Periodic and quasi–periodic motions for the spatial isosceles 3–body problem

Montserrat Corbera Subirana

Memòria presentada per a aspirar al grau de doctor en Ciències Matemàtiques.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
Bellaterra, juny de 1999.

Certifico que la present memòria ha estat realitzada per la Montserrat Corbera Subirana, i dirigida per mi, al Departament de Matemàtiques de la Universitat Autònoma de Barcelona.

Bellaterra, juny de 1999

Dr. Jaume Llibre i Saló

Als meus pares.

Agraïments

Vull expressar el més sincer agraïment al Dr. Jaume Llibre per la gran dedicació i interès amb què ha dirigit aquest treball, i per descomptat, per la formació i el tracte que he rebut durant tot aquest temps.

Un agraïment a la Universitat de Vic i al Departament de Matemàtiques de la Universitat Autònoma de Barcelona per posar a la meva disposició els recursos i la infraestructura necessaris per a la realització d'aquest treball.

Vull donar les gràcies als companys de l'Escola Politècnica Superior de la Universitat de Vic i, de manera especial, als del Departament de Matemàtica i Física aplicades, per la valuosa ajuda que m'han donat i per la seva comprensió durant tota la realització d'aquest treball. He d'esmentar també el servei d'informàtica per la puntualitat amb què han resolt els problemes tècnics que han anat apareixent. Vull agrair també als companys del Departament de Matemàtiques de la Universitat Autònoma de Barcelona l'acollida que m'han donat durant aquest temps.

Agraeixo a la família i als amics el suport que m'han donat en tot moment, malgrat que els he dedicat poc temps.

Finalment vull donar les gràcies d'una manera molt especial als meus pares per tot el que han fet per mi. Sense la seva ajuda aquest treball no hauria estat possible.

Contents

Introduction **4**

1 The $n-$body problem **12**
1.1 Basic definitions and notation 12
1.2 The n–body problem 14
1.2.1 The ten classical integrals 15
1.2.2 Lagrangian and Hamiltonian formulation 17
1.2.3 Symmetries and reduction of the $n-$body problem 17
1.3 The $2-$body problem 19
1.3.1 Solutions of the Kepler problem 21
1.3.2 Elliptic orbits of the Kepler problem 24
1.3.3 Barycentric orbits 26
1.4 Restricted $3-$body problems 27
1.5 Sitnikov problem 29
1.5.1 Circular Sitnikov problem 31
1.5.2 Poincaré map 35

2 Isosceles problem **53**
2.1 Coordinates and equations of motion 53
2.2 Reduced isosceles problem 56
2.3 Symmetries 58
2.4 Restricted isosceles problems 62
2.4.1 Reduced circular Sitnikov problem 63
2.4.2 Reduced elliptic Sitnikov problem 65
2.5 Periodic orbits of the reduced isosceles problem 81
2.5.1 Lyapunov periodic orbits 82

3 Continuation method. Variational equations **85**
3.1 Analytic continuation method 85
3.1.1 Analytic systems of differential equations 85

3.1.2 The continuation method 87

3.2 Variational equations of the reduced restricted isosceles problems 102

3.2.1 Variational equations of $\ddot{x} = f(x)$ 107

3.2.2 Variational equations of the Kepler problem 109

3.2.3 Variational equations of the circular Sitnikov problem . . . 114

3.2.4 Variational equations of the elliptic Sitnikov problem for small values of the eccentricity 122

4 Continuation of symmetric periodic solutions 125

4.1 From reduced circular Sitnikov to reduced isosceles 127

4.1.1 Continuation to double–symmetric periodic solutions . . . 127

4.1.2 Continuation to $r-$symmetric periodic solutions 137

4.1.3 Continuation to $t-$symmetric periodic solutions 139

4.1.4 Periodic orbits 141

4.2 From reduced circular Sitnikov to reduced elliptic Sitnikov 142

4.2.1 Continuation to double–symmetric periodic solutions . . . 143

4.2.2 Continuation to $r-$symmetric periodic solutions 146

4.2.3 Continuation to $t-$symmetric periodic solutions 148

4.2.4 Periodic orbits 151

4.3 From reduced elliptic Sitnikov to reduced isosceles 158

4.3.1 Continuation to double–symmetric periodic solutions . . . 159

4.3.2 Continuation to $r-$symmetric periodic solutions 162

4.3.3 Continuation to $t-$symmetric periodic solutions 172

4.3.4 Periodic orbits 180

4.4 Continuation in two steps 182

4.5 Summary 190

5 Continuation of periodic solutions 193

5.1 From reduced circular Sitnikov to reduced isosceles 194

5.1.1 Continuation from the differential system 195

5.1.2 Continuation from the Poincaré map 199

5.2 From reduced elliptic Sitnikov to reduced isosceles 201

5.2.1 Continuation from the differential system 201

5.2.2 Continuation from the Poincaré map 209

6 Invariant tori for the isosceles problem 214

6.1 Characterization of the relative sets 214

6.2 Relative sets of the restricted isosceles problems 219

6.2.1 Relative sets of the circular restricted isosceles problem . . 219

6.2.2 Relative sets of the elliptic restricted isosceles problem . . 221

6.2.3 On the dynamics of the restricted isosceles problem 225

6.3 Relative sets of the isosceles problem 230

Appendices **235**
Appendix 1. Properties of the Jacobian elliptic functions 235
Elliptic Integrals 235
Jacobian elliptic functions 238
Derivatives 240
Appendix 2. Simplification of (3.2.24) 242
Appendix 3. Simplification of (3.2.36) 243
Appendix 4. Relation between the real time t and the new time ν 245
Appendix 5. Numerical computation of F_1 and $\overline{F}_2$ 250
Numerical computation of F_1 250
Numerical computation of $\overline{F}_2$ 252
Appendix 6. Example of continuation of periodic solutions without using symmetric conditions 255
Verification of condition (A.6.1) 255
Verification of condition (A.6.2) 264
Appendix 7. Stability of some of the known periodic points of the Poincaré map ϕ_e 270
Appendix 8. Continuation using transversal arguments 272
Transversality 272
Continuation of symmetric periodic solutions 273
From reduced circular Sitnikov problem to reduced elliptic Sitnikov problem 279

Bibliography **283**

Index **288**

Introduction

The main objective of classical Celestial Mechanics is the study of the n–body problem, which consists of describing the motion of n point masses moving in the Euclidean 3−dimensional space under the action of their mutual newtonian gravitational forces. The formulation of the $n-$body problem appears at first time in the *Philosophiae Naturalis Principia Mathematica* of Newton (1687). It is in this treatise where the laws of mechanics and the universal gravitational attraction law allowed to formulate the $n-$body problem as a system of differential equations.

Up to the *Mèthodes Nouvelles de la Mécanique Céleste* of Poincaré (1899) the differential equations that appear in Celestial Mechanics problems were treated from a quantitative point of view. Poincaré left the classical methods of integration and quadrature aside and he initiated qualitative methods in order to give a complete description of the orbits on the phase space (the space where the differential equation is defined). We can say that Poincaré started the modern qualitative theory of differential equations.

The 2−body problem is integrable in the classical sense. Using the first integrals of the energy h and the angular momentum c, we can classify all possible orbits of the 2−body problem in the following way. If $c \neq 0$, then the motion is confined on a plane and we have: circular or elliptic orbits when $h < 0$; parabolic orbits when $h = 0$; and hyperbolic orbits when $h > 0$. If $c = 0$, then the motion is confined on a straight line and we have: elliptic collision orbits when $h < 0$, parabolic collision orbits when $h = 0$ and hyperbolic collision orbits when $h > 0$.

The n–body problem for $n > 2$ has resisted all attempts to be solved; indeed it is believed that the problem cannot be integrated in the classical sense, in fact there are partial results in this direction. Over the years many special types of solutions have been found by using distinct mathematical techniques, but really not many things can be said about the behaviour of the solutions. The 3–body problem is the most studied of the n–body problems; and, in particular, special cases of the 3−body problem, the restricted 3−body problems (i.e. when one of the masses is small enough so that its influence on the other two is negligible).

The study of the restricted $3-$body problems is a first step in order to understand the dynamics of the full $3-$body problem.

The restricted $3-$body problems consist of describing the motion of an infinitesimal mass that moves under the influence of the gravitational attraction of two bodies, called primaries, which describe a solution of the $2-$body problem. We can classify the restricted $3-$body problems depending on the kind of motion of the primaries and the dimension of the space where takes place the motion of the infinitesimal mass. In this way we have thirty different restricted $3-$body problems, the most studied of which is undoubtedly the planar circular restricted $3-$body problem followed by the planar elliptic restricted $3-$body problem. These two restricted $3-$body problems have a lot of interest in Celestial Mechanics because they have many applications to different kind of motions in the solar system, binary stars, etc.

Two different restricted $3-$body problems that have got a lot of interest from a mathematical point of view are the circular and elliptic Sitnikov problems, which are special cases of the spatial circular (respectively elliptic) restricted $3-$body problem. The Sitnikov problems are characterized by two equally massive bodies moving on circular (or elliptic) orbits and an infinitesimal mass, the motion of which is confined to the axis perpendicular to the plane of motion of the primaries that passes through their center of mass. In the historical development the circular Sitnikov problem was studied first. MacMillan (1913) presented it like an example of an integrable case of restricted $3-$body problem. But the importance of the Sitnikov problems arise when Sitnikov in 1960 used the elliptic Sitnikov problem to show, for the first time, the possibility of the existence of oscillatory motions in the $3-$body problem. The existence of this kind of motions was predicted by Chazy in 1922–32, who gave a classification of the final evolutions of the $3-$body problem. Later on Alekseev (1968-69) used the elliptic Sitnikov problem to prove that all possible combinations of final evolutions in the sense of Chazy were realized. Moser (1973) gave a new presentation of Alekseev's results relying on a geometric point of view. Since then many other authors have studied the elliptic Sitnikov problem. We know around thirty papers studying the circular or elliptic Sitnikov problems. It is possible to find some analytical and numerical results on periodic orbits of the circular and elliptic Sitnikov problems, see for additional information Section 1.4.

In this work we study an especial case of the spatial $3-$body problem, the *spatial isosceles* $3-$*body problem*. This problem consists of describing the motion of two equally massive bodies $m_1 = m_2$ having initial conditions and velocities symmetric with respect to a straight line which pass through their center of mass, and a third body, with mass $m_3 = \mu$, having initial position and velocity on this straight line. That problem is called isosceles problem because the three bodies form an isosceles triangle at any time, eventually degenerated to a segment.

The most interesting application of the spatial isosceles 3−body problem is given by Xia in [62]. Xia used two spatial isosceles 3−body problems to prove that five bodies can escape to infinity in a finite time without collision. Other works on the spatial isosceles 3−body problem can be [45], see the references inside. If in the spatial isosceles 3−body problem the initial positions and velocities of the three bodies are contained in a plane, then the motion remains always in this plane, and we have the so called *planar isosceles 3−body problem.* There are a lot of papers about the planar isosceles 3−body problem, for instance, [46], [22],...

When the third body of the spatial isosceles 3−body problem has infinitesimal mass (i.e. $\mu = 0$) then we obtain the *restricted isosceles 3−body problems.* Depending on the motion of the primaries we have seven different cases for the spatial restricted isosceles 3−body problems. Here we only consider, due to its richness in periodic orbits, the cases in which the primaries move in circular or elliptic orbits of the 2−body problem, the circular and elliptic restricted isosceles problems; also called the circular and elliptic Sitnikov problems.

The isosceles problem and the restricted isosceles problems possess the first integral of the angular momentum. Using this first integral we reduce in two dimensions (an angle an its derivative) the phase space of these problems obtaining the *reduced isosceles problem* and the *reduced restricted isosceles problems* respectively. We note that the circular and elliptic Sitnikov problems that appear in the literature are essentially our reduced circular and elliptic Sitnikov problems.

In appropriate coordinates we will see that the periodic orbits of the reduced circular Sitnikov problem give 2–dimensional invariant tori on the phase space of the restricted isosceles 3−body problem. These tori are formed by union of either periodic or quasi–periodic orbits, and they are not KAM tori. The main result of this work is to prove that such invariant tori persist when we pass from the restricted isosceles 3−body problem to the isosceles 3−body problem for $\mu > 0$ sufficiently small. Consequently these tori persist inside the general spatial 3−body problem. The main tool for proving this result will be the classical Poincaré's analytic continuation method of periodic orbits.

This memoir is divided in six chapters and eight appendices. Chapters, sections, and subsections are numbered in arabic numbers. Thus, 3.1.4 refers to subsection 4 in Section 1 of Chapter 3. Appendices are also numbered in arabic numbers and they appear at the end of the memoir. The numeration of theorems, propositions, lemmas, corollaries, etc., will be reset in each section and they are numbered in the order that appear inside the section. So a typical theorem reference might be Theorem 2.4.2 meaning the second theorem in Section 4 of Chapter 2. Formulas, figures, and tables are numbered like theorems, but independent on theorems, propositions, ... Hence Figure 2.4.3 is the third figure in Section 4 of Chapter 2. We will use the symbol ■ to denote the end of the proof of a Theorem, Proposition, ... and the symbol □ to denote the end of a remark. The numeration in appendices is quite different. A typical reference of an appendix is for instance Lemma A.4.1, meaning the first lemma of Appendix 4. Now we give a summary of the contents and the main results of each one of these chapters.

In Chapter 1 we give the basic definitions and results about the $n-$body problem, the $2-$body problem, and the restricted $3-$body problems that we need in this work. We put more emphasis summarizing the known results about the circular and elliptic Sitnikov problem that are needed for the development of this memoir.

In Chapter 2 we define the isosceles $3-$body problem and the restricted isosceles $3-$body problems ($\mu = 0$). We prove that the phase portrait of these problems on each level of angular momentum c with $c \neq 0$ is the same. Notice that the angular momentum $c = 0$ contains the triple collision orbits, and collision orbits are not treated in this work. Fixed a value of the angular momentum $c \neq 0$, we reduce in two units the dimension of the phase space of the isosceles problem introducing the reduced isosceles problem and the reduced restricted isosceles problems.

The purpose of this work is to find periodic orbits of the reduced isosceles problem for $\mu > 0$. Results of Alekseev [2] for the reduced isosceles problem when $\mu > 0$ is small enough show the existence of infinitely many periodic orbits, with larger periods, that are close to a heteroclinic loop formed by two parabolic orbits of the elliptic Sitnikov problem. Here we are interested in periodic orbits that are not necessarily close to the parabolic ones.

We analyze the symmetries of the reduced isosceles problem: the $r-$symmetry

$$(t, r, \dot{r}, z, \dot{z}) \longmapsto (-t, r, -\dot{r}, -z, \dot{z}) \ ,$$

and the $t-$symmetry

$$(t, r, \dot{r}, z, \dot{z}) \longmapsto (-t, r, -\dot{r}, z, -\dot{z}) \ .$$

These symmetries will be used in Chapter 4 in order to find *$r-$symmetric* and *$t-$symmetric periodic orbits* of the isosceles problem for $\mu > 0$ small. We still distinguish another type of symmetric periodic orbits, the *double–symmetric periodic orbits*, which are simultaneously $r-$symmetric and $t-$symmetric periodic orbits, see Chapter 2 for precise definitions. We also present some results about symmetric periodic orbits of the reduced circular and elliptic Sitnikov problems. In particular, we prove the following theorem.

Theorem A *The following statements hold.*

(a) All periodic orbits of the reduced circular Sitnikov problem are double–symmetric.

(b) For all $e \in (0,1)$ except perhaps for a discrete set of values of e, there exist four different types of periodic orbits of the reduced elliptic Sitnikov problem: non–symmetric periodic orbits, double–symmetric periodic orbits, and $r-$symmetric and $t-$symmetric periodic orbits that are not double–symmetric.

Finally we give some results about the existence of periodic orbits of the reduced isosceles problem for $\mu > 0$ (not necessarily small), near the Euler equilibrium point, by using the Lyapunov Center Theorem.

In Chapter 3 we develop the Poincaré's analytic continuation method. The idea of this method is to use a known periodic solution and, by small changes of the parameters and of the initial conditions, continue it. In this chapter we also compute analytically the solution of the variational equations of the Kepler problem and of the circular Sitnikov problem along a given periodic orbit. The knowledge of an analytic expression for the solution of those variational equations is a key point in the development of this work.

In Chapter 4 we apply the Poincaré's analytic continuation method to continue the periodic orbits (double–symmetric) of the reduced circular Sitnikov problem to symmetric periodic orbits of the reduced isosceles problem for small values of $\mu > 0$. We continue those periodic orbits in two different ways. The first one goes directly from the reduced circular Sitnikov problem to the reduced isosceles problem. The second one uses two steps, first we continue the periodic orbits from the reduced circular Sitnikov problem to symmetric periodic orbits of the reduced elliptic Sitnikov problem for small values of the eccentricity e, and after we continue those symmetric periodic orbits of the reduced elliptic Sitnikov problem to the reduced isosceles problem for small values of $\mu > 0$. Both ways do not give the same information.

The main result of this chapter is the following theorem.

Theorem B *Let γ be a periodic orbit of the reduced circular Sitnikov problem with period $T > \pi/\sqrt{2}$. Then γ can be continued to the following families of periodic orbits of the reduced isosceles problem with angular momentum $c = 1/4$ and $\mu > 0$ sufficiently small.*

(a) Case $T = 2\pi\omega$ with $\omega > 1/(2\sqrt{2})$ an irrational number.

(i) γ can be continued to one 2–parameter family (that depends on μ and τ) of double–symmetric periodic orbits with period τ sufficiently close to T.

(b) Case $T = 2\pi p/q$ for some $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$.

(i) p odd:

(1) γ can be continued to one 2–parameter family (that depends on μ and τ) of double–symmetric periodic orbits with period τ sufficiently close to T.

(2) γ can be continued to two 2–parameter families (that depend on μ and e) of $r-$symmetric periodic orbits with period $2\pi p\sqrt{(1-e^2)^3} = qT\sqrt{(1-e^2)^3}$ where $e > 0$ is sufficiently small.

(3) γ can be continued to two 2–parameter families (that depend on μ and e) of $t-$symmetric periodic orbits with period $2\pi p\sqrt{(1-e^2)^3} = qT\sqrt{(1-e^2)^3}$ where $e > 0$ is sufficiently small.

(ii) p even and $q = 1$:

(1) γ can be continued to two 2–parameter families (that depend on μ and e) of double–symmetric periodic orbits of period $2\pi p\sqrt{(1-e^2)^3} = qT\sqrt{(1-e^2)^3}$ where $e > 0$ is sufficiently small.

(iii) p even and $q \neq 1$:

(1) γ can be continued to one 2–parameter family (that depends on μ and τ) of double–symmetric periodic orbits with period τ sufficiently close to T.

(2) γ can be continued to two 2–parameter families (that depend on μ and e) of double–symmetric periodic orbits of period $2\pi p\sqrt{(1-e^2)^3} = qT\sqrt{(1-e^2)^3}$ where $e > 0$ is sufficiently small.

In this chapter we also give sufficient conditions to continue an arbitrary symmetric periodic orbit of the reduced elliptic Sitnikov problem to the reduced isosceles problem for small values of $\mu > 0$.

An important by–product result in this chapter is the continuation of some periodic orbits of the reduced circular Sitnikov problem to symmetric periodic orbits of the reduced elliptic Sitnikov problem, which is summarized as follows.

Theorem C *The periodic orbits of the reduced circular Sitnikov problem with period $T = 2\pi p/q$, for given $p, q \in \mathbb{N}$ coprime $p > q/(2\sqrt{2})$, can be continued to:*

(a) two families of $r-$symmetric periodic orbits and two families of $t-$symmetric periodic orbits (that are not double–symmetric) of the reduced elliptic Sitnikov problem, with period $2\pi p = qT$, for $e > 0$ sufficiently small, when p is odd;

(b) two families of double–symmetric periodic orbits of the reduced elliptic Sitnikov problem, with period $2\pi p = qT$, for $e > 0$ sufficiently small, when p is even.

In Appendix 8 we present different proofs of the basic results of Chapter 4 by using transversality arguments instead of the Implicit Function Theorem arguments based in the Poincaré's analytic continuation method. We note that somehow the transversality arguments can be thought like a more general version of the Implicit Function Theorem for arbitrary manifolds.

In Chapter 5 we give sufficient conditions to continue an arbitrary periodic orbit of the reduced elliptic Sitnikov problem (symmetric or not) to the reduced isosceles problem for $\mu > 0$ small enough. The only periodic orbits of the reduced elliptic Sitnikov problem that we know analytically are the ones given by Theorem B. Then, using the results of Chapter 4, these periodic orbits can be continued to symmetric periodic orbits of the reduced isosceles problem for $\mu > 0$ sufficiently small. If we continue one of these symmetric periodic orbits without using conditions of symmetry, then we will obtain again the symmetric periodic orbits given in Chapter 4. Nevertheless in Appendix 6, using numerical computations, we show that some of these periodic orbits can also be continued without using conditions of symmetry, and we see the difficulties of this kind of continuation in the reduced isosceles problem.

Finally, in Chapter 6, we see that, adding the angular variable, each periodic orbit of the reduced isosceles problem gives a $2-$dimensional invariant torus on the phase space of the isosceles problem that can be filled with either periodic or quasi–periodic orbits. Then we summarize the results about periodic orbits of the reduced isosceles problem obtained from Chapters 1 to 4, translated in the language of tori for the isosceles problem. The main result of this chapter can be summarized as follows.

Theorem D *Let γ be a periodic orbit of the reduced circular Sitnikov problem with period $T > \pi/\sqrt{2}$. Then γ gives a $2-$dimensional invariant torus Π_T of the circular restricted isosceles problem. This torus can be continued to the following families of $2-$dimensional tori of the isosceles problem with $\mu > 0$ sufficiently small. These tori are filled of either periodic or quasi–periodic orbits.*

(a) Case $T = 2\pi\omega$ with $\omega > 1/(2\sqrt{2})$ an irrational number.

(i) Π_T can be continued to one 2–parameter family (that depends on μ and τ with τ sufficiently near T) of $2-$dimensional tori.

(b) Case $T = 2\pi p/q$ for some $p, q \in \mathbb{N}$ coprime with $p > q/(2\sqrt{2})$.

(i) p odd:

(1) Π_T can be continued to one 2–parameter family (that depends on μ and τ with τ sufficiently near T) of $2-$dimensional tori.

(2) Π_T can be continued to four 2–parameter families (that depend on μ and e with $e > 0$ sufficiently small) of $2-$dimensional tori.

(ii) p *even and* $q = 1$*:*

- *(1)* Π_T *can be continued to two 2–parameter families (that depend on* μ *and* e *with* $e > 0$ *sufficiently small) of* $2-$*dimensional tori.*

(iii) p *even and* $q \neq 1$*:*

- *(1)* Π_T *can be continued to one 2–parameter family (that depends on* μ *and* τ *with* τ *sufficiently near* T*) of* $2-$*dimensional tori.*
- *(2)* Π_T *can be continued to two 2–parameter families (that depend on* μ *and* e *with* $e > 0$ *sufficiently small) of* $2-$*dimensional tori.*