Physica D 238 (2009) 699–705

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Symmetric periodic orbits near a heteroclinic loop in $\mathbb{R}^3$ formed by two singular points, a semistable periodic orbit and their invariant manifolds

Montserrat Corbera [a], Jaume Llibre [b], Marco Antonio Teixeira [c,*]

[a] Departament de Tecnologies Digitals i de la Informació, Universitat de Vic, Laura 13, 08500 Vic, Barcelona, Catalonia, Spain

[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 – Bellaterra, Barcelona, Catalonia, Spain

[c] Departamento de Matemática, Universidade Estadual de Campinas, Caixa Postal 6065, 13083–970 Campinas SP, Brazil

ARTICLE INFO

Article history:
Received 9 May 2008
Received in revised form
11 December 2008
Accepted 5 January 2009
Available online 19 January 2009
Communicated by B. Sandstede

Keywords:
Heteroclinic loop
Symmetric periodic orbits
Polynomial vector fields

ABSTRACT

In this paper, we consider $\mathcal{C}^1$ vector fields X in $\mathbb{R}^3$ having a "generalized heteroclinic loop" $\mathcal{L}$ which is topologically homeomorphic to the union of a 2–dimensional sphere $\mathbb{S}^2$ and a diameter Γ connecting the north with the south pole. The north pole is an attractor on $\mathbb{S}^2$ and a repeller on Γ. The equator of the sphere is a periodic orbit unstable in the north hemisphere and stable in the south one. The full space is topologically homeomorphic to the closed ball having as boundary the sphere $\mathbb{S}^2$. We also assume that the flow of X is invariant under a topological straight line symmetry on the equator plane of the ball. For each $n \in \mathbb{N}$, by means of a convenient Poincaré map, we prove the existence of infinitely many symmetric periodic orbits of X near $\mathcal{L}$ that gives n turns around $\mathcal{L}$ in a period. We also exhibit a class of polynomial vector fields of degree 4 in $\mathbb{R}^3$ satisfying this dynamics.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

In this paper, we shall analyze the dynamics of $\mathcal{C}^1$ vector fields X in $\mathbb{R}^3$ having a "generalized heteroclinic loop" $\mathcal{L}$. It is well known that the dynamics of a vector field in a neighborhood of a homoclinic or a heteroclinic loop may be very rich and complex. So, from a dynamical point of view, the study of the behavior of a vector field near these objects is very interesting.

Usually the heteroclinic loops that have been studied in the literature come from heteroclinic loops having a transversal intersection of a stable invariant manifold and the unstable one along their orbits different from singular points, see for instance [1–3]. These heteroclinic loops generically have the Bernoulli shift as a subsystem and consequently the vector field possesses infinitely many periodic orbits near the heteroclinic loop. The mechanism that generates the infinitely many periodic orbits near these heteroclinic loops is the transversality, whereas for the generalized heteroclinic loops studied in this paper the mechanism is the symmetry. The origin of the transversality and of the Bernoulli shift can be found in [3], but one of the best applications of this dynamics can be found in [2] where the author uses a heteroclinic transversal loop in order to study the dynamics near the parabolic orbits of the Sitnikov problem.

In this paper, our generalized heteroclinic loop has no transversal intersection between the stable and unstable invariant manifolds, these invariant manifolds coincide. So it does not posses the Bernoulli shift as a subsystem, see for more details [2,4]. Nevertheless it still conserves infinitely many periodic orbits near it. There are very few articles in the literature studying heteroclinic loops that have no transversal intersection between their stable and unstable invariant manifolds and that these coincide, see for instance [5,6]. Moreover, usually the invariant manifolds that appear associated to the heteroclinic loops studied in the literature have the same dimension. In our heteroclinic loop these objects have different dimension. From the point of view of having invariant manifolds with different dimension our heteroclinic loop can recall to the homoclinic loops of Silnikov type (see for example [7–10]), but the main difference with these kind of loops is that their invariant manifolds cannot coincide as in our heteroclinic loop.

Of course there are many other kinds of phenomena related with homoclinic and heteroclinic loops, as for instance the ones related with blue sky catastrophes (see for instance [11,12]), or the ones related with homoclinic snaking (for example see [13–16]), and several others. But all these other phenomena are different to our generalized heteroclinic loops.

Modulo a diffeomorphism we assume that $\mathcal{L}$ is the union of the sphere $\mathbb{S}^2$ of radius 1 centered at the origin of $\mathbb{R}^3$, and the open diameter Γ along the z-axis.

Let $X = (f(x, y, z), g(x, y, z), h(x, y, z))$ be a $\mathcal{C}^1$ vector field defined on the closed ball $\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1\}$.

* Corresponding author. Tel.: +55 19 32891851; fax: +55 19 32895766.
E-mail addresses: montserrat.corbera@uvic.cat (M. Corbera), jllibre@mat.uab.cat (J. Llibre), teixeira@ime.unicamp.br (M.A. Teixeira).

0167-2789/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2009.01.002