1(h)=\ldots=M_{k-1}(h)\equiv 0$, *then*

$$M_k(h)=\int_{H=h}\Omega_k,$$

where

$$\Omega_k=\frac{\omega_k}{G}+\sum_{i=1}^{k-1} r_{k-i}\frac{\omega_i}{G},$$

and r_{k-i} is determined successively by $\Omega_i = dS_i + r_i dH$ for $i = 1, \dots, k-1$.

3. Proof of Theorem 1

In order to proof Theorem 1 we first prove the following lemma where we look at the r_i in Lemma 6 in more detail.

Lemma 8. *Assume $M_1(h) = \dots = M_{k-1}(h) \equiv 0$ for some $k \geq 2$, and define $p_0 = 1$. For $i = 1, \dots, k-1$ let p_i be the polynomial such that the function r_i of Lemma 6 is $r_i = p_i/G^{2i}$, and let Q_i and q_i defined satisfying $p_{i-1}\omega = dQ_i + q_i dH$, see Lemma 3. Then p_i is a polynomial of degree $i(2n-3)+im$ given by $p_i = q_i G + (2i-1)Q_i J$.*

Proof. We know by Lemma 6 that

$$M_1(h) = \int_{H=h} \Omega_1 = \int_{H=h} \frac{\omega}{G} = \frac{1}{\Gamma(h)} \int_{H=h} \omega,$$

and by Lemma 3 that

$$\omega = dQ_1 + q_1\, dH + \alpha(H) y\, dx,$$

where Q_1, q_1 and α are polynomials of degree $m+1$, $m-1$ and $[(m-1)/2]$, respectively. Using this information and induction on k we shall prove that p_i is a polynomial of degree $i(2n-3)+im$ given by $p_i = q_i G + (2i-1)Q_i J$ for $i = 1, \dots, k-1$.

Let $k = 2$. Then we have $M_1(h) \equiv 0$, which means $\omega = dQ_1 + q_1 dH$ due to Corollary 4. By Lemma 6 we have $\Omega_2 = r_1\omega/G$ where $\Omega_1 = dS_1 + r_1\, dH$. Then by Corollary 5 we obtain

$$r_1 = \frac{q_1 G + Q_1 J}{G^2} = \frac{p_1}{G^2},$$

and consequently $\deg(p_1) = 2n - 3 + m$. Hence the statement is true for $k = 2$.

Now assume that the statement holds for some arbitrary $k > 2$. Then we have $M_1(h) = \dots M_{k-1}(h) \equiv 0$ and

$$r_i = \frac{p_i}{G^{2i}} = \frac{q_i G + (2i-1)Q_i J}{G^{2i}}, \tag{7}$$

for $i = 1, \dots, k-1$, where $\deg(p_i) = i(2n-3)+im$. To show that the statement holds for $k+1$, we further assume that $M_k(h) = \int_{H=h} \Omega_k \equiv 0$. Then by Lemma 6 we get

$$\Omega_k = r_{k-1}\frac{\omega}{G} = dS_k + r_k\, dH,$$

for some functions S_k and r_k. We know from (7) that

$$\Omega_k = \frac{p_{k-1}\omega}{G^{2k-1}},$$

where $p_{k-1}\omega$ is a polynomial one–form of degree $(k-1)(2n-3)+km$, which can be expressed as $dQ_k + q_k\, dH$, where $\deg(Q_k) = (k-1)(2n-3)+km+1$ and $\deg(q_k) = (k-1)(2n-3)+km-1$, by Lemma 3 because $M_k(h) \equiv 0$. Hence by Lemma 6 and Corollary 5 we obtain

$$\Omega_k = d\left(\frac{Q_k}{G^{2k-1}}\right) + \frac{q_k G + (2k-1)Q_k J}{G^{2k}} dH.$$

Therefore we get

$$r_k = \frac{p_k}{G^{2k}} = \frac{q_k G + (2k-1)Q_k J}{G^{2k}}$$

where $\deg(p_k) = k(2n-3)+km$. □

Proof of Theorem 1. As a result of Lemma 8, we have that if for some $k \geq 2$ $M_1(h) = \ldots = M_{k-1}(h) \equiv 0$ then

$$M_k(h) = \frac{1}{(\Gamma(h))^{2k-1}} \int_{H=h} \big(q_{k-1}G + (2k-3)Q_{k-1}J\big)\omega$$

has at most $[k(m-1)/2] + (k-1)(n-1)$ positive zeros taking into account their multiplicities due to Corollary 4, concluding the proof of Theorem 1. □

4. Proof of Theorem 2

In order to proof Theorem 2 we first prove the following lemma where we look at the r_i in Lemma 7 in more detail.

Lemma 9. *Assume that $M_1(h) = \ldots = M_{k-1}(h) \equiv 0$ for some $k \geq 2$. Then Ω_k as in Lemma 7 can be written as $\Omega_k = \tau_k/G^{2k-1}$ where τ_k is a polynomial of degree*

$$\deg(\tau_k) = \begin{cases} m + 4(k-1)(n-1), & \text{if } m \leq 2n-1, \\ km + (k-1)(2n-3), & \text{if } m \geq 2n-1. \end{cases}$$

Proof. First we will prove by induction on k that for $k \geq 2$ we have that the r_i for $i = 1, \ldots, k-1$ given in Lemma 6 satisfies

$$r_i = \frac{q_i G + (2i-1)Q_i J}{G^{2i}} := \frac{p_i}{G^{2i}}, \tag{8}$$

where q_i and Q_i are polynomials such that

$$\deg(q_i) = \begin{cases} m + 4(i-1)(n-1) - 1, & \text{if } m \leq 2n-1, \\ im + (i-1)(2n-3) - 1, & \text{if } m \geq 2n-1, \end{cases} \tag{9}$$

and $\deg(Q_i) = \deg(q_i) + 2$.

Let $k = 2$. Then we have

$$M_1(h) = \int_{H=h} \Omega_1 = \frac{1}{\Gamma(h)} \int_{H=h} \omega_1 \equiv 0,$$

which means

$$\omega_1 = dQ_1 + q_1\, dH$$

where Q_1 and q_1 are polynomials of degree $m+1$ and $m-1$, respectively, due to Lemma 3 and Corollary 4. Then by Corollary 5 we get

$$\Omega_1 = \frac{\omega_1}{G} = d\left(\frac{Q_1}{G}\right) + \frac{q_1 G + Q_1 J}{G^2}\, dH,$$

and thus

$$r_1 = \frac{q_1 G + Q_1 J}{G^2},$$

which prove the induction hypothesis for $k = 2$.

Now assume that the lemma holds for some arbitrary $k > 2$. Then by(8) we have

$$\begin{aligned} \Omega_k &= \frac{\omega_k}{G} + \sum_{i=1}^{k-1} r_{k-i}\frac{\omega_i}{G} = \frac{\omega_k}{G} + \sum_{i=1}^{k-1} \frac{p_{k-i}\omega_i}{G^{2k-2i+1}} \\ &= \frac{\omega_k G^{2k-2} + \sum_{i=1}^{k-1} p_{k-i}\omega_i G^{2i-2}}{G^{2k-1}} := \frac{\tau_k}{G^{2k-1}}. \end{aligned} \tag{10}$$

We see that $\deg(\omega_k G^{2k-2}) = m + 4(k-1)(n-1)$ and by the induction hypothesis (see (8) and (9)) we obtain

$$\deg(p_{k-i}\omega_i G^{2i-2}) = \begin{cases} 2m + 4(k-1)(n-1) - (2n-1), & \text{if } m \le 2n-1, \\[2ex] km + (k-1)(2n-3) & \\ \quad + (1-i)\big(m - (2n-1)\big), & \text{if } m \ge 2n-1. \end{cases}$$

In particular the degree of τ_k is

$$\deg(\tau_k) = \begin{cases} m + 4(k-1)(n-1), & \text{if } m \le 2n-1, \\ km + (k-1)(2n-3), & \text{if } m \ge 2n-1. \end{cases} \tag{11}$$

and the lemma is proved if we end the induction process.

To end the induction process, i.e. that (8) and (9) hold for $k+1$, we further assume that $M_k(h) \equiv 0$. Then we have $\tau_k = dQ_k + q_k\, dH$. This together with (11) completes the induction process. □

Proof of Theorem 2. As a result of Lemmas 7 and 9, we have that if $M_1(h) = \ldots = M_{k-1}(h) \equiv 0$ for some $k \ge 2$, then

$$M_k(h) = \frac{1}{(\Gamma(h))^{2k-1}} \int_{H=h} \tau_k.$$

Now the proof of Theorem 2 follows from Lemma 9 and Corollary 4. □

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