

Four-body co-circular central configurations

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Abstract

We classify the set of central configurations lying on a common circle in the Newtonian four-body problem. Using mutual distances as coordinates, we show that the set of four-body co-circular central configurations with positive masses is a two-dimensional surface, a graph over two of the exterior side-lengths. Two symmetric families, the kite and isosceles trapezoid, are investigated extensively. We also prove that a co-circular central configuration requires a specific ordering of the masses and find explicit bounds on the mutual distances. In contrast to the general four-body case, we show that if any two masses of a four-body co-circular central configuration are equal, then the configuration has a line of symmetry.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The study of central configurations (c.c.s) in the Newtonian n -body problem has become a vibrant sub-field of celestial mechanics. While much is known about specific cases, usually involving symmetry or assuming one or more bodies is infinitesimally small, less is known about the general structure of the set of c.c.s. For example, finiteness of c.c.s (up to symmetry) was only recently established by Hampton and Moeckel [12] in the four-body problem, and by Albouy and Kaloshin [2] (except for masses in a codimension two subvariety) in the five-body problem.

In this work, we strive to deepen our understanding of the general set of c.c.s by fully classifying the four-body *co-circular central configurations* (c.c.c.s), that is, those four-body c.c.s which also lie on a common circle (see figure 1). According to Hampton [10], this

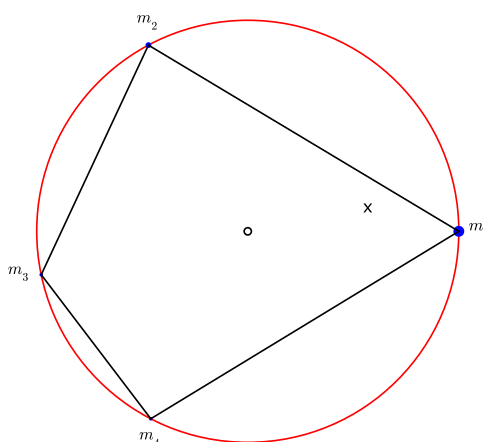


Figure 1. An example of a co-circular c.c. The centre of the circumscribing circle is marked with a $\circ$ and the centre of mass is denoted by $\times$.

problem was first posed by Alain Albouy. Requiring the bodies to lie on a common circle effectively reduces the problem of finding four-body c.c.s by one dimension, allowing for a more substantial investigation. Although such a restriction may seem somewhat arbitrary, it is hoped that this work will prove useful in analysing the general structure of four-body convex c.c.s, that is, those c.c.s forming a convex quadrilateral.

Co-circular c.c.s are significant since they individually form the base of any spatial, pyramidal c.c. [1, 9]. Such a three-dimensional configuration contains five bodies, with four co-planar bodies lying on a common circle and the fifth body (top of the pyramid) lying on the normal line through the centre of the common circle. Albouy showed that the four co-planar bodies must be a c.c. of the planar four-body problem [1]. Therefore, our work also serves to classify the set of pyramidal c.c.s.

Our approach uses only the mutual distances as coordinates. The fact that Ptolemy's theorem is so nicely expressed in these coordinates is critical to our results. Our main result is proving that the set of positions that yield a c.c.c. with positive masses is a two-dimensional surface, the graph of a differentiable function f over two of the exterior side-lengths. The boundaries of this surface correspond to three important symmetric cases: a kite, an isosceles trapezoid and a degenerate case where three bodies lie at the vertices of an equilateral triangle and the fourth body of the quadrilateral has zero mass. This last case corresponds to c.c.s of the planar, circular, restricted four-body problem [13]. We also prove that for any c.c.c., the masses must be ordered in a precise fashion, with the largest body lying on the vertex between the two longest exterior sides, the smallest body opposite the largest, and the two largest (smallest, respectively) bodies lying on the longest (smallest, respectively) exterior side (see section 4.2). In contrast to the famous result of MacMillan and Bartky [15] for convex 4-body c.c.s, we demonstrate that not all choices of positive masses lead to a co-circular c.c.

Moreover, we prove the somewhat surprising result that if just two masses of a four-body c.c.c. are equal, then the configuration is symmetric, either a kite or an isosceles trapezoid. These symmetric cases are explored in great detail in section 3, in part to aid in the proof of our main result demonstrating the two-dimensional surface of c.c.c.s. In section 5.1, we prove sharp bounds for each exterior side-length, each diagonal, the ratio of the diagonals and the ratio of the second- and third-largest exterior side-lengths. We finish with some geometric results relating the positions of the bodies to the circumscribing circle.

Any planar c.c., given the correct initial velocities, leads to a relative equilibrium solution in which the bodies rigidly rotate about their centre of mass. If the c.c. is also co-circular, the circumcircle containing the bodies will also rotate about the centre of mass. In [10], Hampton shows that only in the case of four equal masses positioned at the vertices of square, does the centre of mass coincide with the centre of the circumcircle, answering a question posed by Alain Chenciner. Some recent work in [18] proves the existence of a family of co-circular kite c.c.s using Cartesian coordinates. A similar result also appears in [23], where Dziobek's equations are utilized, and the family is shown to exist for homogeneous force laws that include the Newtonian n -body problem as well as the n -vortex problem.

2. Co-circular central configurations

Suppose we have n bodies with mass and position given by m_i and $\mathbf{q}_i \in \mathbb{R}^2$, respectively, for $i = 1, \dots, n$. Denote $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ as the distance between the i th and j th bodies and let $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n) \in \mathbb{R}^{2n}$. The centre of mass of the system is $\mathbf{c} = M^{-1} \sum_{i=1}^n m_i \mathbf{q}_i$ where $M = m_1 + \dots + m_n$ is the total mass. The equations of motion can be described by the Newtonian potential function

$$U(\mathbf{q}) = \sum_{i < j}^n \frac{m_i m_j}{r_{ij}}$$

as simply

$$m_i \ddot{\mathbf{q}}_i = \frac{\partial U}{\partial \mathbf{q}_i}, \quad i \in \{1, 2, \dots, n\}.$$

Denote $I(\mathbf{q})$ as the *moment of inertia*,

$$I(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^n m_i \|\mathbf{q}_i - \mathbf{c}\|^2 = \frac{1}{2} \sum_{i=1}^n m_i \|\mathbf{q}_i\|^2 - \frac{1}{2} M \|\mathbf{c}\|^2$$

a characterization of the size of the system as measured from the centre of mass.

A planar c.c. of the n -body problem is a configuration $\mathbf{x} \in \mathbb{R}^{2n}$ which satisfies the algebraic equations

$$\nabla U(\mathbf{x}) + \lambda \nabla I(\mathbf{x}) = 0 \quad (1)$$

for some value of λ (see [19–21, 26] or [29] for more details). Viewing λ as a Lagrange multiplier, a c.c. is simply a critical point of U subject to the constraint I equals a constant.

The i th component in (1) is given by

$$\sum_{j \neq i} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^3} + \lambda m_i (\mathbf{x}_i - \mathbf{c}) = 0.$$

From this equation, it follows that if $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ is a planar c.c., then

$$k\mathbf{x} = (k\mathbf{x}_1, k\mathbf{x}_2, \dots, k\mathbf{x}_n)$$

and

$$R\mathbf{x} = (R\mathbf{x}_1, R\mathbf{x}_2, \dots, R\mathbf{x}_n)$$

are also planar c.c.s for any constant $k \neq 0$ and any $R \in SO(2)$. In the first case, the value of the multiplier λ changes to $\lambda/|k|^3$ whereas the second case requires no change in λ . When counting c.c.s it is standard to fix the size (a unique choice of k) and identify any configurations which are rotationally equivalent via the equivalence relation $\mathbf{x} \sim R\mathbf{x}$ for $R \in SO(2)$. Any planar c.c. leads to a two-parameter family of homographic solutions where each body travels along the same (scaled) solution of the Kepler problem.

2.1. Mutual distances

In this section we derive the equations for a c.c.c. using the mutual distances r_{ij} as coordinates. We make use of *Ptolemy's theorem* to give a derivation for c.c.c.s that mirrors the approach used by Schmidt [24].

Denote $\mathbf{r} = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \in \mathbb{R}^{+6}$ as a vector of mutual distances. It is a straightforward calculation to check that

$$I = \frac{1}{2M} \sum_{i < j} m_i m_j r_{ij}^2.$$

For the case $n = 4$, the mutual distances are not independent; they generically describe a tetrahedron in $\mathbb{R}^3$ rather than a configuration in the plane. Locating planar four-body c.c.s requires an additional constraint found by setting the volume of the tetrahedron to zero. This constraint arises from the Cayley–Menger determinant

$$V(\mathbf{r}) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix}.$$

Assuming the r_{ij} correspond to an actual tetrahedron, then the square of the volume of this tetrahedron is $V/288$ [24]. However, it is important to note that V may be positive for a given choice of 6 mutual distances, but that does not imply a configuration exists which realizes those particular distances. For instance, $\mathbf{r} = (2, 4, 1, 7, 4, 1)$ gives $V(\mathbf{r}) = 3118$ but there is no physical configuration with these distances since $r_{12} = 2$, $r_{13} = 4$, $r_{23} = 7$ does not satisfy the triangle inequality. Conversely, it is possible for all the triangle inequalities to be satisfied, yet have $V(\mathbf{r}) < 0$. One such example is the one-parameter family $\mathbf{r}_t = (1, t, 1, 1, 2/t, 1)$ which gives $V(\mathbf{r}_t) = -8t^{-2}(t^2 - 2)^2$ which is strictly negative for all $t > 0$ except at the unit square $t = \sqrt{2}$.

To ensure a given vector $\mathbf{r}$ corresponds to an actual configuration of four bodies, either in the plane (with no three bodies collinear) or in space, it is necessary and sufficient that all strict triangle inequalities be satisfied and $V(\mathbf{r}) \geq 0$. Sufficiency follows from theorem 9.7.3.4 and its proof in [6]. Specifically, let

$$\mathcal{G} = \{\mathbf{r} \in \mathbb{R}^{+6} : V(\mathbf{r}) \geq 0 \text{ and } r_{ij} + r_{jk} > r_{ik} \text{ for all triples of indices } (i, j, k) \text{ where } i \neq j \neq k\}.$$

We say that $\mathbf{r} \in \mathbb{R}^{+6}$ is *geometrically realizable* if $\mathbf{r} \in \mathcal{G}$. Any geometrically realizable vector of mutual distances corresponds to a four-body planar, non-collinear configuration if $V(\mathbf{r}) = 0$ and a tetrahedron if $V(\mathbf{r}) > 0$.

We say that the bodies are *ordered sequentially* if they are numbered consecutively while traversing the boundary of the quadrilateral (either clockwise or counter-clockwise). For the rest of this work, we will assume that any cyclic quadrilateral is ordered sequentially so that r_{12}, r_{23}, r_{34} and r_{14} are the lengths of the exterior sides of the quadrilateral and r_{13} and r_{24} are the lengths of the diagonals. Denote

$$P(\mathbf{r}) = r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24}.$$

If a quadrilateral is ordered sequentially and is cyclic, then Ptolemy's theorem states that $P = 0$. Moreover, for any convex quadrilateral ordered sequentially or for any tetrahedron, we have that $P \geq 0$, with equality if and only if the four bodies lie on a circle [5].

Let $\mathcal{P} \subset \mathcal{G}$ denote the set of all geometrically realizable r satisfying $P(r) = 0$. The c.c.s. we seek to classify lie inside $\mathcal{P}$. We note that the codimension one level surfaces $\{V = 0\}$ and $\{P = 0\}$ meet tangentially at any point in $\mathcal{P}$. To prove this, we will show that ∇V and ∇P are parallel at any point in $\mathcal{P}$.

Let Δ_i be the oriented area of the triangle whose vertices contain all bodies except for the i th body. For a quadrilateral ordered sequentially, we have $\Delta_1, \Delta_3 > 0$ and $\Delta_2, \Delta_4 < 0$. To find ∇V restricted to $\mathcal{P}$, we use an important formula concerning the Cayley–Menger determinant,

$$\frac{\partial V}{\partial r_{ij}^2} = -32 \Delta_i \Delta_j. \quad (2)$$

This formula is only valid when restricting to planar configurations. The minus sign in equation (2) is not included in Dziobek's original paper [8] nor in several later works on four-body c.c.s that utilize the Cayley–Menger determinant. However, checking equation (2) on the square configuration indicates the need for the minus sign. The correct formula appears in the doctoral thesis of Hampton [11] and has also been confirmed by Schmidt [25].

Let $\delta = r_{12}r_{13}r_{14}r_{23}r_{24}r_{34}$. Then, for any $r \in \mathcal{P}$, we have

$$\begin{aligned} \frac{\partial V}{\partial r_{ij}}(r) &= \frac{\partial V}{\partial r_{ij}^2} \cdot \frac{d(r_{ij}^2)}{dr_{ij}} \\ &= -64r_{ij} \Delta_i \Delta_j \\ &= -64r_{ij} \cdot \frac{\pm 1}{16r_c^2} \cdot r_{jk}r_{kl}r_{jl} \cdot r_{ik}r_{il}r_{kl} \\ &= \frac{\pm 4\delta}{r_c^2} r_{kl} \end{aligned}$$

where r_c is the circumradius of the cyclic quadrilateral and k and l are the two distinct indices in $\{1, 2, 3, 4\}$ different from both i and j . Here we have used the formula $|\Delta_i| = r_{jk}r_{kl}r_{jl}/(4r_c)$ for the area of a triangle inscribed in a circle of radius r_c . Using the same ordering for indices as we used for the mutual distances, the signs of the partials of V are $(+, -, +, +, -, +)$, which are identical to the signs of $\nabla P = (r_{34}, -r_{24}, r_{23}, r_{14}, -r_{13}, r_{12})$. We have proven the following important lemma.

Lemma 2.1. *For any $r \in \mathcal{P}$,*

$$\nabla V(r) = \left(\frac{4}{r_c^2} \prod_{i < j} r_{ij} \right) \nabla P(r).$$

In other words, on the set of geometrically realizable vectors for which both V and P vanish, the gradients of these two functions are parallel.

We note that the above lemma does not hold in general and relies heavily on the fact that we are restricting to the family of cyclic quadrilaterals. Also, if the sign in equation (2) were not present, then $\nabla V(r)$ and $\nabla P(r)$ would point in opposite directions, violating the fact that any tetrahedron must have $P > 0$. Lemma 2.1 shows that ∇V is superfluous when using the Lagrange multiplier method to locate c.c.s. However, the Cayley–Menger determinant is still required to ensure the configuration is geometrically realizable. For example, the family $r_t = (1, t, 1, 1, 2/t, 1)$ mentioned earlier satisfies $V(r_t) < 0$ (except for the unit square at $t = \sqrt{2}$), but also yields $P(r_t) = 0$. Such a family needs to be excluded from the set of solutions.

Corollary 2.2. *Assuming a sequential ordering, a four-body c.c.c. $\mathbf{r}$ is a critical point of the function*

$$U + \lambda M(I - I_0) + \sigma P \quad (3)$$

satisfying $I = I_0$, $P = 0$ and $V = 0$.

The choice of λM as the first Lagrange multiplier in corollary 2.2 simplifies the resulting equations. Using the six mutual distances as variables, we find

$$m_1 m_2 (r_{12}^{-3} - \lambda) = \sigma \frac{r_{34}}{r_{12}}, \quad m_3 m_4 (r_{34}^{-3} - \lambda) = \sigma \frac{r_{12}}{r_{34}}, \quad (4)$$

$$m_1 m_3 (r_{13}^{-3} - \lambda) = -\sigma \frac{r_{24}}{r_{13}}, \quad m_2 m_4 (r_{24}^{-3} - \lambda) = -\sigma \frac{r_{13}}{r_{24}}, \quad (5)$$

$$m_1 m_4 (r_{14}^{-3} - \lambda) = \sigma \frac{r_{23}}{r_{14}}, \quad m_2 m_3 (r_{23}^{-3} - \lambda) = \sigma \frac{r_{14}}{r_{23}}. \quad (6)$$

Identical equations are obtained using V instead of P in the function (3) and simplifying the resulting area formulae by assuming the bodies lie on a common circle. Following Schmidt's derivation of the equations for a c.c. of the planar four-body problem [24], we have grouped the above equations in pairs so that the product of the right-hand sides is simply σ^2 . This yields the well-known relation of Dziobek [8]

$$(r_{12}^{-3} - \lambda)(r_{34}^{-3} - \lambda) = (r_{13}^{-3} - \lambda)(r_{24}^{-3} - \lambda) = (r_{14}^{-3} - \lambda)(r_{23}^{-3} - \lambda) \quad (7)$$

which is required of any planar 4-body c.c. (not just co-circular ones). Solving each of the three pairs of equations in (7) for λ yields

$$\lambda = \frac{r_{12}^{-3} r_{34}^{-3} - r_{13}^{-3} r_{24}^{-3}}{r_{12}^{-3} + r_{34}^{-3} - r_{13}^{-3} - r_{24}^{-3}} = \frac{r_{13}^{-3} r_{24}^{-3} - r_{14}^{-3} r_{23}^{-3}}{r_{13}^{-3} + r_{24}^{-3} - r_{14}^{-3} - r_{23}^{-3}} = \frac{r_{14}^{-3} r_{23}^{-3} - r_{12}^{-3} r_{34}^{-3}}{r_{14}^{-3} + r_{23}^{-3} - r_{12}^{-3} - r_{34}^{-3}}. \quad (8)$$

If we set

$$\begin{aligned} s_1 &= r_{12}^{-3} + r_{34}^{-3}, & p_1 &= r_{12}^{-3} r_{34}^{-3}, \\ s_2 &= r_{13}^{-3} + r_{24}^{-3}, & p_2 &= r_{13}^{-3} r_{24}^{-3}, \\ s_3 &= r_{14}^{-3} + r_{23}^{-3}, & p_3 &= r_{14}^{-3} r_{23}^{-3}, \end{aligned}$$

then equation (8) can be written as

$$\lambda = \frac{p_1 - p_2}{s_1 - s_2} = \frac{p_2 - p_3}{s_2 - s_3} = \frac{p_3 - p_1}{s_3 - s_1}$$

which means that (s_1, p_1) , (s_2, p_2) , (s_3, p_3) , viewed as points in $\mathbb{R}^{+2}$, must all lie on the same line with slope λ . This in turn, is equivalent to

$$\begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = 0,$$

a representation that leads to the particularly nice factorization

$$(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) = (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3). \quad (9)$$

Equation (9) is necessary and sufficient for a four-body planar c.c. given that the six mutual distances determine a geometrically realizable planar configuration. However, it does *not* ensure positivity of the masses. This equation can also be derived from the mass ratios given by Schmidt (equations (9)–(14)) in [24]. A similar and equivalent equation also appears on page 278 in the classic text of Wintner [29]. Note that equation (9) shows that if two sides with a common vertex are equal, then to be a c.c., there must be another pair of congruent sides sharing a common vertex.

2.2. Mass ratios

The ratios of the masses are easily obtained by dividing appropriate sets of equations from (4), (5) and (6). For example, we have

$$\frac{m_2}{m_1} = \frac{(\lambda - r_{13}^{-3}) r_{13} r_{14}}{(r_{23}^{-3} - \lambda) r_{23} r_{24}} = \frac{(r_{14}^{-3} - \lambda) r_{13} r_{14}}{(\lambda - r_{24}^{-3}) r_{23} r_{24}}, \quad (10)$$

$$\frac{m_3}{m_1} = \frac{(r_{12}^{-3} - \lambda) r_{12} r_{14}}{(r_{23}^{-3} - \lambda) r_{23} r_{34}} = \frac{(r_{14}^{-3} - \lambda) r_{12} r_{14}}{(r_{34}^{-3} - \lambda) r_{23} r_{34}}, \quad (11)$$

$$\frac{m_4}{m_1} = \frac{(r_{12}^{-3} - \lambda) r_{12} r_{13}}{(\lambda - r_{24}^{-3}) r_{24} r_{34}} = \frac{(\lambda - r_{13}^{-3}) r_{12} r_{13}}{(r_{34}^{-3} - \lambda) r_{24} r_{34}}. \quad (12)$$

Requiring positive masses places additional constraints on the mutual distances. First, note that if $\lambda = r_{ij}^{-3}$ for any pair of indices i, j , then equations (4), (5) and (6) and nonzero masses imply that $\sigma = 0$ and all r_{ij} are equal (the regular tetrahedron solution). If $r_{13}^{-3} - \lambda < 0$, then positivity of the masses implies $\sigma > 0$ by (5). This in turn gives

$$r_{12}, r_{14}, r_{23}, r_{34} < \lambda^{-1/3} < r_{13}, r_{24}. \quad (13)$$

If we had assumed that $r_{13}^{-3} - \lambda > 0$, then the inequalities in (13) would be reversed. However, this is impossible, for then $P > r_{13} r_{24} > 0$ and the configuration is not co-circular. Therefore, (13) is the only possibility and each side of the quadrilateral is shorter in length than either diagonal.

It can also be shown that the shortest exterior side must lie opposite the longest. This follows from

$$(r_{12}^{-3} - \lambda)(r_{34}^{-3} - \lambda) = (r_{14}^{-3} - \lambda)(r_{23}^{-3} - \lambda) \quad (14)$$

and inequalities (13). For example, suppose that r_{12} is the largest exterior side-length. Then, since each factor in equation (14) is positive, we have $r_{12} \geq r_{14}$ implies $r_{23} \geq r_{34}$ and $r_{12} \geq r_{23}$ implies $r_{14} \geq r_{34}$. Together, these two implications show that r_{34} is the shortest exterior side-length. The argument works exactly the same regardless of which exterior side is taken to be the longest. Without loss of generality, we label the bodies so that r_{12} is the longest exterior side-length. Then, positivity of the masses implies

$$r_{13}, r_{24} > r_{12} \geq r_{14}, r_{23} \geq r_{34}.$$

We can also assume that $r_{14} \geq r_{23}$ by an appropriate relabelling. More specifically, equations (4), (5) and (6) are invariant under the transformation $r_{13} \leftrightarrow r_{24}, r_{14} \leftrightarrow r_{23}, m_1 \leftrightarrow m_2, m_3 \leftrightarrow m_4$, which corresponds to interchanging bodies one and two, and bodies three and four. This interchange reverses the direction of the sequential ordering around the circle (clockwise to counter-clockwise or vice versa). The choice $r_{14} \geq r_{23}$ imposes the relation $r_{13} \geq r_{24}$ between the two diagonals. To see this, note that for a cyclic quadrilateral ordered sequentially, the lengths of each diagonal r_{13} and r_{24} can be written as functions of the four exterior sides,

$$r_{13} = \sqrt{\frac{ab}{c}} \quad \text{and} \quad r_{24} = \sqrt{\frac{ac}{b}}, \quad (15)$$

where $a = r_{12} r_{34} + r_{14} r_{23}$, $b = r_{12} r_{14} + r_{23} r_{34}$ and $c = r_{12} r_{23} + r_{14} r_{34}$. These can be derived using the Law of Cosines and the fact that opposite interior angles are supplementary [28]. If each equation in (15) holds, one can check that both $P = 0$ and $V = 0$ follows.

We then have

$$\frac{r_{13}}{r_{24}} = \frac{b}{c} = \frac{r_{12} r_{14} + r_{23} r_{34}}{r_{12} r_{23} + r_{14} r_{34}}$$

and $r_{13} \geq r_{24}$ if and only if $b - c \geq 0$ if and only if $(r_{14} - r_{23})(r_{12} - r_{34}) \geq 0$. Since r_{12} is the largest exterior side-length, $r_{14} \geq r_{23}$ if and only if $r_{13} \geq r_{24}$. Our final assumption is to fix the longest exterior side-length to one ($r_{12} = 1$), which specifies a particular choice of scaling. This is equivalent to holding the moment of inertia I constant. Summarizing, without loss of generality, we can restrict to the set of mutual distances

$$\Omega = \{\mathbf{r} \in \mathbb{R}^6 : r_{13} \geq r_{24} > r_{12} = 1 \geq r_{14} \geq r_{23} \geq r_{34}\}.$$

Any c.c.c. with positive masses and our particular choice of labelling and scaling will correspond to some vector $\mathbf{r}$ of mutual distances in Ω . A similar reduction was established by Hampton [10].

Using the appropriate formula for λ produces nice formulae for the mass ratios. For example, substituting $\lambda = (p_2 - p_3)/(s_2 - s_3)$ into the first equation in (10) yields, after some simplification,

$$\frac{m_2}{m_1} = \frac{r_{23}^2 r_{24}^2 (r_{13}^3 - r_{14}^3)}{r_{13}^2 r_{14}^2 (r_{24}^3 - r_{23}^3)}. \quad (16)$$

Similar substitutions in (11) and (12) give, respectively,

$$\frac{m_3}{m_1} = \frac{r_{23}^2 r_{34}^2 (r_{12}^3 - r_{14}^3)}{r_{12}^2 r_{14}^2 (r_{23}^3 - r_{34}^3)}, \quad (17)$$

and

$$\frac{m_4}{m_1} = \frac{r_{24}^2 r_{34}^2 (r_{13}^3 - r_{12}^3)}{r_{12}^2 r_{13}^2 (r_{24}^3 - r_{34}^3)}. \quad (18)$$

All of these ratios are positive and well-defined on Ω , except for m_3/m_1 when $r_{12} = r_{14}$ and $r_{23} = r_{34}$ (a kite configuration). In this exceptional symmetric case, we use a different formula,

$$\frac{m_3}{m_1} = \frac{(r_{12}^{-3} - \lambda)r_{12}^2}{(r_{23}^{-3} - \lambda)r_{23}^2} = \frac{(r_{13}^3 - r_{12}^3)(r_{24}^3 - r_{12}^3)r_{23}^4}{(r_{13}^3 - r_{23}^3)(r_{24}^3 - r_{23}^3)r_{12}^4} \quad (19)$$

obtained using $\lambda = (p_1 - p_2)/(s_1 - s_2)$. This can also be derived using equation (9).

Other mass ratios can be obtained in a similar fashion or by dividing pairs of equations from (16), (17) and (18). Since the masses can be scaled by any positive factor, we fix $m_1 = 1$ without any loss of generality. This choice will be made for the rest of this work.

Summarizing the above, a configuration $\mathbf{r} \in \Omega$ with masses $m_1 = 1, m_2, m_3, m_4$ is a c.c.c. if and only if it is in the common zero set of the following six polynomials:

$$\begin{aligned} F_1 &= r_{13}^2(r_{23} + r_{14}r_{34}) - (r_{14}^2r_{23} + r_{14}r_{34}(r_{23}^2 + 1) + r_{23}r_{34}^2), \\ F_2 &= r_{24}^2(r_{14} + r_{23}r_{34}) - (r_{14}^2r_{23}r_{34} + r_{14}(r_{34}^2 + r_{23}^2) + r_{23}r_{34}), \\ F_3 &= (r_{13}^3 - 1)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) - (1 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3), \\ F_4 &= m_2r_{13}^2r_{14}^2(r_{24}^3 - r_{23}^3) - r_{23}^2r_{24}^2(r_{13}^3 - r_{14}^3), \\ F_5 &= m_3r_{14}^2(r_{23}^3 - r_{34}^3) - r_{23}^2r_{34}^2(1 - r_{14}^3), \\ F_6 &= m_4r_{13}^2(r_{24}^3 - r_{34}^3) - r_{24}^2r_{34}^2(r_{13}^3 - 1). \end{aligned}$$

We will refer to this system of six equations as system I. The polynomials F_1 and F_2 are easily obtained from the two equations for r_{13} and r_{24} given in (15). F_3 is a restatement of equation (9) and the remaining three polynomials are easily derived from equations (16), (17) and (18), respectively.

There are two ways of approaching the problem of classifying co-circular c.c.s. First, given three positive masses m_2, m_3 and m_4 , one can try to simultaneously solve system I.

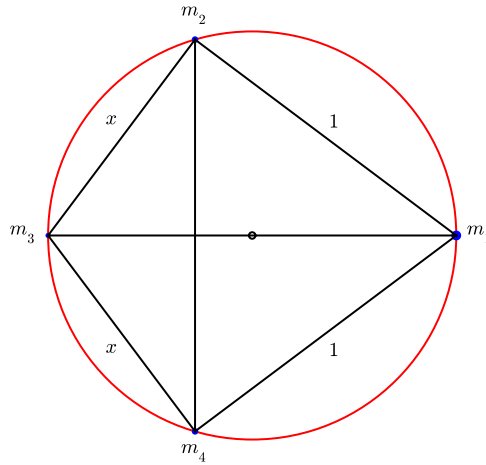


Figure 2. An example of a co-circular kite c.c. with $m_2 = m_4$. The centre of the circumscribing circle is marked with a $\circ$.

Since this is six equations in five unknowns, there is typically no solution. On the other hand, there is the *inverse* approach: for what set of mutual distances does there correspond a set of positive masses that make the configuration central? This is the approach pursued here. It involves a simpler analysis, restricting to the set of solutions to $\{F_1 = 0, F_2 = 0, F_3 = 0\}$ lying in Ω .

3. Symmetric cases

Before studying the full set of c.c.c.s, we analyse the two symmetric cases: a kite configuration and an isosceles trapezoid. These symmetric cases occur on the boundaries of the space Ω , sharing a common point corresponding to the square c.c. with equal masses. There is a one-parameter family for each case.

3.1. Kite configurations

We call a convex quadrilateral a *kite configuration* if two opposite bodies lie on an axis of symmetry of the configuration (see figure 2). Using the bounds that define Ω , it is straightforward to show that a co-circular c.c. is a kite if and only if one pair of opposite bodies lie on a diameter of the circumscribing circle. To see that *any* kite configuration is automatically a c.c., note that equation (9) is immediately satisfied when $r_{12} = r_{14} = 1$ and $r_{23} = r_{34}$. This gives a kite c.c. with an axis of symmetry between bodies 1 and 3. A similar kite exists with symmetry axis between bodies 2 and 4, but is excluded here by our choice that $r_{14} \geq r_{23}$.

We set $r_{23} = r_{34} = x$ and the diagonals $r_{13} = c$ and $r_{24} = 2x/c$ where $x \in (0, 1]$ is a parameter and $c = \sqrt{1 + x^2}$. These distances, together with $r_{12} = r_{14} = 1$, describe a planar ($V = 0$), kite c.c. that also satisfies Ptolemy's relation $P = 0$. It only remains to check whether the mass ratios are positive. First note that under these assumptions, $m_2 = m_4$, a result we expect from symmetry. Expressions (16), (18) and (19) with $m_1 = 1$ give $m_2 = m_4 = m$ and $m_3 = \alpha m$, where

$$m = \frac{4x(c^3 - 1)}{c(8 - c^3)} \quad \text{and} \quad \alpha = \frac{c(8x^3 - c^3)}{4(c^3 - x^3)}.$$

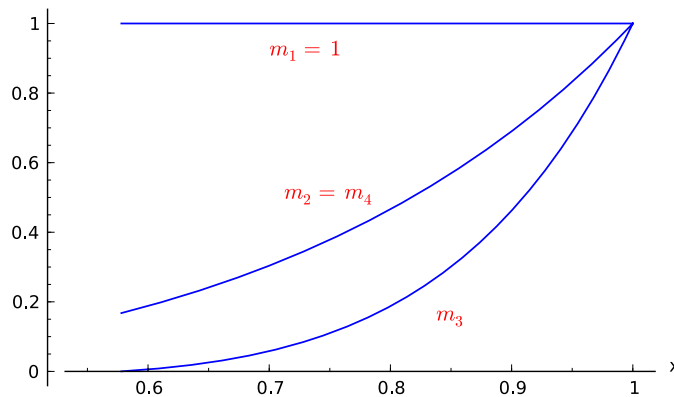


Figure 3. The values of the masses for the co-circular kite configurations.

The value for $m_2 = m_4$ is positive since $1 < c \leq \sqrt{2}$. For $x \in (0, 1]$, α , and consequently m_3 , is positive if and only if $1/\sqrt{3} < x \leq 1$. We have $m_3 = 0$ and $r_{24} = 1$ when $x = 1/\sqrt{3}$. In this degenerate case, bodies 1, 2 and 4 form an equilateral triangle and the configuration is a c.c. of the planar, circular, restricted four-body problem. At the other endpoint, setting $x = 1$ yields a square and all masses equal to one, as expected. Figure 3 displays the values of the masses as a function of the parameter x .

Theorem 3.1. *There exists a one-parameter family of co-circular kite c.c.s with bodies one and three lying on the diameter of the circumscribing circle. The masses are $m_1 = 1$, $m_2 = m_4 = m$ and $m_3 = \alpha m$ and are ordered $m_1 \geq m_2 = m_4 \geq m_3$ with equality if and only if the configuration is a square. At one end of the family, $x = 1/\sqrt{3}$, is a c.c. of the planar, circular, restricted four-body problem, with bodies 1, 2 and 4 forming an equilateral triangle and $m_3 = 0$. At the other end, $x = 1$, is the square with equal masses.*

Proof. The only item remaining to show is the ordering of the masses, which is clear from figure 3, but can be shown rigorously with straightforward analysis. We first show $m_3 \leq m_2$ by verifying that $\alpha \leq 1$. On the interval $1/\sqrt{3} \leq x \leq 1$, this is equivalent to showing $c(8x^3 - c^3) \leq 4(c^3 - x^3)$ or

$$c(8x^3 - 4x^2 - 4) \leq (x^2 + 1)^2 - 4x^3.$$

This inequality becomes an equality at $x = 1$. Assuming $x < 1$, we can divide both sides by the common positive factor $1 - x$ to obtain

$$-4c(2x^2 + x + 1) < 4x + (1 - x)^3$$

which is clearly valid for $1/\sqrt{3} \leq x < 1$.

Next, we show that $m \leq 1$ for $1/\sqrt{3} \leq x \leq 1$. This is equivalent to $4x(c^3 - 1) \leq c(8 - c^3)$ or

$$(x^2 + 1)^2 - 4x \leq 4c(2 - x(x^2 + 1)).$$

This inequality also becomes an equality at $x = 1$. As before, assuming $x < 1$, we can divide both sides by the common positive factor $1 - x$ to obtain

$$-(x^3 + x^2 + 3x - 1) < 4c(x^2 + x + 2).$$

It is straightforward to show that the polynomial on the left-hand side of this last inequality is negative for $1/\sqrt{3} \leq x < 1$ while the right-hand side is clearly positive. The above arguments

also show that the relationships between the masses are strict inequalities if $x < 1$. This completes the proof. $\square$

Remark.

- (1) While our work was being completed, a similar theorem appeared in both [18] and [23].
- (2) It can be shown with straightforward calculus that both m and α are strictly increasing functions of x for $1/\sqrt{3} \leq x \leq 1$. Thus, m_2, m_3 and m_4 are strictly increasing functions of the common side-length $r_{23} = r_{34}$.
- (3) The centre of mass for the kite lies closest to the largest body (m_1) for $x < 1$ and is equivalent to the circumcentre at $x = 1$. If the circumcentre is taken to be the origin, and m_1 is positioned on the positive x -axis, then a short calculation shows that the centre of mass for $x = 1/\sqrt{3}$ is $((41\sqrt{3} - 21)/104, 0) \approx (0.4809, 0)$. By the main result of Hampton [10] and by continuity, the centre of mass has coordinates $(\bar{x}, 0)$ with $\bar{x} > 0$ for $1/\sqrt{3} \leq x < 1$. Moreover, a graph of the first coordinate of the centre of mass shows that it is a strictly decreasing function of x .

Denote θ_{ij} as the arc along the circumscribing circle between bodies i and j . Since we have specified $r_{12} = 1$ as the longest exterior side, θ_{12} will be the largest arc between any two consecutive bodies. For the kite family, the circumradius r is simply $r = c/2$ and θ_{12} can be expressed in terms of the parameter x as $\theta_{12} = 2 \arctan(1/x)$, a monotonically decreasing function. The maximum value of θ_{12} is 120° ($x = 1/\sqrt{3}, m_3 = 0$) while the minimum value is 90° ($x = 1$, square). For the smallest arc between any two consecutive bodies, we have $\theta_{23} = \theta_{34} = 2 \arctan(x)$, an increasing function of x ranging from 60° to 90° for $1/\sqrt{3} < x \leq 1$. Note also that the maximum interior angle $\angle_{234}$ of the quadrilateral is equivalent to θ_{12} while the minimum interior angle $\angle_{214}$ is equivalent to θ_{23} . Therefore, the largest interior angle of a co-circular kite c.c. has a supremum of 120° ($m_3 = 0$) and a minimum of 90° (square). Furthermore, the arc length along the circumscribing circle between bodies 1 and 2 is given by

$$\text{Arc}(x) = \theta_{12} \cdot \frac{c}{2} = \sqrt{x^2 + 1} \arctan\left(\frac{1}{x}\right).$$

From this formula, it is clear that $\text{Arc}(x)$ is a decreasing function, with a supremum of $2\pi/(3\sqrt{3}) \approx 1.2092$ ($m_3 = 0$) and a minimum attained at the square configuration of $2^{-3/2}\pi \approx 1.1107$.

3.2. The isosceles trapezoid

In this section we verify the existence of a one-parameter family of isosceles trapezoids. Some of the results depend on symbolic calculations where Maple [16] is used to compute Gröbner bases and/or Sturm sequences. A Maple worksheet containing all the pertinent computations is available at <http://mathcs.holycross.edu/~groberts/Papers/papers.html>.

Any isosceles trapezoid, that is, a trapezoid whose legs are congruent, can be circumscribed in a circle with centre lying on the axis of symmetry. In our setup, the isosceles trapezoids correspond to the case where $r_{14} = r_{23}$ and $r_{13} = r_{24}$. Equation (16) then implies that $m_1 = m_2$. From equations (17), (18) and (9), we have

$$\frac{m_3}{m_4} = \frac{r_{13}^2 r_{23}^2 (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)}{r_{14}^2 r_{24}^2 (r_{23}^3 - r_{34}^3)(r_{13}^3 - r_{12}^3)} = \frac{r_{13}^2 r_{23}^2 (r_{24}^3 - r_{14}^3)}{r_{14}^2 r_{24}^2 (r_{13}^3 - r_{23}^3)}, \quad (20)$$

so that $m_3 = m_4$ follows as well. The equality of the two pairs of masses is expected from the symmetry of the configuration. This fact appears in the well-known paper of MacMillan and

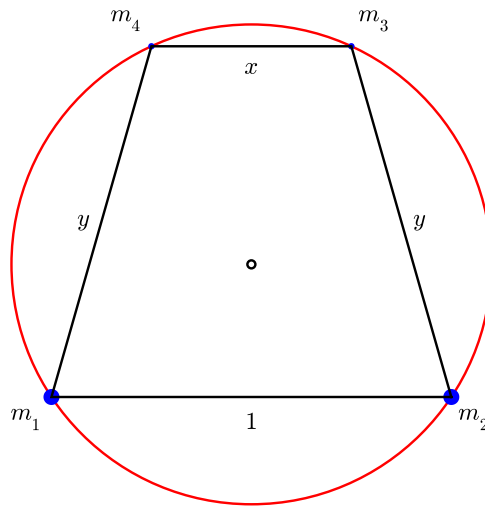


Figure 4. An example of an isosceles trapezoid c.c. The centre of the circumscribing circle is marked with a $\circ$.

Bartky [15]. It also turns out that if the c.c. is assumed to be co-circular, then equality of two pairs of adjacent masses implies the configuration is an isosceles trapezoid. It is unknown if the same fact holds for general four-body convex c.c.s.

Lemma 3.2. *If $m_1 = m_2$ and $m_3 = m_4$, then the corresponding c.c.c. must be an isosceles trapezoid.*

Proof. If $m_1 = m_2$ and $m_3 = m_4$, then equations (16) and (20) imply

$$\begin{aligned} r_{23}^2 r_{24}^2 (r_{13}^3 - r_{14}^3) - r_{13}^2 r_{14}^2 (r_{24}^3 - r_{23}^3) &= 0, \\ r_{13}^2 r_{23}^2 (r_{24}^3 - r_{14}^3) - r_{14}^2 r_{24}^2 (r_{13}^3 - r_{23}^3) &= 0. \end{aligned}$$

Taking the difference of these two equations and factoring the result yields

$$(r_{13} - r_{24})(r_{13}^2 r_{14}^2 r_{24}^2 + r_{13}^2 r_{23}^2 r_{24}^2 + r_{13} r_{14}^3 r_{23}^2 + r_{13} r_{14}^2 r_{23}^3 + r_{14}^3 r_{23}^2 r_{24} + r_{14}^2 r_{23}^3 r_{24}) = 0,$$

which implies $r_{13} = r_{24}$ since the mutual distances are real, positive numbers. Since the configuration is on a circle, it follows that $r_{14} = r_{23}$, and the configuration is an isosceles trapezoid. $\square$

Remark. A slightly stronger result exists. In fact, for c.c.c.s, only *one* pair of adjacent masses need be equal (e.g. $m_1 = m_2$ or $m_3 = m_4$) to imply the configuration is an isosceles trapezoid (see corollary 4.7).

To describe the family of isosceles trapezoid c.c.s, we set $r_{12} = 1$, $r_{34} = x$, $r_{14} = r_{23} = y$ where $0 \leq x \leq 1$ and $x \leq y \leq 1$ are dependent parameters (see figure 4). Using Ptolemy's theorem, we have $r_{13} = r_{24} = \sqrt{x + y^2}$. One can then check that $V = 0$ is satisfied for this family. As before, we fix $m_1 = 1$ so that $m_2 = 1$. Then, by equations (17) and (20), $m_3 = m_4 = m$, where

$$m = \frac{x^2(1 - y^3)}{y^3 - x^3}.$$

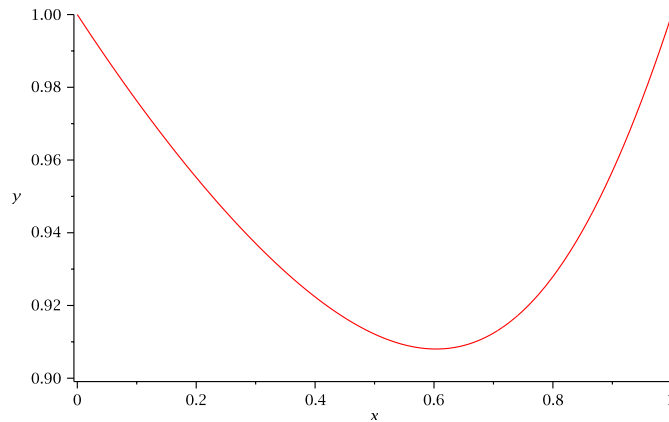


Figure 5. The relationship between the two distances $x = r_{34}$ and $y = r_{14} = r_{23}$ in the isosceles trapezoid family of c.c.s.

In order for the trapezoid to be a c.c., equation (9) must also be satisfied. Unlike the kite family, this yields a non-trivial condition. Letting

$$T(x, y) = (y^2 + x)^{3/2}(2y^3 - x^3 - 1) - y^3 - x^3y^3 + 2x^3, \quad (21)$$

the constraint $T = 0$ must be satisfied in order for the isosceles trapezoid to be a c.c.

A graph of the level curve $T = 0$ in the xy -plane for $0 \leq x \leq 1$, $x \leq y \leq 1$ is shown in figure 5. Note that while the smallest side of the trapezoid (parallel to the base) can range from 0 to 1, the length of the congruent legs is considerably constrained between approximately 0.908 and 1. At the point $(0, 1)$ (left endpoint of the curve), we have an equilateral triangle ($r_{34} = 0$, $r_{23} = r_{24} = r_{14} = r_{12} = 1$) and $m_3 = m_4 = 0$, corresponding to an equilibrium point of the planar, circular, restricted, three-body problem where two infinitesimal masses are located at the same Lagrange point. At the point $(1, 1)$ (right endpoint), we have the equal mass square c.c.

Next we show that $T(x, y) = 0$ implicitly defines a differentiable function $y = \tau(x)$ for $x \in [0, 1]$. Consequently, there exists a one-parameter family of co-circular isosceles trapezoid c.c.s parametrized by $r_{34} = x$. The following lemma and its proof will be important in the verification of theorem 4.3, establishing the surface of co-circular c.c.s.

Lemma 3.3. *For each value of $x \in (0, 1]$, there exists a unique value of $y \in [x, 1]$ such that $T(x, y) = 0$. Moreover, the distance parameter y can be written as a differentiable function, $y = \tau(x)$.*

Proof. Since $r_{13} = r_{24} > 1$ on Ω , we have that $\sqrt{x + y^2} > 1$ or $y > \sqrt{1 - x}$. Since $1 \geq r_{23} \geq r_{34}$ on Ω , we also have that $x \leq y \leq 1$. Thus we can restrict to the subset Λ of the xy -plane defined as

$$\Lambda = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x \leq y \leq 1 \text{ and } y \geq \sqrt{1 - x}\}.$$

For $x \geq 0$, the curves $y = \sqrt{1 - x}$ and $y = x$ intersect at $x = (-1 + \sqrt{5})/2$. Thus, on Λ , $y \geq (-1 + \sqrt{5})/2$.

Note that $T(x, 1) = (1 - x^3)((1 + x)^{3/2} - 1) \geq 0$ for $0 \leq x \leq 1$ with equality only at $x = 0$ or $x = 1$. Furthermore,

$$T(x, \sqrt{1 - x}) = (1 - x^3)((1 - x)^{3/2} - 1),$$

$$T(x, x) = (1 - x^3)(x^3 - (x^2 + x)^{3/2}),$$

each of which are negative for $0 < x < 1$. Fix an $x \in (0, 1)$ and consider $T(x, y) = T_x(y)$ as a function in the variable y . It follows that $T_x < 0$ at the lower boundary of Λ and $T_x > 0$ at the upper boundary of Λ . Since T is continuous, there is a solution to $T(x, y) = 0$ inside Λ for each $x \in (0, 1)$. To see that it is unique, we compute that

$$\frac{\partial^2 T}{\partial y^2} = 3 \left[-2y(x^3 + 1) + \sqrt{x + y^2} (16y^3 + 4xy - x^3 - 1) + \frac{y^2(4y^3 + 2xy - x^3 - 1)}{\sqrt{x + y^2}} \right].$$

The quantity $4y^3 + 2xy - x^3 - 1$ is strictly positive on Λ . To see this, note that the inequality $x + y^2 > 1$ implies that $x + 2y^2 > 1 + y^2$. Then, we have

$$\begin{aligned} 4y^3 + 2xy - x^3 - 1 &= 2y(x + 2y^2) - x^3 - 1 \\ &> 2y(1 + y^2) - x^3 - 1 \\ &\geq 2y(1 + y^2) - y^3 - 1 \quad (\text{since } x \leq y) \\ &= y^3 + 2y - 1 \\ &\geq \frac{(-1 + \sqrt{5})^3}{8} + \sqrt{5} - 2 \quad (\text{since } y \geq (-1 + \sqrt{5})/2) \\ &> 0. \end{aligned}$$

Next, using the inequalities $\sqrt{x + y^2} > 1$ and $x \leq 1$, we have

$$\begin{aligned} \frac{\partial^2 T}{\partial y^2} &> 3[-2y(x^3 + 1) + (x + y^2)^{1/2}(16y^3 + 4xy - x^3 - 1)] \\ &> 3[-2y(x^3 + 1) + 16y^3 + 4xy - x^3 - 1] \\ &= 3[4y(4y^2 + x) - (2y + 1)(x^3 + 1)] \\ &> 3[4y(1 + 3y^2) - 2(2y + 1)] \\ &= 6(6y^3 - 1) \\ &> 0 \quad (\text{since } y \geq (-1 + \sqrt{5})/2 > \sqrt[3]{1/6}). \end{aligned}$$

This proves that $\frac{\partial^2 T}{\partial y^2} > 0$ on Λ . Then, since $T_x < 0$ on the lower boundary of Λ , and $T_x > 0$ on the upper boundary of Λ , it follows that

$$\frac{d}{dy} (T_x(y))|_{y=\hat{y}} > 0, \quad (22)$$

where $\hat{y}$ satisfies $T_x(y) = 0$ and is the smallest such y -value to satisfy this equation. Since $\frac{\partial^2 T}{\partial y^2} > 0$, $y = \hat{y}$ is the unique solution to $T_x(y) = 0$. This proves there exists a function $y = \tau(x)$ for $0 < x < 1$ satisfying $T(x, \tau(x)) = 0$ on Λ .

Finally, since $T(1, 1) = 0$, set $\tau(1) = 1$. We compute that $\frac{\partial T}{\partial y}(1, 1) = 6(2\sqrt{2} - 1) > 0$. Applying the implicit function theorem, this fact and inequality (22) proves that $y = \tau(x)$ is a differentiable function of x for $0 < x \leq 1$. $\square$

Remark.

- (1) For $x = 1$ (the square), we have $\tau'(1) = 1/2$ by implicit differentiation. Although $x = r_{34} = 0$ is not a true physical solution, we can extend $\tau(x)$ to the closed interval $[0, 1]$ by defining $\tau(0) = 1$. Another computation gives $T(0, 1) = 0$, $\frac{\partial T}{\partial y}(0, 1) = 6 > 0$ and $\tau'(0) = -1/4$.
- (2) It can be shown that $0.9 < \tau(x) < 1$ for $0 < x < 1$. The fact that $\tau(x) < 1$ for $0 < x < 1$ follows quickly from $T(x, 1) = (1 - x^3)((1 + x)^{3/2} - 1)$. To show that $\tau(x) > 0.9$, we introduce the new variable $z = \sqrt{x + y^2}$ and use Maple to compute a Gröbner basis for

the ideal generated by the two polynomials $T(x, y = 9/10)$ and $x + (9/10)^2 - z^2$ [7]. This produces a polynomial in z of degree 9 which, using Sturm's theorem [27], contains no real roots between 0 and 2. Consequently, since $\tau(x)$ is a continuous function and $\tau(0) = \tau(1) = 1$, it follows that $\tau(x) > 0.9$. While this is a computer-assisted argument, it only depends on symbolic and exact integer computations.

- (3) The minimum of $\tau(x)$ can be found by simultaneously solving the equations $T(x, y) = 0$ and $\frac{\partial T}{\partial x} = 0$ on Λ . As above, introducing the variable $z = \sqrt{x + y^2}$, we use Maple to compute a Gröbner basis of the relevant ideal. Numerically approximating the roots of two one-variable polynomials yields a unique minimum at $x = 0.603\,538\,1491$, $y = 0.908\,025\,9298$. This calculation was also confirmed using the free software Sage [22] and agrees with the data given in table II in [15].

In [15], MacMillan and Bartky show that a unique isosceles trapezoid c.c. exists for any choice of masses $m_1 = m_2$ and $m_3 = m_4$. They use a different parametrization than ours, setting $r_{23} = \kappa \sqrt{r_{12}r_{34}}$, with κ as a parameter. In our notation, $\kappa = y/\sqrt{x}$. The same result is proven by Albouy [3]. For completeness sake, we provide another proof using symbolic computation with Maple.

Theorem 3.4. *The mass parameter m is an increasing function of x satisfying $m(0) = 0$ and $m(1) = 1$. Consequently, for each choice of the mass parameter m , there exists a unique isosceles trapezoid c.c. Moreover, the largest pair of equal masses in the isosceles trapezoid c.c. lies on the larger base.*

Proof. By lemma 3.3 and its proof, there exists a differentiable function $\tau(x)$ for $x \in [0, 1]$ satisfying $T(x, y = \tau(x)) = 0$. Moreover, $x < \tau(x) < 1$ for $x \in [0, 1)$ and $\tau(1) = 1$. Thus, the mass parameter m is non-negative on $x \in [0, 1]$ and is given by

$$m(x) = \begin{cases} \frac{x^2(1 - (\tau(x))^3)}{(\tau(x))^3 - x^3} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Using $\tau(1) = 1$ and $\tau'(1) = 1/2$, a straightforward application of L'Hôpital's rule gives

$$\lim_{x \rightarrow 1^-} m(x) = 1,$$

so that $m(x)$ is a continuous function on $[0, 1]$. We compute that, for $x \in [0, 1)$,

$$\frac{dm}{dx} = \frac{x[(1 - (\tau(x))^3)(2(\tau(x))^3 + x^3) - 3x(\tau(x))^2(1 - x^3)\tau'(x)]}{((\tau(x))^3 - x^3)^2}.$$

We claim that this quantity is positive for $0 < x < 1$. Note that since $\tau(0) = 1$ and $\tau(x) < 1$ for $x < 1$, we have that $\tau'(x) < 0$ for x near 0. Consequently, $dm/dx > 0$ for x sufficiently close to 0. To show that $dm/dx > 0$ for all $x \in (0, 1)$, we use Gröbner bases and Sturm's theorem [27]. Using the fact that $\tau'(x) = -\frac{\partial T}{\partial x} / \frac{\partial T}{\partial y}$ and inequality (22), it is sufficient to show that

$$m_x = (1 - y^3)(2y^3 + x^3) \frac{\partial T}{\partial y} + 3xy^2(1 - x^3) \frac{\partial T}{\partial x}$$

is strictly positive on the curve $y = \tau(x)$ for $x \in (0, 1)$. As with previous arguments, introduce the variable $z = \sqrt{x + y^2}$. Computing a Gröbner basis with respect to the lexicographic ordering $z > y > x$ for the ideal generated by

$$\begin{aligned} T &= z^3(2y^3 - x^3 - 1) - y^3 - x^3y^3 + 2x^3, \\ m_x &= (1 - y^3)(2y^3 + x^3) \frac{\partial T}{\partial y} + 3xy^2(1 - x^3) \frac{\partial T}{\partial x}, \\ Z &= x + y^2 - z^2, \end{aligned}$$

yields a polynomial in x of the form

$$x^4(x-1)^2(x^2+x+1)^2 \cdot \phi(x)$$

where $\phi(x)$ is degree 48. Using the Maple commands `sturmseq` and `sturm`, we apply Sturm's theorem to conclude that $\phi(x)$ has no real roots in $[0, 1]$. Consequently, the function m_x does not change sign along the curve $y = \tau(x)$. Since $m_x > 0$ for x sufficiently close to 0, it follows that $m_x > 0$ on the curve $y = \tau(x)$ for all $x \in (0, 1)$. $\square$

Next, consider the arc θ_{12} along the circumscribing circle between bodies 1 and 2. We have $\theta_{12} = 2 \arcsin(\frac{1}{2r_c})$ where r_c is the circumradius

$$r_c = y \sqrt{\frac{x+y^2}{4y^2-(1-x)^2}}.$$

At $(0, 1)$ (degenerate equilateral triangle), we have $r_c = 1/\sqrt{3}$ and at $(1, 1)$ (square), we have $r_c = \sqrt{2}/2$, as expected. The maximum value of θ_{12} will occur at the minimum value of the circumradius. We consider the quantity r_c^2 and claim that $\frac{dr_c^2}{dx}$ is positive on $0 < x \leq 1$. This is equivalent to showing that

$$y(2y^2 + 2xy^2 + x^2 - 1) \frac{\partial T}{\partial y} - (8y^4 - 2(1-x)^2(x+2y^2)) \frac{\partial T}{\partial x} \quad (23)$$

is strictly positive on $0 < x \leq 1$. This is verified with Maple using the same symbolic computing techniques as in the proof of theorem 3.4. In this case, we obtain a polynomial in y which, by Sturm's theorem, has no roots between 0.9 and 1. This means that expression (23) does not change sign along the curve $y = \tau(x)$. Since expression (23) evaluates to $48(2\sqrt{2} - 1) > 0$ at the square $x = 1, y = 1$, it must be positive along all of $y = \tau(x)$. Thus, the infimum of the circumradius is $r = 1/\sqrt{3}$ and θ_{12} is a strictly decreasing function of x .

Next, consider the arc θ_{14} between bodies 1 and 4. In this case, we have $\theta_{14} = 2 \arcsin(\frac{y}{2r_c})$. A similar argument as above, using Maple, shows that the quantity $\frac{y}{2r_c}$ is a strictly decreasing function for $0 \leq x \leq 1$. Consequently, θ_{14} is also a strictly decreasing function of x . This shows that the largest interior angle, $\angle_{234}$, given by $(\theta_{12} + \theta_{14})/2$, is a strictly decreasing function of x . We have given a computer-assisted proof of the following theorem.

Theorem 3.5. *The largest arc θ_{12} in the isosceles trapezoid family is a strictly decreasing function of the smallest side-length $r_{34} = x$. The supremum of θ_{12} is 120° , from the degenerate equilateral triangle ($m_3 = m_4 = 0$ and $r_{34} = 0$), while the minimum is 90° , attained at the square configuration. Similarly, the largest interior angle of the trapezoid is a strictly decreasing function of x , with a supremum of 120° (degenerate equilateral triangle) and minimum of 90° (square) while the smallest interior angle strictly increases from 60° to 90° .*

Remark.

- (1) These bounds are the same as those for the co-circular kite family studied in section 3.1.
- (2) In [15], a theorem stating that a unique isosceles trapezoid c.c. exists for each interior angle between 60° and 120° is presented, although no formal proof is given.

4. The surface of c.c.c.s

We now analyse the full set of c.c.c.s and show it is equivalent to a graph over two of the exterior side-lengths. The set of interest is all vectors $\mathbf{r} = (1, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}) \in \Omega$ satisfying equation (9), $P = 0$ and $V = 0$. Recall that for a cyclic quadrilateral, the diagonals can be expressed in terms of the exterior sides. Expanding the expressions in equation (15) yields

$$r_{13} = \left(\frac{r_{14}^2 r_{23} + r_{14} r_{34} (r_{23}^2 + 1) + r_{23} r_{34}^2}{r_{23} + r_{14} r_{34}} \right)^{1/2}, \quad (24)$$

$$r_{24} = \left(\frac{r_{14}^2 r_{23} r_{34} + r_{14} (r_{34}^2 + r_{23}^2) + r_{23} r_{34}}{r_{14} + r_{23} r_{34}} \right)^{1/2}, \quad (25)$$

which are equivalent to $F_1 = 0$ and $F_2 = 0$, respectively. If equations (24) and (25) hold, one can check that both $P = 0$ and $V = 0$ follow. This avoids the need to use the Cayley–Menger determinant $V = 0$ and also serves to eliminate the variables r_{13} and r_{24} . Substituting relations (24) and (25) into equation (9) and setting $r_{12} = 1$ yields

$$F(r_{14}, r_{23}, r_{34}) = (r_{13}^3 - 1)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) - (1 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3) = 0$$

where r_{13} and r_{24} are understood to be functions of r_{14}, r_{23}, r_{34} given by equations (24) and (25), respectively.

The relations on the mutual distances in Ω , coupled with the fact that the bodies are on a common circle, lead to further restrictions on the variables r_{14}, r_{23}, r_{34} . Since $r_{13} > 1$, equation (24) implies that

$$r_{14}^2 r_{23} + r_{14} r_{34} (r_{23}^2 + 1) + r_{23} r_{34}^2 > r_{23} + r_{14} r_{34}$$

which simplifies to

$$r_{14}^2 + r_{34}^2 + r_{14} r_{34} r_{23} > 1. \quad (26)$$

Similarly, $r_{24} > 1$ and equation (25) imply that

$$r_{23}^2 + r_{34}^2 + r_{14} r_{34} r_{23} > 1. \quad (27)$$

Note that since we are assuming $r_{14} \geq r_{23}$, inequality (26) follows directly from inequality (27). Also, the calculations above can be reversed, so that inequalities (26) and (27) imply $r_{13} > 1$ and $r_{24} > 1$, respectively. We have proven the following lemma.

Lemma 4.1. *Let*

$$\mathcal{C} = \{(r_{14}, r_{23}, r_{34}) \in \mathbb{R}^{+3} : 1 \geq r_{14} \geq r_{23} \geq r_{34} \text{ and } r_{23}^2 + r_{34}^2 + r_{14} r_{34} r_{23} > 1\}, \text{ and}$$

$$\Gamma = \{s = (r_{14}, r_{23}, r_{34}) \in \mathbb{R}^{+3} : s \in \mathcal{C} \text{ and } F(s) = 0\}.$$

Any point in Γ corresponds to a four-body c.c.c. with positive masses and up to a relabelling and rescaling, Γ contains all such configurations.

4.1. Γ as a graph

We now show that Γ is a surface by proving that it is a graph $r_{14} = f(r_{23}, r_{34})$ (see figure 6). First note that the boundaries of Γ consist of a line corresponding to the kites (K), a curve containing the isosceles trapezoids (T) and a curve corresponding to degenerate c.c.s (D) with $m_3 = 0$. Since they contain only positive masses, (K) and (T) are included in Γ while (D) is excluded. The kites lie on the line $r_{14} = 1, r_{23} = r_{34} = x$ with $1/\sqrt{3} < x \leq 1$. The

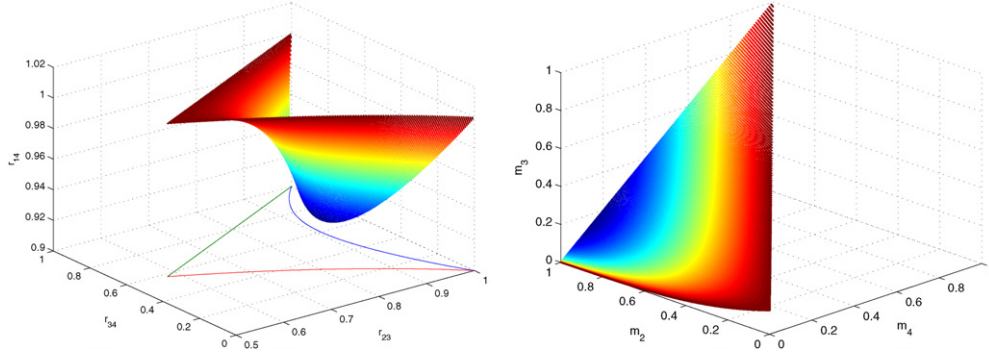


Figure 6. On the left is the surface Γ of c.c.c.s in $r_{34}r_{23}r_{14}$ -space. The outline of the projection onto the $r_{34}r_{23}$ -plane is shown plotted in the plane $r_{14} = 0.9$. This figure was generated with Matlab [17] using a bisection algorithm. On the right is the image of Γ in $m_2m_3m_4$ -space under equations (16), (17) and (18) with $m_1 = 1$. This figure shows the full set of masses for which a co-circular c.c. exists.

trapezoids lie on a curve in the plane $r_{14} = r_{23}$ and as shown in section 3.2, r_{23} can be written as a differentiable function of r_{34} . The degenerate c.c. with $m_3 = 0$ are critical points of the planar, circular, restricted four-body problem. Here, bodies 1, 2 and 4 lie on an equilateral triangle, with $r_{12} = r_{24} = r_{14} = 1$. It is straightforward to show that if $r_{24} = 1$, that is, when the only strict inequality defining $\mathcal{C}$ becomes an equality, then $F = 0$ can only be satisfied on $\mathcal{C}$ if $r_{14} = 1$. Therefore, the boundary of Γ contains only the cases (K), (T) or (D).

Lemma 4.2. On $\mathcal{C}$, both $\frac{\partial r_{13}}{\partial r_{14}}$ and $\frac{\partial r_{24}}{\partial r_{14}}$ are strictly positive.

Proof. A straightforward calculation shows that

$$\frac{\partial r_{13}}{\partial r_{14}} = \frac{r_{23}[r_{34}(r_{14}^2 + r_{23}^2 - r_{34}^2 + 1) + 2r_{14}r_{23}]}{2r_{13}(r_{23} + r_{14}r_{34})^2}$$

which is strictly positive on $\mathcal{C}$ since $r_{23} \geq r_{34}$. A similar calculation yields

$$\frac{\partial r_{24}}{\partial r_{14}} = \frac{r_{23}r_{34}(r_{14}^2 + r_{23}^2 + r_{34}^2 + 2r_{14}r_{23}r_{34} - 1)}{2r_{24}(r_{14} + r_{23}r_{34})^2}$$

which is strictly positive due to inequality (27). $\square$

Recall that for the isosceles trapezoid family, the exterior side r_{23} can be written as a differentiable function of r_{34} , denoted by $r_{23} = \tau(r_{34})$ for $0 \leq r_{34} \leq 1$ and that $\tau < 1$ except for $r_{34} = 0, 1$.

Theorem 4.3. The set of c.c.c.s Γ is the graph of a differentiable function $r_{14} = f(r_{34}, r_{23})$ over the two variables r_{34} and r_{23} . The domain of this function is the region

$$\mathcal{D} = \{(r_{34}, r_{23}) \in \mathbb{R}^+ : 1 \geq r_{23} \geq r_{34}, r_{23} \leq \tau(r_{34}) \text{ and } r_{23}^2 + r_{34}^2 + r_{34}r_{23} > 1\}.$$

Proof. We will show that the projection of Γ onto the $r_{34}r_{23}$ -plane equals $\mathcal{D}$ (see figure 7) and that r_{14} can be written as a differentiable function of r_{34} and r_{23} over $\mathcal{D}$, that is, $\frac{\partial f}{\partial r_{34}}$ and $\frac{\partial f}{\partial r_{23}}$ exist and are continuous on the interior of $\mathcal{D}$.

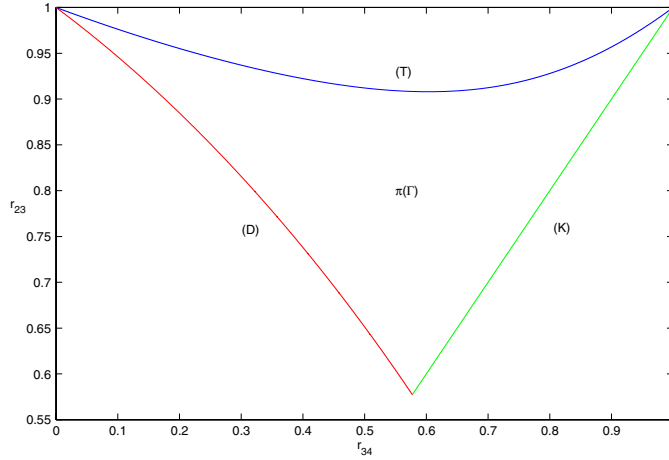


Figure 7. The projection $\pi(\Gamma) = \mathcal{D}$ of the surface of co-circular c.c.s onto the $r_{34}r_{23}$ -plane.

According to the definition of $\mathcal{C}$, the distance r_{14} must satisfy both

$$r_{14} \geq r_{23} \quad \text{and} \quad r_{14} > \frac{1 - r_{23}^2 - r_{34}^2}{r_{23}r_{34}}. \quad (28)$$

Therefore, define

$$\hat{z}(r_{34}, r_{23}) = \max \left\{ r_{23}, \frac{1 - r_{23}^2 - r_{34}^2}{r_{23}r_{34}} \right\}.$$

We will show that for a given point (r_{34}, r_{23}) in $\mathcal{D}$, $F = 0$ has a unique solution for some r_{14} satisfying $\hat{z} \leq r_{14} \leq 1$. This point will clearly lie in Γ .

The intersection of the two surfaces $r_{14} = r_{23}$ and $r_{14} = (1 - r_{23}^2 - r_{34}^2)/(r_{23}r_{34})$ projected onto the first quadrant of the $r_{34}r_{23}$ -plane is a piece of the parabola $r_{23}^2 = 1 - r_{34}$. This curve divides $\mathcal{D}$ into two regions, denoted D_1 and D_2 . On D_1 , defined as the set of points in the interior of $\mathcal{D}$ for which $r_{23} < \sqrt{1 - r_{34}}$, we have that $\hat{z} = (1 - r_{23}^2 - r_{34}^2)/(r_{23}r_{34})$. On D_2 , defined as the set of points in the interior of $\mathcal{D}$ for which $r_{23} \geq \sqrt{1 - r_{34}}$, we have that $\hat{z} = r_{23}$.

The three curves defining the boundaries of $\mathcal{D}$ correspond to the kites ($r_{23} = r_{34}$), the isosceles trapezoid family ($r_{23} = \tau(r_{34})$) and the degenerate case with $m_3 = 0$ ($r_{24} = r_{14} = 1$). If $r_{14} = 1$ and we restrict to the interior of $\mathcal{D}$, then inequalities (26) and (27) show that $r_{13} > 1$ and $r_{24} > 1$. It then follows that

$$F(r_{14} = 1, r_{23}, r_{34}) = (r_{13}^3 - 1)(r_{24}^3 - 1)(r_{23}^3 - r_{34}^3)$$

is strictly positive on the interior of $\mathcal{D}$.

Next, we claim that $F(r_{14} = \hat{z}, r_{23}, r_{34})$ is strictly negative on the interior of $\mathcal{D}$. To see this, consider a point in the region D_1 and suppose that $r_{14} = \hat{z} = (1 - r_{23}^2 - r_{34}^2)/(r_{23}r_{34})$. Then, inequality (27) becomes an equality and $r_{24} = 1$. Moreover, $0 < r_{23} \leq r_{14} < 1$ by the definition of $\hat{z}$ and the fact that (r_{34}, r_{23}) is in the interior of $\mathcal{D}$. We also have, using equation (24), that

$$r_{13}^2 - r_{34}^2 = \frac{r_{14}(r_{14}r_{23} + r_{34} + r_{34}(r_{23}^2 - r_{34}^2))}{r_{23} + r_{14}r_{34}} > 0$$

since $r_{23} > r_{34}$. It follows that

$$F(r_{14} = \hat{z}, r_{23}, r_{34}) = -(1 - r_{14}^3)(1 - r_{23}^3)(r_{13}^3 - r_{34}^3)$$

is strictly negative on D_1 .

Now consider a point in D_2 and suppose that $r_{14} = r_{23}$. On this plane, we also have that $r_{13} = r_{24}$ since the quadrilateral is cyclic (or by examining equations (24) and (25)). A short computation gives

$$F(r_{14} = r_{23}, r_{23}, r_{34}) = (r_{13}^3 - r_{23}^3) \cdot T(x = r_{34}, y = r_{23})$$

where T is given by equation (21) from the isosceles trapezoid family. As demonstrated in the proof of lemma 3.3, the value of T on D_2 is strictly negative since T vanishes only on the upper boundary of $\mathcal{D}$ and $\frac{\partial T}{\partial y}(x, y = \tau(x)) > 0$. Since $r_{14} = r_{23}$ implies that $r_{13} = \sqrt{r_{34}^2 + r_{23}^2} > r_{23}$, we have that $F(r_{14} = r_{23}, r_{23}, r_{34})$ is strictly negative. This shows that $F(r_{14} = \hat{z}, r_{23}, r_{34})$ is strictly negative on the interior of $\mathcal{D}$. By the intermediate value theorem, it follows that for each point (r_{34}, r_{23}) in the interior of $\mathcal{D}$, there exists a solution to $F = 0$ for some r_{14} satisfying $\hat{z} < r_{14} < 1$.

To see that this solution is unique, we show that $\frac{\partial F}{\partial r_{14}} > 0$ on Γ . We compute that

$$\frac{\partial F}{\partial r_{14}} = 3r_{13}^2 \frac{\partial r_{13}}{\partial r_{14}} \alpha_1 + 3r_{24}^2 \frac{\partial r_{24}}{\partial r_{14}} \alpha_2 + 3r_{14}^2 \alpha_3$$

where

$$\begin{aligned} \alpha_1 &= (r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) - (1 - r_{14}^3)(r_{24}^3 - r_{34}^3), \\ \alpha_2 &= (r_{23}^3 - r_{34}^3)(r_{13}^3 - 1) - (1 - r_{14}^3)(r_{13}^3 - r_{23}^3), \\ \alpha_3 &= (r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3) - (r_{23}^3 - r_{34}^3)(r_{13}^3 - 1). \end{aligned}$$

Using $F = 0$, each α_i restricted to Γ simplifies to

$$\begin{aligned} \alpha_1 &= \frac{(1 - r_{14}^3)(r_{24}^3 - r_{34}^3)(1 - r_{23}^3)}{r_{13}^3 - 1}, \\ \alpha_2 &= \frac{(1 - r_{14}^3)(r_{13}^3 - r_{23}^3)(r_{14}^3 - r_{34}^3)}{r_{24}^3 - r_{14}^3}, \\ \alpha_3 &= \frac{(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3)(r_{24}^3 - 1)}{r_{24}^3 - r_{14}^3}. \end{aligned}$$

It is clear that α_3 is strictly positive on Γ while α_1 and α_2 are non-negative. Taken together with lemma 4.2, this proves that $\frac{\partial F}{\partial r_{14}} > 0$ on Γ . Hence, there exists a function $r_{14} = f(r_{34}, r_{23})$ implicitly defined by $F = 0$ over the interior of $\mathcal{D}$.

Finally, on the boundary of $\mathcal{D}$ defined by $r_{23} = r_{34}$, the only possible solution to $F = 0$ that lies inside $\mathcal{C}$ is $r_{14} = 1$ (the kite configurations). Similarly, on the boundary $r_{23} = \tau(r_{34})$, since $\frac{\partial F}{\partial r_{14}} > 0$, the only possible solution to $F = 0$ that lies inside $\mathcal{C}$ is $r_{14} = r_{23}$ (the trapezoid family). Thus, we define $f(r_{34}, r_{23} = r_{34}) = 1$ and $f(r_{34}, r_{23} = \tau(r_{34})) = r_{23}$ to extend f to all of $\mathcal{D}$. The implicit function theorem, which is applicable on the boundaries (K) and (T) as well, and the fact that $F = 0$ is algebraic, then shows that f is continuous on all of $\mathcal{D}$ and differentiable on the interior of $\mathcal{D}$.

From the proof of lemma 3.3, any point (r_{34}, r_{23}) lying above the curve τ has $T > 0$. Since $\frac{\partial F}{\partial r_{14}} > 0$ on Γ , it follows that the solution to $F = 0$ must satisfy $r_{14} < r_{23}$, which lies outside $\mathcal{C}$. Likewise, any point (r_{34}, r_{23}) lying on or below the curve $r_{23}^2 + r_{34}^2 + r_{34}r_{23} = 1$ will have $r_{24} \leq 1$, which also lies outside $\mathcal{C}$. It follows that the projection of Γ onto the $r_{34}r_{23}$ -plane is precisely equal to $\mathcal{D}$. $\square$

4.2. Ordering of the masses

Remarkably, it is not difficult to show that for any c.c.c., the masses must be ordered in a precise fashion. The arguments that follow rely mostly on the fact that the four bodies lie on a common circle, as well as our ordering of the mutual distances.

Theorem 4.4. *Any c.c.c. in Γ satisfies*

$$m_3 \leq m_4 \leq m_2 \leq m_1 = 1.$$

In other words, the largest body is located at the vertex between the two longest exterior sides, and the smallest body is opposite the largest one. In addition, the two largest bodies lie on the longest side while the two smallest bodies lie on the smallest side.

Before proving this theorem, we verify some important inequalities relating the two diagonals with the second and third-longest exterior sides.

Lemma 4.5. *Any c.c.c. in Γ satisfies*

$$r_{13} - r_{24} \leq (r_{14} - r_{23})(1 - r_{34}) \quad \text{and} \quad (29)$$

$$r_{13} - r_{24} \leq r_{14} - r_{23}. \quad (30)$$

Proof. Using equation (15), we have

$$r_{13} - r_{24} = \sqrt{\frac{a}{bc}}(b - c) = \sqrt{\frac{a}{bc}}(r_{14} - r_{23})(1 - r_{34}).$$

Then, since

$$\begin{aligned} bc &= (r_{14} + r_{23}r_{34})(r_{23} + r_{14}r_{34}) \\ &= r_{14}r_{23} + r_{34}(r_{14}^2 + r_{23}^2 + r_{14}r_{23}r_{34}) \\ &> r_{14}r_{23} + r_{34} \quad (\text{by the definition of } \mathcal{C}) \\ &= a, \end{aligned}$$

we have $\frac{a}{bc} < 1$, which verifies inequality (29). Since $0 < r_{34} \leq 1$, inequality (30) immediately follows from (29). We note that equality is also needed in both (29) and (30) due to the isosceles trapezoid family where both sides vanish. In fact, these inequalities are strict for all other co-circular c.c.s. $\square$

Lemma 4.6. *Any c.c.c. in Γ satisfies*

$$\frac{r_{13}}{r_{24}} \leq \frac{r_{14}}{r_{23}}.$$

Proof. Since $r_{23} \leq r_{14}$, we have

$$\frac{r_{13}}{r_{24}} = \frac{r_{14} + r_{23}r_{34}}{r_{23} + r_{14}r_{34}} \leq \frac{r_{14}(1 + r_{34})}{r_{23}(1 + r_{34})} = \frac{r_{14}}{r_{23}}. \quad \square$$

Proof of theorem 4.4. We begin by showing

$$m_2 = \frac{r_{23}^2 r_{24}^2 (r_{13}^3 - r_{14}^3)}{r_{13}^2 r_{14}^2 (r_{24}^3 - r_{23}^3)} \leq 1.$$

Since $r_{13} \geq r_{24}$ and $r_{14} \geq r_{23}$, it suffices to show that $r_{24}^3 - r_{23}^3 \geq r_{13}^3 - r_{14}^3$ or equivalently,

$$r_{13}^3 - r_{24}^3 \leq r_{14}^3 - r_{23}^3. \quad (31)$$

Cubing both sides of inequality (30) yields

$$r_{13}^3 - r_{24}^3 + (-3r_{13}^2 r_{24} + 3r_{13} r_{24}^2 + 3r_{14}^2 r_{23} - 3r_{14} r_{23}^2) \leq r_{14}^3 - r_{23}^3.$$

Inequality (31) will then follow, provided $-r_{13}^2 r_{24} + r_{13} r_{24}^2 + r_{14}^2 r_{23} - r_{14} r_{23}^2$ is non-negative. Factoring, this is equivalent to showing

$$r_{13} r_{24} (r_{24} - r_{13}) + r_{14} r_{23} (r_{14} - r_{23}) \geq 0. \quad (32)$$

Beginning with $1 \leq r_{13} r_{24} = r_{23} r_{14} + r_{34}$, we have the following sequence of implications:

$$\begin{aligned} 1 \leq r_{34} + r_{14} r_{23} &\implies r_{34} - r_{34}^2 - r_{34} r_{14} r_{23} \leq 0 \\ &\implies (1 - r_{34})(r_{14} r_{23} + r_{34}) \leq r_{14} r_{23} \\ &\implies r_{13} r_{24} (1 - r_{34}) \leq r_{14} r_{23} \\ &\implies r_{13} r_{24} (r_{14} - r_{23})(1 - r_{34}) \leq r_{14} r_{23} (r_{14} - r_{23}). \end{aligned}$$

Then, using inequality (29), we have

$$r_{13} r_{24} (r_{13} - r_{24}) \leq r_{13} r_{24} (r_{14} - r_{23})(1 - r_{34}) \leq r_{14} r_{23} (r_{14} - r_{23})$$

which verifies inequality (32) and shows that $m_2 \leq 1$.

Next, we verify that $m_4 \leq m_2$. Using equations (16) and (18), we have

$$\frac{m_2}{m_4} = \frac{r_{23}^2}{r_{14}^2 r_{34}^2} \cdot \frac{r_{13}^3 - r_{14}^3}{r_{13}^3 - 1} \cdot \frac{r_{24}^3 - r_{34}^3}{r_{24}^3 - r_{23}^3}. \quad (33)$$

Each of the three fractions in equation (33) are greater than or equal to 1. This follows since $r_{23} \geq r_{34}$ and $1 \geq r_{14}$. Consequently, $m_2/m_4 \geq 1$, as desired.

Finally, using equations (17), (18) and (9), we compute

$$\frac{m_4}{m_3} = \frac{r_{24}^2 r_{14}^2 (r_{13}^3 - 1)(r_{23}^3 - r_{34}^3)}{r_{13}^2 r_{23}^2 (r_{24}^3 - r_{34}^3)(1 - r_{14}^3)} = \frac{r_{24}^2 r_{14}^2 (r_{13}^3 - r_{23}^3)}{r_{13}^2 r_{23}^2 (r_{24}^3 - r_{14}^3)}.$$

By lemma 4.6, $r_{24} r_{14} \geq r_{13} r_{23}$. Since $r_{13} \geq r_{24}$ and $r_{14} \geq r_{23}$, we also have $r_{13}^3 - r_{23}^3 \geq r_{24}^3 - r_{14}^3$. This proves that $m_3 \leq m_4$. $\square$

Somewhat surprisingly, theorem 4.4 and its proof give strong implications if just two of the masses are equal.

Corollary 4.7. *If just two bodies of a c.c.c. have equal mass, then the configuration is symmetric, either a kite or an isosceles trapezoid. Specifically, for any co-circular c.c. in Γ , if either $m_1 = m_2$ or $m_3 = m_4$, then the configuration is an isosceles trapezoid and the other pair of masses are necessarily equal. If $m_2 = m_4$, the configuration is a kite. If any three masses are equal, then the configuration is a square and all four masses are necessarily equal.*

Proof. From the proof for $m_2 \leq 1$, we have that

$$m_2 = \frac{r_{23}^2}{r_{14}^2} \cdot \frac{r_{24}^2}{r_{13}^2} \cdot \frac{r_{13}^3 - r_{14}^3}{r_{24}^3 - r_{23}^3} \quad (34)$$

is the product of three positive numbers less than or equal to one. If $m_2 = m_1 = 1$, it follows that each of the fractions in equation (34) is equal to one. This quickly yields $r_{14} = r_{23}$ and $r_{13} = r_{24}$, yielding an isosceles trapezoid. Then, by equation (20), $m_3 = m_4$ follows.

Next, suppose that $m_3 = m_4$. Then, from the proof for $m_3 \leq m_4$, we must have $r_{13}^3 - r_{23}^3 = r_{24}^3 - r_{14}^3$. This implies that

$$r_{13}^3 - r_{24}^3 = r_{23}^3 - r_{14}^3. \quad (35)$$

If $r_{23} < r_{14}$, then the right-hand side of equation (35) is negative, contradicting the fact that $r_{13} \geq r_{24}$. Thus, $r_{23} = r_{14}$ and then equation (35) gives $r_{13} = r_{24}$ and the configuration is, once again, an isosceles trapezoid. Then, by equation (16), $m_1 = m_2$ follows.

Finally, if $m_2 = m_4$, a similar argument using equation (33) shows that $r_{14} = 1$ and $r_{23} = r_{34}$, yielding a kite configuration. If three of the masses are equal (either $m_3 = m_4 = m_2$ or $m_4 = m_2 = m_1$), then the co-circular c.c. is simultaneously both an isosceles trapezoid and a kite. This means it is a square and that all four masses must be equal. $\square$

Remark.

- (1) Corollary 4.7 shows that if one pair of masses on either the longest or shortest exterior side are equal, then the other pair is also equal and the configuration must be an isosceles trapezoid. One would expect that with the same assumption, such a strong conclusion does not hold for generic four-body convex c.c.s.
- (2) The fact that $m_2 = m_4$ implies the configuration is a kite is already a consequence of the main theorem in [4], where it is shown that this fact is true for any convex c.c., not just co-circular ones.
- (3) Corollary 4.7 and our work on the symmetric cases shows that there are positive masses for which no c.c.c. exists. For example, if we consider a set of masses where $m_i = m_j$ for some choice of indices, then the configuration must be symmetric. However, our work in section 3 shows that there is a specific relationship between the other two masses to ensure that the configuration is a c.c.c. Thus, generically choosing the remaining two masses will not yield a c.c.c., regardless of relabelling or rescaling.
- (4) In the four-body problem, for a given choice and ordering of the masses, there exists a convex c.c. [15, 30]. It is not known whether this configuration is unique. If we restrict to co-circular c.c.s, we conjecture that the configuration is indeed unique. However, a global argument on all of Γ seems technically quite challenging. We present some visual evidence for uniqueness in figures 8 and 9. By contradiction, if there existed a choice of masses for which two distinct c.c.c.s existed, then one of the curves in figure 9 would have to intersect itself or collapse to a point. This does not appear to be the case.
- (5) The right-hand graph in figure 6 shows the surface of masses in $m_2m_3m_4$ -space for which a co-circular c.c. exists. Attempts using Gröbner bases and symbolic computation software to find a single polynomial expression relating the three masses were unsuccessful. However, if $I = \langle F_1, F_2, F_3 \rangle \subset \mathbb{C}[r_{13}, r_{14}, r_{23}, r_{24}, r_{34}]$ is the ideal generated by the first three equations in system I, then it is possible (using Maple) to compute a Gröbner basis for I using a graded reverse lexicographic order. From this calculation, we deduce that the dimension of the algebraic variety of I is two. Then, since the equations for m_2, m_3 and m_4 are rational functions of the distance variables, it follows that the image of Γ under the three mass functions is contained in a two-dimensional algebraic variety in $\mathbb{R}[m_2, m_3, m_4]$. Consequently, for a generic choice of positive masses, there does not exist a c.c.c. We thank John Little for providing the details for this argument.

5. Some geometric facts about co-circular c.c.s

We now provide some precise bounds on the mutual distances as well as some interesting geometric facts about four-body c.c.c.s.

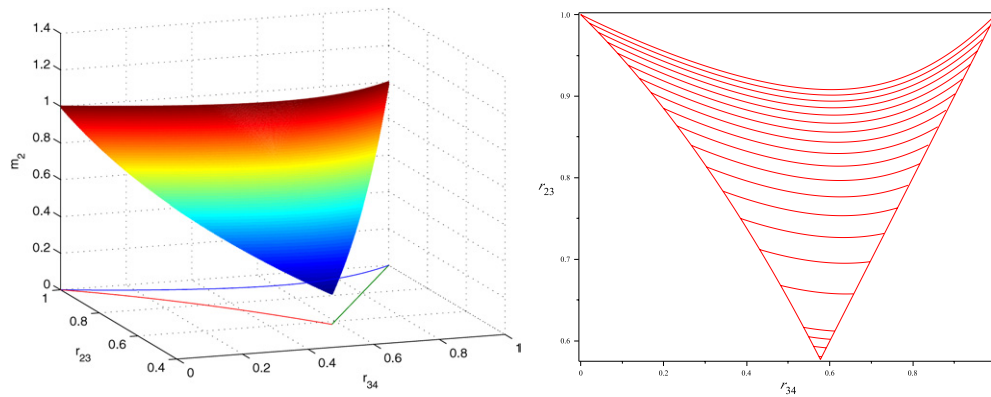


Figure 8. Surface and contour plots of the mass m_2 as a function of the exterior side-lengths r_{23} and r_{34} .

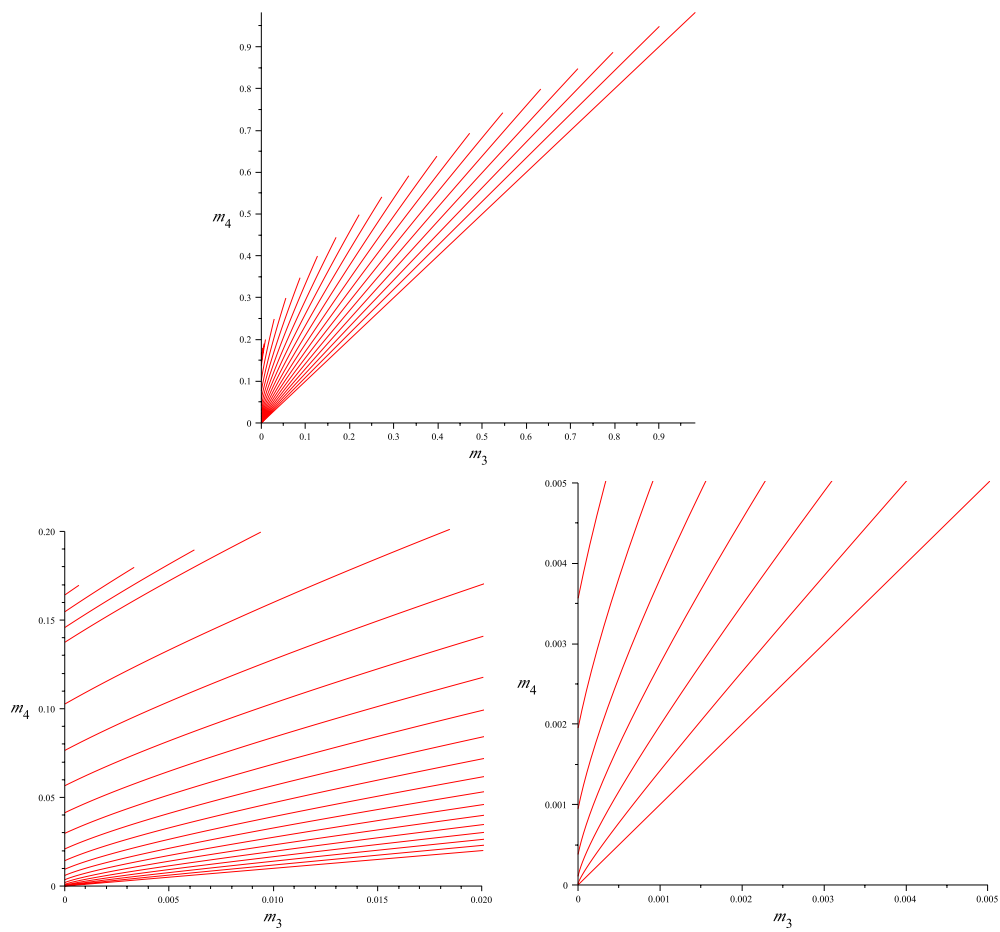


Figure 9. Each curve shows the relationship between m_3 and m_4 along a particular level curve of $m_2 = \text{constant}$. The bottom two figures are magnifications near the origin of the top figure.

5.1. Bounds on the mutual distances

Let $\hat{\tau}$ be the minimum value of the function $\tau(x)$ defined in section 3.2 for the isosceles trapezoid family. From remark 3 after lemma 3.3, $\hat{\tau} \approx 0.908\,026$. It turns out that this value is also the minimum value for r_{14} on all of Γ .

Proposition 5.1. *The exterior sides and the diagonals of a c.c.c. in Γ are restricted by*

$$0 < r_{34} \leq 1, \quad \frac{1}{\sqrt{3}} < r_{23} \leq 1, \quad \hat{\tau} \leq r_{14} \leq 1, \quad 1 < r_{13}, r_{24} \leq \sqrt{2}.$$

These bounds are sharp.

Proof. The bounds for the distances r_{34} and r_{23} follow directly from theorem 4.3. Using a similar argument as in the proof of theorem 4.3, we have that $\frac{\partial F}{\partial r_{23}} > 0$ on the interior of $\mathcal{D}$. By the implicit function theorem, it follows that $\frac{\partial r_{14}}{\partial r_{23}} < 0$ on the interior of $\mathcal{D}$. Since $r_{14} = 1$ on the lower boundaries of $\mathcal{D}$ (the degenerate equilateral triangle or kite families), it follows that the minimum value for r_{14} must occur on the upper boundary of $\mathcal{D}$ given by the curve $r_{23} = \tau(r_{34})$. However, on this curve we have $r_{14} = r_{23}$ (the isosceles trapezoid case). Therefore, the minimum value of r_{14} occurs at the minimum value of r_{23} on τ . This is precisely the number $\hat{\tau} \approx 0.908\,026$.

The lower bounds on the diagonals come directly from the definition of Γ and follow from inequality (27). To see that they are sharp, note that as (r_{34}, r_{23}) approaches the lower-left boundary curve of $\mathcal{D}$, $r_{34}^2 + r_{23}^2 + r_{34}r_{23} = 1$, r_{14} approaches one due to the second inequality in (28). Then, inequality (27) becomes an equality and r_{24} approaches one. The diagonal r_{13} can also become arbitrarily close to one in the upper-left corner of $\mathcal{D}$. Here, we have r_{34} approaching zero and both r_{23} and r_{14} approaching one. Equation (24) then shows that r_{13} also approaches one.

Next, by equation (15), $r_{13} \leq \sqrt{2}$ is equivalent to $ab \leq 2c$ or

$$(r_{34} + r_{14}r_{23})(r_{14} + r_{23}r_{34}) \leq 2(r_{23} + r_{14}r_{34}).$$

This, in turn, can be written as

$$r_{14}r_{34}(1 - r_{23}^2) \geq r_{23}(r_{14}^2 + r_{34}^2 - 2). \quad (36)$$

Since $r_{14} \leq 1$ and $r_{34} \leq 1$, the right-hand side of (36) is non-positive, while $r_{23} \leq 1$ implies that the left-hand side of (36) is non-negative. This verifies (36) and proves that $r_{13} \leq \sqrt{2}$. Since $r_{24} \leq r_{13}$, the upper bound on r_{24} is also established. These bounds are attained, but only at the square configuration, since $r_{23} = r_{34} = r_{14} = 1$ is the only possible way to make inequality (36) an equality. $\square$

For a general four-body convex c.c. with diagonals r_{13} and r_{24} , and $r_{13} \geq r_{24}$, a straightforward geometric argument shows that $1 \leq \frac{r_{13}}{r_{24}} < \sqrt{3}$. This is part of theorem 4.1 in [24]. Requiring the bodies to lie on a common circle reduces this bound considerably. More precisely, the ratio r_{13}/r_{24} of the diagonals satisfies the following sharp bounds:

$$1 \leq \frac{r_{13}}{r_{24}} < \frac{2}{\sqrt{3}} \approx 1.1547.$$

This follows because $\partial(\frac{r_{13}}{r_{24}})/\partial r_{23} < 0$ on the interior of $\mathcal{D}$. The minimum value of the ratio of the diagonals (one) occurs at the upper boundary of $\mathcal{D}$, at the isosceles trapezoid case (T), while the maximum value occurs at $r_{34} = r_{23} = 1/\sqrt{3}$, the point of intersection of the kite family (K) and the family of degenerate c.c.s (D).

Furthermore, the ratio of the two opposite exterior sides r_{14} and r_{23} satisfies the following sharp bounds:

$$1 \leq \frac{r_{14}}{r_{23}} < \sqrt{3}.$$

This follows since $\partial(\frac{r_{14}}{r_{23}})/\partial r_{23} < 0$ on the interior of $\mathcal{D}$. Once again, the minimum value of the ratio in question occurs at the upper boundary of $\mathcal{D}$. This value is one, since $r_{14} = r_{23}$ on the boundary (T). The maximum occurs at the lower boundary of $\mathcal{D}$ where $r_{14} = 1$ for both (K) and (D). Thus, $\frac{r_{14}}{r_{23}} = \frac{1}{r_{23}}$ has a supremum of $\sqrt{3}$, since the infimum of r_{23} is $1/\sqrt{3}$.

It is interesting to note that while the largest values of $\frac{r_{13}}{r_{24}}$ and $\frac{r_{14}}{r_{23}}$ occur very close to the point of intersection between the kite and degenerate cases, the larger values of the ratio $\frac{r_{12}}{r_{34}} = \frac{1}{r_{34}}$ occur near the intersection of the trapezoid and degenerate cases (upper-left corner of $\mathcal{D}$). However, this ratio has no upper bound since r_{34} can be arbitrarily small.

5.2. Interior angles, arcs and semi-circles

According to Long [14], any interior angle of a four-body convex c.c. must lie between 60° and 150° . Since opposite angles of a cyclic quadrilateral are supplementary, we can amend this bound for our problem. In fact, the maximum interior angle of a four-body c.c.c. lies between 90° and 120° while the minimum interior angle lies between 60° and 90° . These bounds are sharp as they are identical to those obtained for the kite and trapezoid families.

A few restrictions on the possible shape and angles of a c.c.c. follow nicely from the *perpendicular bisector theorem* due to Conley. Suppose that x is a planar c.c. For any pair of bodies x_i, x_j , consider the perpendicular bisector of the line segment joining the two bodies. Taken with the line passing through x_i and x_j , a coordinate system is formed dividing the plane into four quadrants. The union of two opposite quadrants (taking the first with the third quadrant or the second with the fourth) excluding the axes forms an open cone. The perpendicular bisector theorem states that if one of these cones contains a body of the configuration, then so must the other cone. An elegant proof can be found in Moeckel's nice paper on c.c.s [20].

As in section 3, denote θ_{12} as the arc along the circumscribing circle between bodies 1 and 2. By construction, this is the largest possible arc between consecutive bodies. Using the perpendicular bisector theorem and some simple geometry, it is possible to show that an upper bound for θ_{12} is 144° . However, numerical calculations suggest that this bound can be decreased to 120° and that the value of θ_{12} decreases as r_{23} increases through $\mathcal{D}$. The maximum θ_{12} -value occurs at a minimum of the circumradius. Both the kite and trapezoid families have a supremum of 120° . Since θ_{12} is the largest arc along the circumscribing circle, it follows that the minimum value of θ_{12} is 90° , attained at the square configuration.

Moreover, using the perpendicular bisector theorem, it is straightforward to see that no c.c.c. can lie entirely in a semi-circle. This result easily extends to any number of bodies, not just four-body c.c.c.s. Finally, using the law of cosines and the definition of Γ , we find that three bodies of a c.c.c. cannot lie on the same half of the circumscribing circle as the longest side r_{12} unless the configuration is a kite.

6. Conclusion

We have used mutual distances as coordinates to fully classify the set of co-circular c.c.s in the planar four-body problem. This set is a two-dimensional surface, a graph over the two

shortest exterior side-lengths, whose boundaries are important symmetric families consisting of a kite, an isosceles trapezoid and an equilateral triangle with a zero mass off the triangle. A specific ordering of the masses has been demonstrated to hold for any co-circular c.c. Somewhat surprisingly, if just two masses of a co-circular c.c. are equal, then the configuration is symmetric, either a kite or an isosceles trapezoid. The set of positive masses which yield a co-circular c.c. lies in a two-dimensional algebraic variety. We conjecture that for those masses for which a co-circular c.c. exists, the configuration is unique.

By restricting the central configuration to lie on a circle, we have been able to utilize the resulting geometry as well as Ptolemy's theorem to obtain many results. It is our hope that this may spark similar efforts to use constraints in an attempt to classify general convex, four-body c.c.s. For example, is it useful to search for c.c.s under the constraint $P = c$, where $c \neq 0$? Are there other feasible geometric constraints whose level sets are tangent to $V = 0$? Could these methods help find a foliation of the space of convex c.c.s and suggest a technique for proving the uniqueness of such configurations? Finally, are any of the relative equilibria corresponding to the co-circular c.c.s on Γ linearly stable and if so, is it possible to rigorously prove this is the case? We hope to explore some of these interesting questions in future work.

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