PERIODIC ORBITS OF PERTURBED NON–AXIALLY SYMMETRIC POTENTIALS IN 1:1:1 AND 1:1:2 RESONANCES

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ABSTRACT. We analytically study the Hamiltonian system in $\mathbb{R}^6$ with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) + \varepsilon(az^3 + z(bx^2 + cy^2)),$$

being $a, b, c \in \mathbb{R}$ with $c \neq 0$, ε a small parameter, and ω_1, ω_2 and ω_3 the unperturbed frequencies of the oscillations along the x, y and z axis, respectively. For $|\varepsilon| > 0$ small, using averaging theory of first and second order we find periodic orbits in every positive energy level of H whose frequencies are $\omega_1 = \omega_2 = \omega_3/2$ and $\omega_1 = \omega_2 = \omega_3$, respectively (the number of such periodic orbits depends on the values of the parameters a, b, c). We also provide the shape of the periodic orbits and their linear stability.

1. Introduction and statement of the main results

Over the last half century dynamical systems perturbing a harmonic oscillator in dimension 2 or 3 have been used extensively to study the local motion around equilibrium points or periodic orbits and their stability. This kind of studies are relevant in many physical, chemical,... problems of the sciences. The study of these motions has been made mainly using several numerical techniques, see for instance [1, 3, 4, 5, 6, 7, 8, 11, 14, 15, 21, 22] to cite just a few.

We consider the following potential

$$V = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) + \varepsilon(az^3 + z(bx^2 + cy^2)),$$

of a three-dimensional dynamical system composed of perturbed oscillators, where $a, b, c \in \mathbb{R}$ are parameters, ω_1, ω_2 and ω_3 are the unperturbed frequencies of the oscillators along the x, y and the z axes respectively, and ε is the small perturbation parameter.

The Hamiltonian associated to the potential V is

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2}{2} + \varepsilon(az^3 + z(bx^2 + cy^2)), \tag{1}$$

and the corresponding Hamiltonian system is

$$\begin{array}{ll} \dot{x} = p_x, & \dot{p_x} = -\omega_1^2 x - 2\varepsilon bxz, \\ \dot{y} = p_y, & \dot{p_y} = -\omega_2^2 y - 2\varepsilon cyz, \\ \dot{z} = p_z, & \dot{p_z} = -\omega_3^2 z - \varepsilon(bx^2 + cy^2 + 3az^2). \end{array} \tag{2}$$

As usual the dot denotes derivative with respect to the time $t \in \mathbb{R}$. Due to the physical meaning the frequencies ω_1, ω_2 and ω_3 are all positive.

The objective of this paper is to study analytically the existence of periodic orbits of the Hamiltonian system (2) and their linear stability. The study of periodic orbits plays a key role in understanding the orbital structure of a given differential system. The motion in neighborhood of a periodic orbit can be determined by their kind of stability. More precisely, the stable periodic orbits explain the dynamics of bounded regular motion, while the unstable ones helps to understand the possible chaotic motion of the system.

The Hamiltonian here studied has been used for modeling the motion in a central region of a galaxy. It is a particular Hamiltonian of the class of Hamiltonians denoted by some authors generalized Hénon–Heiles Hamiltonian in dimension 3. There are several papers studying the dynamics of these class of Hamiltonians. Now we shall mention some of the closer papers to the Hamiltion (1) here studied. In 1998 Ferrer et al. studied this Hamiltonian in the particular case $a = -1/3$, $b = c = 1$ in [9], where they proved numerically the existence of

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