PERIODIC ORBITS OF PERTURBED NON–AXIALLY SYMMETRIC POTENTIALS IN 1:1:1 AND 1:1:2 RESONANCES

MONTSERRAT CORBERA[1], JAUME LLIBRE[2] AND CLAUDIA VALLS[3]

ABSTRACT. We analytically study the Hamiltonian system in $\mathbb{R}^6$ with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) + \varepsilon(az^3 + z(bx^2 + cy^2)),$$

being $a, b, c \in \mathbb{R}$ with $c \neq 0$, ε a small parameter, and ω_1, ω_2 and ω_3 the unperturbed frequencies of the oscillations along the x, y and z axis, respectively. For $|\varepsilon| > 0$ small, using averaging theory of first and second order we find periodic orbits in every positive energy level of H whose frequencies are $\omega_1 = \omega_2 = \omega_3/2$ and $\omega_1 = \omega_2 = \omega_3$, respectively (the number of such periodic orbits depends on the values of the parameters a, b, c). We also provide the shape of the periodic orbits and their linear stability.

1. Introduction and statement of the main results

Over the last half century dynamical systems perturbing a harmonic oscillator in dimension 2 or 3 have been used extensively to study the local motion around equilibrium points or periodic orbits and their stability. This kind of studies are relevant in many physical, chemical,... problems of the sciences. The study of these motions has been made mainly using several numerical techniques, see for instance [1, 3, 4, 5, 6, 7, 8, 11, 14, 15, 21, 22] to cite just a few.

We consider the following potential

$$V = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) + \varepsilon(az^3 + z(bx^2 + cy^2)),$$

of a three-dimensional dynamical system composed of perturbed oscillators, where $a, b, c \in \mathbb{R}$ are parameters, ω_1, ω_2 and ω_3 are the unperturbed frequencies of the oscillators along the x, y and the z axes respectively, and ε is the small perturbation parameter.

The Hamiltonian associated to the potential V is

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2}{2} + \varepsilon(az^3 + z(bx^2 + cy^2)), \tag{1}$$

and the corresponding Hamiltonian system is

$$\begin{aligned} \dot{x} &= p_x, & \dot{p_x} &= -\omega_1^2 x - 2\varepsilon bxz, \\ \dot{y} &= p_y, & \dot{p_y} &= -\omega_2^2 y - 2\varepsilon cyz, \\ \dot{z} &= p_z, & \dot{p_z} &= -\omega_3^2 z - \varepsilon(bx^2 + cy^2 + 3az^2). \end{aligned} \tag{2}$$

As usual the dot denotes derivative with respect to the time $t \in \mathbb{R}$. Due to the physical meaning the frequencies ω_1, ω_2 and ω_3 are all positive.

The objective of this paper is to study analytically the existence of periodic orbits of the Hamiltonian system (2) and their linear stability. The study of periodic orbits plays a key role in understanding the orbital structure of a given differential system. The motion in neighborhood of a periodic orbit can be determined by their kind of stability. More precisely, the stable periodic orbits explain the dynamics of bounded regular motion, while the unstable ones helps to understand the possible chaotic motion of the system.

The Hamiltonian here studied has been used for modeling the motion in a central region of a galaxy. It is a particular Hamiltonian of the class of Hamiltonians denoted by some authors generalized Hénon–Heiles Hamiltonian in dimension 3. There are several papers studying the dynamics of these class of Hamiltonians. Now we shall mention some of the closer papers to the Hamiltion (1) here studied. In 1998 Ferrer et al. studied this Hamiltonian in the particular case $a = -1/3$, $b = c = 1$ in [9], where they proved numerically the existence of

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some periodic orbits of the corresponding Hamiltonian system, and also showed analytically the existence of three circular periodic orbits. Four years later in [10] Ferrer et al. improved these previous results for the Hamiltonian (1) with $b = c$, we remark that in this case scaling the variables it is not restrictive to take $b = c = 1$. Haffmann and van der Meer [13] also in 2002 studied the Hamiltonian (1) for the values of the parameters $a = 1$ and $b = c$. In these three quoted previous papers the authors take equal frequencies, i.e. $\omega_1 = \omega_2 = \omega_3$. Also in 2002 Lanchares et al. [16] studied the Hamiltonian (1) with $\omega_1 = \omega_2 \neq \omega_3$ and for fixed values of the parameters $a, b = c$. We remark that in this mentioned four papers all the Hamiltonians studied have an axial symmetry and the major part of the results on their periodic orbits are numerical.

When $c = 0$ the Hamiltonian becomes $H = H_1 + H_2$ where

$$H_1 = \frac{1}{2}p_y^2 + \frac{\omega_2^2 y^2}{2}, \qquad H_2 = \frac{1}{2}(p_x^2 + p_z^2) + \frac{\omega_1^2 x^2 + \omega_3^2 z^2}{2} + \varepsilon(az^3 + bx^2 z).$$

The Hamiltonian H_1 is the well known harmonic oscillator and the Hamiltonian H_2 is a generalized Hénon Heiles Hamiltonian in dimension 2. Note that the Hamiltonian system associated to $H = H_1 + H_2$ splits into two separate Hamiltionian systems, both are widely studied by several authors, see for instance [12, 14, 17, 19]. So in this work we do not consider the case $c = 0$.

Without loss of generality we can assume that $c = 1$. Indeed, taking the change of variables

$$X = c\,x,\ Y = c\,y,\ Z = c\,z,\ P_X = c\,p_x,\ P_Y = c\,p_y,\ P_Z = c\,p_z,$$

we transform system (2) into system

$$\begin{array}{ll} \dot{X} = P_X, & \dot{P_X} = -\omega_1^2 X - 2\varepsilon\tilde{b}XZ, \\ \dot{Y} = P_Y, & \dot{P_Y} = -\omega_2^2 Y - 2\varepsilon YZ, \\ \dot{Z} = P_Z, & \dot{P_Z} = -\omega_3^2 Z - \varepsilon(\tilde{b}X^2 + Y^2 + 3\tilde{a}Z^2), \end{array} \tag{3}$$

in which $\tilde{a} = a/c$ and $\tilde{b} = b/c$. From now on in order to avoid heavy notation we denote again $(X, Y, Z, P_X, P_Y, P_Z, \tilde{a}, \tilde{b})$ as $(x, y, z, p_x, p_y, p_z, a, b)$, i.e. we work with system (2) with $c = 1$.

In this paper we will study the periodic orbits and their linear stability of the Hamiltonian system (2) by using the averaging theory of first and second order described in Section 2. In order to apply the averaging theory for computing periodic orbits of a differential system we must overcome the following steps:

(a) Find a convenient change of variables which allows to write the differential systems into the normal form of the averaging theory.
(b) Compute for the differential system in normal form its averaged function. For doing that some integrals must be computed.
(c) Compute the simple zeros of the averaged function, each one of these zeros provides a periodic orbit of the initial differential system.
(d) In the case of Hamiltonian differential systems its periodic orbits usually belong to families of periodic orbits depending on the energy and consequently they are not isolated in the set of periodic orbits. Hence for such systems it is necessary to apply the averaging theory in each energy level fixing the Hamiltonian.

With the normal form that we use for applying the averaging theory we only can study the periodic orbits of the Hamiltonian system (2) with $c = 1$ having frequencies either $\omega_2 = \omega_1$, $\omega_3 = 2\omega_1$ or $\omega_1 = \omega_2 = \omega_3$. These periodic orbits are the unique ones which come from the simple zeros of the averaged function associated to the Hamiltonian system (2) with $c = 1$. Thus our main results are the following.

Theorem 1. *The following statements hold for the Hamiltonian system* (2) *with* $c = 1$*.*

(i) *If* $\omega_2 = \omega_1$ *and* $\omega_3 = 2\omega_1 > 0$*, using the averaging theory of first order for* $|\varepsilon| \neq 0$ *sufficiently small at every positive energy level* $H = h$ *we find the following periodic solutions* $(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t))$*:*
  (a) *for each* $b \in (-1, 0) \cup (0, 1)$ *the two unstable periodic solutions*

$$\frac{\sqrt{h}}{\sqrt{3}}\Big(2\cos(\omega_1 t), 0, (-1)^j \frac{\cos(2\omega_1 t)}{\sqrt{2}}, -2\sin(\omega_1 t), 0, (-1)^{j+1}\sqrt{2}\sin(2w_1 t)\Big) + O(\varepsilon),$$

  with $j = 0, 1$*;*
  (b) *for each* $b \in \mathbb{R} \setminus \{0\}$ *the one unstable periodic solution*

$$\frac{\sqrt{h}}{\sqrt{2}}\Big(0, 0, \sin(2\omega_1 t), 0, 0, 2\cos(2\omega_1 t)\Big) + O(\varepsilon);$$

(c) *and for each* $b \in (-\infty,-1)\cup(1,+\infty)$ *the four unstable periodic solutions*

$$\frac{\sqrt{h}}{\sqrt{3}}\Big(0, 2\cos(\tilde\omega(t)), \frac{(-1)^j}{\sqrt{2}}\cos(2\tilde\omega(t)), 0, -2\sin(\tilde\omega(t)), (-1)^{j+1}\sqrt{2}\sin(2\tilde\omega(t))\Big) + O(\varepsilon),$$

with $j=0,1,$ *where* $\tilde\omega(t)=\omega_1 t-(-1)^k\arccos(1/b)/2$ *with* $k=0,1$.

(ii) *If* $\omega_3=2\omega_2$ *and* $\omega_3\neq 2\omega_1>0$, *or* $\omega_3\neq 2\omega_2$ *and* $\omega_3=2\omega_1>0$ *the normal form of the Hamiltonian system* (2) *with* $c=1$ *that we use for applying the averaging theory of first does not provide any information about the periodic orbits of the system.*

(iii) *If* $\omega_3\neq 2\omega_2$ *and* $\omega_3\neq 2\omega_1$ *then the averaging of first order is identically zero, and we can apply the averaging theory of second order.*

Theorem 2. *Consider the Hamiltonian system* (2) *with* $c=1$. *If* $\omega_3\neq 2\omega_2$ *and* $\omega_3\neq 2\omega_1$ *and* $(\omega_2-\omega_3)^2+(\omega_1-\omega_3)^2\neq 0$ *the normal form of the Hamiltonian system* (2) *with* $c=1$ *that we use for applying the averaging theory of second order does not provide any information about the periodic orbits of the system, and we cannot go to the third order averaging theory because the second averaged function is not identically zero.*

Note that the condition $\omega_3\neq 2\omega_2$ and $\omega_3\neq 2\omega_1$ in Theorem 2 is statement (iii) of Theorem 1. The condition $(\omega_2-\omega_3)^2+(\omega_1-\omega_3)^2=0$ corresponds to $\omega_1=\omega_2=S\omega_3$ and we shall see this is the unique case where we can apply second order averaging so we treat it in Theorem 4.

We introduce some notation. Let

$$\begin{aligned}
A_{a,b} &= \arccos\Big(\frac{-45a^2+18ab+4b^2}{3b(a-2b)}\Big), \quad B_b=\arccos\Big(\frac{2(b^2+3b-5)}{b(1-3b)}\Big),\\
C_{a,b} &= \arccos\Big(-\frac{2(7b^2+3b+9a(b-2)+1)}{b(1-18a+21b)}\Big),\\
\overline{r} &= \frac{\overline{r}_1}{(3a-7)\overline{r}_2}, \quad \overline{R}=\frac{6\left(1-9a^2\right)h}{\overline{r}_2}, \quad \overline{\rho}=\frac{2(3a-1)\overline{\rho}_1 h}{(3a-7)\overline{r}_2}\\
\overline{r}_1 &= -72(a-2)(3a+1)(6a+1)h, \quad \overline{r}_2=63a^3-159a^2-83a-17,\\
\overline{\rho}_1 &= 63a^3-96a^2-5a-26,\\
\tilde r &= \frac{\tilde r_1}{(b-1)\tilde r_2}, \quad \tilde R=\frac{\tilde R_1}{\tilde r_2}, \quad \tilde\rho=\frac{2bh(2b-a)\tilde\rho_1}{(b-1)\tilde r_2},\\
c_a &= \frac{c_{1A}}{2c_{2A1}c_{2A2}}, \quad c_b=\frac{c_{1B}}{6c_{2B}c_{2A2}},\\
\tilde r_1 &= 2(2-a)(9a^2(2+b)+2b(1+11b)-a(1+40b+18b^2))h,\\
\tilde r_2 &= 18a^3(b+1)-b(4+53b+4b^2)-a^2(37+80b+37b^2)\\
&\quad +2a(1+45b+45b^2+b^3),\\
\tilde R_1 &= 2b(-3a^2-5b+6a(b+1))h,\\
\tilde\rho_1 &= -18a^2b-9a^2+ab^2+40ab+18a-2b^2-22b,\\
C_A &= \frac{239-1936a+3703a^2+1896a^3+9153a^4-12096a^5+3969a^6}{4(1+3a)(1+6a)(-26-5a-96a^2+63a^3)},\\
C_B &= \frac{33-283a+274a^2+18a^3+945a^4-1323a^5}{12(1+3a)^2(1+6a)},\\
c_{1A} &= 162a^4(2+b)(1+2b)-18a^3(1+b)(2+b(173+2b))-16ab(1+b)(16\\
&\quad +b(145+16b))+b^2(263+b(626+263b))+a^2(1+b(1718+b(6116\\
&\quad +b(1718+b)))),\\
c_{2A1} &= 2b(11+b)+9a^2(1+2b)-a(18+b(40+b)),\\
c_{2A2} &= 9a^2(2+b)+2b(1+11b)-a(1+2b(20+9b)),\\
c_{1B} &= 162a^4(2+b)-15b^2(17+47b)+18a^3(-2+11b+34b^2)+8ab(31+302b\\
&\quad +166b^2)-a^2(-1+1413b+2601b^2+647b^3),\\
c_{2B} &= 3a^2+5b-6a(1+b).
\end{aligned} \tag{4}$$

Moreover, we also set

$$
\begin{aligned}
D_{1,a,b} &= 1 + b + b^2 + 3a(1+b), \quad D_{2,a,b} = 1 + 54a^2 + 7b + b^2 - 21a(1+b)),\\
D_{3,a,b} &= 1 + 54a^2 + 7b + 36b^2 - 35b^3 + 25b^4 + 3a(-7 - 7b - 35b^2 + 25b^3),\\
D_{4,a,b} &= -a + 18a^2 + 9b - 80ab + 36a^2b + 39b^2 - 37ab^2,\\
D_{5,a,b} &= 25 - 35b + 36b^2 + 54a^2b^2 + 7b^3 + b^4 - 3a(-25 + 35b + 7b^2 + 7b^3),\\
D_{6,a,b} &= 37a - 36a^2 - 39b + 80ab - 18a^2b - 9b^2 + ab^2,\\
D_{7,a,b} &= 36a^2b + 18a^2 - 37ab^2 - 80ab - a + 39b^2 + 9b,\\
D_{8,a,b} &= 18a^2b + 36a^2 - ab^2 - 80ab - 37a + 9b^2 + 39b,
\end{aligned}
$$

Finally, we define the domains

$$
\begin{aligned}
S_1 = &\Big\{(a,b)\in\mathbb{R}^2 : a\in(-1/3,2/15)\cup(1/3,2/3),\\
&\frac{(3a-2b)(3a+b)}{3(a-2b)b}>0,\ \frac{(15a-2b)(3a-b)}{3(a-2b)b}<0\Big\},\\
S_2 = &\Big\{(a,b)\in\mathbb{R}^2 : a\in(-\infty,-1/3)\cup(-1/3,2/3), b\in(-\infty,-2)\cup(1,2)\cup(5,+\infty)\Big\},\\
S_3 = &\Big\{(a,b)\in\mathbb{R}^2 : a<2/15,\\
&\frac{-2+36a-7b-35b^2}{b(-21b+18a-1)}>0,\ \frac{(b-1)(-2+36a+7b)}{b(-21b+18a-1)}>0\Big\},\\
S_4 = &\Big\{(a,b)\in\mathbb{R}^2 : \frac{(1+3a)(2+b)}{(b-1)D_{1,a,b}}<0,\ \frac{b}{D_{1,a,b}}<0,\\
&\frac{b(3a+b)(1+2b)}{(b-1)D_{1,a,b}}>0, a\neq\frac{b}{b+1}\Big\},\\
S_5 = &\Big\{(a,b)\in\mathbb{R}^2 : \frac{(-1+3a)(-2+36a-7b)}{(b-1)D_{2,a,b}}<0,\ \frac{b}{D_{2,a,b}}>0,\\
&\frac{b(3a-b)(-7+36a-2b)}{(b-1)D_{2,a,b}}>0, a-18a^2+5b+ab\neq 0\Big\},\\
S_6 = &\Big\{(a,b)\in\mathbb{R}^2 : \frac{(3a-1)(-2+36a-7b-35b^2)}{D_{3,a,b}}>0,\ \frac{b(3a+b)(2b-1)(5b-1)}{D_{3,a,b}}>0,\\
&\frac{b(7-36a+32b-35b^2))}{D_{3,a,b}}>0, (b-1)D_{4,a,b}\neq 0\Big\},\\
S_7 = &\Big\{(a,b)\in\mathbb{R}^2 : \frac{(3a+1)(b-2)(b-5)}{D_{5,a,b}}>0,\ \frac{b(3a-b)(-35-7b+36ab-2b^2)}{D_{5,a,b}}>0,\\
&\frac{b(-35-4(-8+9a)b+7b^2)}{D_{5,a,b}}>0, (b-1)D_{6,a,b}\neq 0\Big\},\\
S_8 = &\Big\{(a,b)\in\mathbb{R}^2 : b\in(-1/2,1/5)\cup(1/2,1),\ (3a-2b)(a-b)>0,\ b(b-a)>0,\\
&a\neq 2b,\ 3a\neq -b\ \Big\},\\
S_9 = &\Big\{(a,b)\in\mathbb{R}^2 : \frac{(b-1)(-7+36a-2b)}{-21+18a-b}<0,\ \frac{-35-7b+36ab-2b^2}{-21+18a-b}>0,\\
&b(3b-5a)>0, (15a-2b)(5a-3b)>0,\ a\neq b/3, a\neq 2b\Big\},\\
S_{10} = &\Big\{(a,b)\in\mathbb{R}^2 : b=(3a-1)/6, 1/21<a<1/3\Big\},\\
S_{11} = &\Big\{(a,b)\in\mathbb{R}^2 :\ b\neq(3a-1)/6,\ b\neq 3(a^2-2a)/(6a-5),\ \widetilde{r}\geqslant 0,\\
&\widetilde{R}>0,\ \widetilde{\rho}\geqslant 0,\ \frac{(18a^2-ab-a-5b)(ab+a-b)}{c_{2A1}c_{2A2}}<0,\ \frac{D_{7,a,b}D_{8,a,b}}{c_{2A1}c_{2A2}}>0,\\
&\frac{(ab+a-b)D_{7,a,b}}{c_{2B}c_{2A2}}>0,\ \frac{(18a^2-ab-a-5b)D_{8,a,b}}{c_{2B}c_{2A2}}<0\Big\}.
\end{aligned}
\tag{5}
$$

The domains S_i are not empty and they are plotted in the Appendix.

Theorem 3. *Consider the Hamiltonian system (2) with $c=1$ and $\omega_1=\omega_2=\omega_3$. Using averaging theory of second order for $|\varepsilon|\neq 0$ sufficiently small at every positive energy level $H=h$ we find the following periodic solutions $(x(t),y(t),z(t),p_x(t),p_y(t),p_z(t))$:*

(i) *for $(a,b)\in S_1$ the two unstable periodic solutions*

$$\frac{\sqrt{h}}{\sqrt{2}}\Big(0,0,\cos(2\omega_1 t-(-1)^j A_{a,b}/2),0,0,-2\sin(\omega_1 t-(-1)^j A_{a,b}/2)\Big)+O(\varepsilon),$$

with $j=0,1$;

(ii) *for* $(a,b)\in S_2$ *the two unstable periodic solutions*

$$\frac{\sqrt{2h}}{\sqrt{3}}\Big(0,\sqrt{\frac{-3a+2}{1-a}}\cos(\omega_1 t-(-1)^j B_b/2),\frac{1}{\sqrt{1-a}}\cos(\omega_1 t-(-1)^j B_b/2),0,$$
$$-\sqrt{\frac{-3a+2}{1-a}}\sin(\omega_1 t-(-1)^j B_b/2),-\frac{1}{\sqrt{1-a}}\sin(\omega_1 t-(-1)^j B_b/2)\Big)+O(\varepsilon),$$

with $j=0,1$*;*

(iii) *for* $(a,b)\in S_3$ *the two unstable periodic solutions*

$$\frac{\sqrt{2h}}{\sqrt{3}}\Big(0,\sqrt{\frac{2-15a}{3-5a}}\sin(\omega_1 t-(-1)^j C_{a,b}/2),\sqrt{\frac{7}{3-5a}}\cos(\omega_1 t-(-1)^j C_{a,b}/2),0,$$
$$\sqrt{\frac{2-15a}{3-5a}}\cos(\omega_1 t-(-1)^j C_{a,b}/2),-\sqrt{\frac{7}{3-5a}}\sin(\omega_1 t-(-1)^j C_{a,b}/2)\Big)+O(\varepsilon),$$

with $j=0,1$*;*

(iv) *for* $(a,b)\in S_4$ *the unstable periodic solution*

$$\sqrt{h}\Big(\sqrt{\frac{R_1}{D_{1,a,b}}}\cos(\omega_1 t),\sqrt{\frac{R_2}{D_{1,a,b}}}\cos(\omega_1 t),\sqrt{\frac{-b}{D_{1,a,b}}}\cos(\omega_1 t),-\sqrt{\frac{R_1}{D_{1,a,b}}}\sin(\omega_1 t),$$
$$-\sqrt{\frac{R_2}{D_{1,a,b}}}\sin(\omega_1 t),-\sqrt{\frac{-b}{D_{1,a,b}}}\sin(\omega_1 t)\Big)+O(\varepsilon),$$

where $R_1=(1+3a)(2+b)/(1-b)$ *and* $R_2=b(3a+b)(1+2b)/(b-1)$*;*

(v) *for* $(a,b)\in S_5$ *the unstable periodic solution*

$$\sqrt{h}\Big(\sqrt{\frac{R_3}{D_{2,a,b}}}\cos(\omega_1 t),\sqrt{\frac{R_4}{D_{2,a,b}}}\cos(\omega_1 t),\sqrt{\frac{5b}{D_{2,a,b}}}\sin(\omega_1 t),$$
$$-\sqrt{\frac{R_3}{D_{2,a,b}}}\sin(\omega_1 t),-\sqrt{\frac{R_4}{D_{2,a,b}}}\sin(\omega_1 t),\sqrt{\frac{5b}{D_{2,a,b}}}\cos(\omega_1 t)\Big)+O(\varepsilon),$$

where $R_3=(3a-1)(-2+36a-7b)/(1-b)$ *and* $R_4=b(3a-b)(-7+36a-2b)/(b-1)$*;*

(vi) *for* $(a,b)\in S_6$ *the periodic solution*

$$\sqrt{h}\Big(\sqrt{R_5}\cos(\omega_1 t),\sqrt{R_6}\sin(\omega_1 t),\sqrt{R_7}\cos(\omega_1 t),$$
$$-\sqrt{R_5}\sin(\omega_1 t),\sqrt{R_6}\cos(\omega_1 t),-\sqrt{R_7}\sin(\omega_1 t)\Big)+O(\varepsilon),$$

where $R_5=(3a-1)(-2+36a-7b-35b^2)/D_{3,a,b}$*,* $R_6=5b(3a+b)(2b-1)(5b-1)/D_{3,a,b}$*,* $R_7=b(7-36a+32b-35b^2)/D_{3,a,b}$*, where for different values of* $(a,b)\in S_6$ *the solution can be either linearly stable or unstable;*

(vii) *for* $(a,b)\in S_7$ *the periodic solution*

$$\sqrt{h}\Big(\sqrt{R_8}\cos(\omega_1 t),\sqrt{R_9}\sin(\omega_1 t),\sqrt{R_{10}}\sin(\omega_1 t),$$
$$-\sqrt{R_8}\sin(\omega_1 t),\sqrt{R_9}\cos(\omega_1 t),\sqrt{R_{10}}\cos(\omega_1 t)\Big)+O(\varepsilon),$$

where $R_8=5(3a+1)(b-2)(b-5)/D_{5,a,b}$*,* $R_9=b(3a-b)(-35-7b+36ab-2b^2)/D_{5,a,b}$*,* $R_{10}=b(-35-4(-8+9a)b+7b^2)/D_{5,a,b}$*, where for different values of* $(a,b)\in S_7$ *the solution can be either linearly stable or unstable;*

(viii) *for* $(a,b)\in S_8$ *the unstable periodic solution*

$$\sqrt{\frac{2h}{3}}\Big(\sqrt{\frac{3a-2b}{a-b}}\cos(\omega_1 t),0,\sqrt{\frac{-b}{a-b}}\cos(\omega_1 t),$$
$$-\sqrt{\frac{3a-2b}{a-b}}\sin(\omega_1 t),0,-\sqrt{\frac{-b}{a-b}}\sin(\omega_1 t)\Big)+O(\varepsilon),$$

(ix) *for $(a,b) \in S_9$ the unstable periodic solution*

$$\sqrt{\frac{2h}{3}}\Big(\sqrt{\frac{15a-2b}{5a-3b}}cos(\omega_1 t), 0, \sqrt{\frac{-7b}{5a-3b}}\sin(\omega_1 t),$$
$$-\sqrt{\frac{15a-2b}{5a-3b}}\sin(\omega_1 t), 0, \sqrt{\frac{-7b}{5a-3b}}\cos(\omega_1 t)\Big) + O(\varepsilon),$$

(x) *for $(a,b) \in S_{10}$ the eight unstable periodic solutions*

$$\Big(\sqrt{\overline{r}}\cos(\omega_1 t), (-1)^{l_1}\overline{\rho}\cos(\overline{\omega}_a(t)), (-1)^{l_2}\sqrt{\overline{R}}\cos(\overline{\omega}_b(t)), -\sqrt{\overline{r}}\sin(\omega_1 t),$$
$$(-1)^{l_1+1}\overline{\rho}\sin(\overline{\omega}_a(t)), (-1)^{l_2+1}\sqrt{\overline{R}}\sin(\overline{\omega}_b(t))\Big) + O(\varepsilon),$$

where $\overline{r}$, $\overline{R}$, $\overline{\rho}$ as in (4), $\overline{\omega}_a(t) = \omega_1 t - (-1)^{j_1}\arccos(C_A)/2$, $\overline{\omega}_b(t) = \omega_1 t + (-1)^{j_1}\arccos(C_B)/2$, C_A and C_B as in (4) and $j_1, l_1, l_2 \in \{0,1\}$.

(xi) *for $(a,b) \in S_{11}$ the eight periodic solutions*

$$\Big(\sqrt{\widetilde{r}}\cos(\omega_1 t), (-1)^{l_1}\widetilde{\rho}\cos(\widetilde{\omega}_a(t)), (-1)^{l_2}\sqrt{\widetilde{R}}\cos(\widetilde{\omega}_b(t)), -\sqrt{\widetilde{r}}\sin(\omega_1 t),$$
$$(-1)^{l_1+1}\widetilde{\rho}\sin(\widetilde{\omega}_a(t)), (-1)^{l_2+1}\sqrt{\widetilde{R}}\sin(\widetilde{\omega}_b(t))\Big) + O(\varepsilon);$$

where $\widetilde{r}$, $\widetilde{R}$, $\widetilde{\rho}$ as in (4), $\widetilde{\omega}_a(t) = \omega_1 t - (-1)^{j_1}\arccos(c_a)/2$, $\widetilde{\omega}_b(t) = \omega_1 t + (-1)^{j_1}\arccos(c_b)/2$, c_a and c_b as in (4), $j_1, l_1, l_2 \in \{0,1\}$. For different values of $(a,b) \in S_{11}$ the solution can be either linearly stable or unstable.

The proof of Theorems 1, 2 and 4 are given in sections 3, 4 and 5, respectively.

In section 2 we present a summary of the results on the averaging theory that we shall need for proving our results.

2. The averaging theory of first and second order

Now we present the averaging theory of second order that we need for proving the results of this paper. This theory provides sufficient conditions for the existence of periodic solutions for a periodic differential system depending on a small parameter. See the paper [2] for more information about these theorem and for the proofs of the results here stated.

Theorem 4. *Consider the non–autonomous differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 G(t,x,\varepsilon), \tag{6}$$

where ε is a small parameter, D is an open subset of $\mathbb{R}^n$, and F_1, $F_2 : \mathbb{R}\times D \to \mathbb{R}^n$, $G : \mathbb{R}\times D\times(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous and T-periodic functions in the first variable. Supose that the following conditions hold.

(i) *$F_1(t,\cdot) \in \mathcal{C}^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, G and $D_x F_1$ are locally Lipschitz with respect to x, and G is differentiable with respect to ε. The functions f_1, $f_2 : D \to \mathbb{R}^n$ are*

$$f_1(z) = \int_0^T F_1(s,z)ds, \tag{7}$$

$$f_2(z) = \int_0^T [D_z F_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)]ds. \tag{8}$$

(ii) *For each $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$ and for $V \subset D$ an open and bounded set, there exist $a \in V$ such that*
  (ii.1) *if $f_1(z) \not\equiv 0$, then $f_1(a) = 0$ and $d_B(f_1, a) \neq 0$ (here $d_B(f_1,a)$ is the Brouwer degree of the function $f_1 : V \to \mathbb{R}^n$ at the fixed point a); and*
  (ii.1) *if $f_1(z) \equiv 0$ and $f_2(z) \not\equiv 0$, then $f_2(a) = 0$ and $d_B(f_2, a) \neq 0$.*

Then for $|\varepsilon| > 0$ small enough, there is a $T-$periodic solution $x(t,\varepsilon)$ of the system such that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$. The kind of linear stability or instability of the periodic solutions $x(t,\varepsilon)$ is given by the eigenvalues of the Jacobian matrix $D_z(f_1(z) + \varepsilon f_2(z))|_{z=a}$.

We recall that a sufficient condition in order that the Brouwer degree of a $\mathcal{C}^1$ function f at a zero a is non–zero, is that the Jacobian of the function f at a is non–zero, see for more details [18].

Under the assumption (ii.1) Theorem 4 provides the *averaging theory of first order*, and it provides the *averaging theory of second order* when assumption (ii.2) holds.

3. Proof of Theorem 1

For proving Theorem 1 we shall use Theorem 4, so the first step is to write system (2) in such a way that conditions of Theorem 4 be satisfied.

We observe that system (2) is invariant by the symmetries

$$(x, y, z, p_x, p_y, p_z) \mapsto (-x, y, z, -p_x, p_y, p_z),$$
$$(x, y, z, p_x, p_y, p_z) \mapsto (x, -y, z, p_x, -p_y, p_z).$$

This implies that if $(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t))$ is a solution of system (2) then

$$(-x(t), y(t), z(t), -p_x(t), p_y(t), p_z(t)),\ (x(t), -y(t), z(t), p_x(t), -p_y(t), p_z(t))$$

are also solutions of system (2).

Without loss of generality we can assume that $\omega_1 = 1$. Indeed, taking the change of variables

$$X = \omega_1 x,\ Y = \omega_1 y,\ Z = \omega_1 z,\ P_x = p_x,\ P_y = p_y,\ P_z = p_z,$$

rescaling the time by $\tau = \omega_1 t$, and proceeding in a similar way as in (3) we transform system (2) with $c = 1$ into the system

$$\begin{array}{ll} X' = P_X, & P'_X = -X - 2\tilde{\varepsilon} bXZ, \\ Y' = P_Y, & P'_Y = -\tilde{\omega}_2^2 Y - 2\tilde{\varepsilon} YZ, \\ Z' = P_Z, & P'_Z = -\tilde{\omega}_3^2 Z - \tilde{\varepsilon}(bX^2 + Y^2 + 3aZ^2), \end{array}$$

where the prime denotes derivative with respect to the variable τ, and in which $\tilde{\varepsilon} = \varepsilon/\omega_1^3$, $\tilde{\omega}_2 = \omega_2/\omega_1$ and $\tilde{\omega}_3 = \omega_3/\omega_1$. From now on in order to avoid heavy notation we denote again $(X, Y, Z, P_X, P_Y, P_Z, \tilde{\varepsilon}, \tilde{\omega}_2, \tilde{\omega}_3)$ as $(x, y, z, p_x, p_y, p_z, \varepsilon, \omega_2, \omega_3)$, i.e. we work with system (2) with $c = 1$ and $\omega_1 = 1$; more precisely with the system

$$\begin{array}{ll} \dot{x} = p_x, & \dot{p_x} = -x - 2\varepsilon bxz, \\ \dot{y} = p_y, & \dot{p_y} = -\omega_2^2 y - 2\varepsilon yz, \\ \dot{z} = p_z, & \dot{p_z} = -\omega_3^2 z - \varepsilon(bx^2 + y^2 + 3az^2). \end{array} \tag{9}$$

The Hamiltonian associated to system (9) is

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{x^2 + \omega_2^2 y^2 + \omega_3^2 z^2}{2} + \varepsilon(az^3 + z(bx^2 + y^2)). \tag{10}$$

First we write system (9) and the Hamiltonian (10) in cylindrical coordinates $(r, \rho, R, \theta, \alpha, \beta)$ defined by

$$\begin{array}{ll} x = r\,\cos\theta, & p_x = r\,\sin\theta, \\ y = \dfrac{\rho\,\cos(\alpha + \omega_2\theta)}{\omega_2}, & p_y = \rho\,\sin(\alpha + \omega_2\theta), \\ z = \dfrac{R\,\cos(\beta + \omega_3\theta)}{\omega_3}, & p_z = R\,\sin(\beta + \omega_3\theta), \end{array} \tag{11}$$

and we get the system of equations

$$
\begin{aligned}
\dot{r} &= -\frac{\varepsilon b r R}{\omega_3}\cos(\beta+\omega_3\theta)\sin(2\theta),\\
\dot{\theta} &= -1-\frac{2\varepsilon b R}{\omega_3}\cos^2\theta\cos(\beta+\omega_3\theta),\\
\dot{\rho} &= -\frac{\varepsilon R\rho}{\omega_2\omega_3}\cos(\beta+\omega_3\theta)\sin(2(\alpha+\omega_2\theta)),\\
\dot{\alpha} &= \frac{2R\varepsilon}{\omega_2\omega_3}\big(b\omega_2^2\cos^2\theta-\cos^2(\alpha+\omega_2\theta)\big)\cos(\beta+\omega_3\theta),\\
\dot{R} &= -\frac{\varepsilon}{\omega_2^2\omega_3^2}\big(br^2\omega_2^2\omega_3^2\cos^2\theta+\rho^2\omega_3^2\cos^2(\alpha+\omega_2\theta)\\
&\quad+3a\omega_2^2R^2\cos^2(\beta+\omega_3\theta)\big)\sin(\beta+\omega_3\theta),\\
\dot{\beta} &= -\frac{\varepsilon\cos(\beta+\omega_3\theta)}{R\omega_2^2\omega_3^2}\big(b(r^2-2R^2)\omega_2^2\omega_3^2\cos^2\theta+\rho^2\omega_3^2\cos^2(\alpha+\omega_2\theta)\\
&\quad+3aR^2\omega_2^2\cos^2(\beta+\omega_3\theta)\big),
\end{aligned} \tag{12}
$$

and the Hamiltonian (10) becomes

$$
\begin{aligned}
H = \frac{1}{2}\left(r^2+R^2+\rho^2\right)+\frac{\varepsilon R\cos(\beta+\omega_3\theta)}{\omega_3}\Bigg(&br^2\cos^2\theta+\frac{\rho^2\cos^2(\alpha+\omega_2\theta)}{\omega_2^2}\\
&+\frac{aR^2\cos^2(\beta+\omega_3\theta)}{\omega_3^2}\Bigg).
\end{aligned}
$$

Note that in system (12) the equations of $\dot{r}$, $\dot{\theta}$ and $\dot{\alpha}$ do not depend on ρ, and the equations of $\dot{R}$ and $\dot{\beta}$ depend on ρ^2 instead of ρ. We thus introduce the new variable $\Gamma=\rho^2$. In this new variable system (12) becomes

$$
\begin{aligned}
\dot{r} &= -\frac{\varepsilon b r R}{\omega_3}\cos(\beta+\omega_3\theta)\sin(2\theta),\\
\dot{\theta} &= -1-\frac{2\varepsilon b R}{\omega_3}\cos^2\theta\cos(\beta+\omega_3\theta),\\
\dot{\Gamma} &= -\frac{2\varepsilon R\Gamma}{\omega_2\omega_3}\cos(\beta+\omega_3\theta)\sin(2(\alpha+\omega_2\theta)),\\
\dot{\alpha} &= \frac{2R\varepsilon}{\omega_2\omega_3}\big(b\omega_2^2\cos^2\theta-\cos^2(\alpha+\omega_2\theta)\big)\cos(\beta+\omega_3\theta),\\
\dot{R} &= -\frac{\varepsilon}{\omega_2^2\omega_3^2}\big(br^2\omega_2^2\omega_3^2\cos^2\theta+\Gamma\omega_3^2\cos^2(\alpha+\omega_2\theta)\\
&\quad+3a\omega_2^2R^2\cos^2(\beta+\omega_3\theta)\big)\sin(\beta+\omega_3\theta),\\
\dot{\beta} &= -\frac{\varepsilon\cos(\beta+\omega_3\theta)}{R\omega_2^2\omega_3^2}\big(b(r^2-2R^2)\omega_2^2\omega_3^2\cos^2\theta+\Gamma\omega_3^2\cos^2(\alpha+\omega_2\theta)\\
&\quad+3aR^2\omega_2^2\cos^2(\beta+\omega_3\theta)\big),
\end{aligned} \tag{13}
$$

and the Hamiltonian (10) becomes

$$
\begin{aligned}
H = \frac{1}{2}\left(r^2+R^2+\Gamma\right)+\frac{\varepsilon R\cos(\beta+\omega_3\theta)}{\omega_3}\Bigg(&br^2\cos^2\theta+\frac{\Gamma\cos^2(\alpha+\omega_2\theta)}{\omega_2^2}\\
&+\frac{aR^2\cos^2(\beta+\omega_3\theta)}{\omega_3^2}\Bigg).
\end{aligned}
$$

Now in system (13) we take as independent variable the angular variable θ and the system becomes

$$
r'=\frac{\dot{r}}{\dot{\theta}},\quad \Gamma'=\frac{\dot{\Gamma}}{\dot{\theta}},\quad \alpha'=\frac{\dot{\alpha}}{\dot{\theta}},\quad R'=\frac{\dot{R}}{\dot{\theta}},\quad \beta'=\frac{\dot{\beta}}{\dot{\theta}}, \tag{14}
$$

where the prime denotes derivative with respect to θ. We note that the right hand part of system (14) is π–periodic in the variables θ and $\omega_2\theta$, and it is 2π-periodic in the variable $\omega_3\theta$. Hence in order to have a periodic orbit we must have

$$
\theta=k_1\pi,\quad \omega_2\theta=k_2\pi\quad\text{and}\quad \omega_3\theta=2k_3\pi\quad\text{for some } k_1,k_2,k_3\in\mathbb{N},
$$

and then

$$\omega_2 = \frac{k_2}{k_1} \quad \text{and} \quad \omega_3 = \frac{2k_3}{k_1}, \quad k_1, k_2, k_3 \in \mathbb{N},$$

with k_1 and k_2 coprime and k_1 and k_3 coprime. So system (14) is $k_1\pi$–periodic in the variable θ.

By fixing an energy level $H = h$ we can compute Γ by solving the equation $H = h$, and we get

$$\begin{aligned}\Gamma &= \frac{\omega_2^2\big((2h - r^2 - R^2)\omega_3^3 - \varepsilon(2br^2R\omega_3^2\cos^2\theta\cos(\beta+\omega_3\theta) + 2aR^3\cos^3(\beta+\omega_3\theta))\big)}{\omega_3^2\big(\omega_2^2\omega_3 + 2\varepsilon R\cos^2(\alpha+\omega_2\theta)\cos(\beta+\omega_3\theta)\big)} \\ &= \Gamma_0 + \Gamma_1\varepsilon + O(\varepsilon^2),\end{aligned} \tag{15}$$

where

$$\begin{aligned}\Gamma_0 &= 2h - r^2 - R^2, \\ \Gamma_1 &= -\frac{2R\cos(\beta+\omega_3\theta)}{\omega_2^2\omega_3^3}\Big(br^2\omega_2^2\omega_3^2\cos^2\theta + (2h - r^2 - R^2)\omega_3^2\cos^2(\alpha+\omega_2\theta) \\ &\quad + aR^2\omega_2^2\cos^2(\beta+\omega_3\theta)\Big).\end{aligned}$$

We substitute the expression of Γ into (14) and we develop the right-hand side in power series of ε up to second order. Therefore at each energy level $H = h$ the equations of motion can be written as

$$\begin{aligned} r' &= \varepsilon F_{11} + \varepsilon^2 F_{21} + O(\varepsilon^3), \quad \alpha' = \varepsilon F_{12} + \varepsilon^2 F_{22} + O(\varepsilon^3), \\ R' &= \varepsilon F_{13} + \varepsilon^2 F_{23} + O(\varepsilon^3), \quad \beta' = \varepsilon F_{14} + \varepsilon^2 F_{24} + O(\varepsilon^3),\end{aligned} \tag{16}$$

where

$$\begin{aligned}
F_{11} &= \frac{brR}{\omega_3}\cos(\beta+\omega_3\theta)\sin(2\theta), \\
F_{12} &= -\frac{2R}{\omega_2\omega_3}\Big(b\omega_2^2\cos^2\theta - \cos^2(\alpha+\omega_2\theta)\Big)\cos(\beta+\omega_3\theta), \\
F_{13} &= \frac{1}{\omega_2^2\omega_3^2}\Big(br^2\omega_2^2\omega_3^2\cos^2\theta + (2h - r^2 - R^2)\omega_3^2\cos^2(\alpha+\omega_2\theta) \\
&\quad + 3aR^2\omega_2^2\cos^2(\beta+\omega_3\theta)\Big)\sin(\beta+\omega_3\theta), \\
F_{14} &= -\frac{1}{R\omega_2^2\omega_3^2}\Big(b(2R^2 - r^2)\omega_2^2\omega_3^2\cos^2\theta + (R^2 + r^2 - 2h)\omega_3^2\cos^2(\alpha+\omega_2\theta) \\
&\quad - 3aR^2\omega_2^2\cos^2(\beta+\omega_3\theta)\Big)\cos(\beta+\omega_3\theta), \\
F_{21} &= -\frac{4b^2rR^2}{\omega_3^2}\sin\theta\cos^3\theta\cos^2(\beta+\omega_3\theta), \\
F_{22} &= \frac{4bR^2}{\omega_2\omega_3^2}\Big(b\omega_2^2\cos^2\theta - \cos^2(\alpha+\omega_2\theta)\Big)\cos^2\theta\cos^2(\beta+\omega_3\theta), \\
F_{23} &= -\frac{R}{\omega_2^4\omega_3^3}\Big(b^2r^2\omega_2^4\omega_3^2\cos^4\theta + (2h - r^2 - R^2)\omega_3^2\cos^4(\alpha+\omega_2\theta) \\
&\quad + aR^2\omega_2^2\cos^2(\alpha+\omega_2\theta)\cos^2(\beta+\omega_3\theta) \\
&\quad + b\omega_2^2\cos^2\theta((2h - R^2)\omega_3^2\cos^2(\alpha+\omega_2\theta) \\
&\quad + 3aR^2\omega_2^2\cos^2(\beta+\omega_3\theta)\Big)\sin(2(\beta+\omega_3\theta)), \\
F_{24} &= -\frac{2}{\omega_2^4\omega_3^3}\Big(b^2(r^2 - 2R^2)\omega_2^4\omega_3^2\cos^4\theta + (2h - r^2 - R^2)\omega_3^2\cos^4(\alpha+\omega_2\theta) \\
&\quad + aR^2\omega_2^2\cos^2(\alpha+\omega_2\theta)\cos^2(\beta+\omega_3\theta) \\
&\quad + b\omega_2^2\cos^2\theta((2h - R^2)\omega_3^2\cos^2(\alpha+\omega_2\theta) \\
&\quad + 3aR^2\omega_2^2\cos^2(\beta+\omega_3\theta))\Big)\cos^2(\beta+\omega_3\theta).
\end{aligned}$$

In order that the differential system (16) be in the normal form (6) for applying the averaging theory, this system must be periodic in the variable θ. Since system (14) is $k_1\pi$–periodic in the variable θ for $\omega_2 = k_2/k_1$ and $\omega_3 = 2k_3/k_1$ with $k_1, k_2, k_3 \in \mathbb{N}$, we have that also system (16) is $k_1\pi$–periodic. Then system (16) is in the normal form (6) for applying the averaging theory with $T = k_1\pi$, $\mathbf{x} = (r, \alpha, R, \beta)$, $t = \theta$, $F_1(\theta, \mathbf{x}) = (F_{11}, F_{12}, F_{13}, F_{14})$, $F_2(\theta, \mathbf{x}) = (F_{21}, F_{22}, F_{23}, F_{24})$ and $\varepsilon^2 G(\theta, \mathbf{x}, \varepsilon)$ is $O(\varepsilon^3)$. We also observe that F and G are $\mathcal{C}^2$ in $\mathbf{x}$ and $k_1\pi$–periodic in θ. After some computations from (7) we get

$$f_1(\mathbf{x}) = \int_0^{k_1\pi} F_1(\theta, \mathbf{x}) d\theta = (f_{11}(\mathbf{x}), f_{12}(\mathbf{x}), f_{13}(\mathbf{x}), f_{14}(\mathbf{x})),$$

where

$$f_{1j}(\mathbf{x}) = \int_0^{k_1\pi} F_{1j}(\theta, \mathbf{x})\, d\theta, \quad j = 1, \dots, 4$$

with

$$f_{11}(\mathbf{x}) = \begin{cases} 0 & k_1 \neq k_3,\ k_2 \neq k_3, \\ -\dfrac{bk_1\pi r R \sin\beta}{4} & k_1 = k_3,\ k_2 \neq k_3, \\ 0 & k_1 \neq k_3,\ k_2 = k_3, \\ -\dfrac{bk_1\pi r R \sin\beta}{4} & k_1 = k_2 = k_3, \end{cases}$$

$$f_{12}(\mathbf{x}) = \begin{cases} 0 & k_1 \neq k_3,\ k_2 \neq k_3, \\ -\dfrac{bk_2\pi R \cos\beta}{4} & k_1 = k_3,\ k_2 \neq k_3, \\ \dfrac{k_1^3\pi R \cos(2\alpha - \beta)}{4k_2^2} & k_1 \neq k_3,\ k_2 = k_3, \\ \dfrac{k_1\pi R(\cos(2\alpha - \beta) - b\cos\beta)}{4} & k_1 = k_2 = k_3, \end{cases}$$

$$f_{13}(\mathbf{x}) = \begin{cases} 0 & k_1 \neq k_3,\ k_2 \neq k_3, \\ \dfrac{bk_1\pi r^2}{4}\sin\beta & k_1 = k_3,\ k_2 \neq k_3, \\ \dfrac{k_1^3\pi(r^2 + R^2 - 2h)\sin(2\alpha - \beta)}{4k_2^2} & k_1 \neq k_3,\ k_2 = k_3, \\ \dfrac{k_1\pi\big((r^2 + R^2 - 2h)\sin(2\alpha - \beta) + br^2\sin\beta\big)}{4} & k_1 = k_2 = k_3, \end{cases}$$

$$f_{14}(\mathbf{x}) = \begin{cases} 0 & k_1 \neq k_3,\ k_2 \neq k_3, \\ \dfrac{bk_1\pi(r^2 - 2R^2)\cos\beta}{4R} & k_1 = k_3,\ k_2 \neq k_3, \\ -\dfrac{k_1^3\pi(r^2 + R^2 - 2h)\cos(2\alpha - \beta)}{4k_2^2 R} & k_1 \neq k_3,\ k_2 = k_3, \\ -\dfrac{k_1\pi\big((r^2 + R^2 - 2h)\cos(2\alpha - \beta) - b(r^2 - 2R^2)\cos\beta\big)}{4R} & k_1 = k_2 = k_3. \end{cases}$$

If $k_1 \neq k_3$ and $k_2 \neq k_3$, then $f_{1j}(\mathbf{x}) = 0$ for $j = 1, \dots, 4$, we need to consider averaging of second order.

When $k_1 \neq k_3$ and $k_2 = k_3$ since $f_{11}(\mathbf{x}) = 0$ and $f_{12}(\mathbf{x})$ are not identically zero, we cannot go to second order in the averaging theory and so we do not get any information on the periodic solutions in this case.

If $k_1 = k_3$, $k_2 \neq k_3$ we first note that the functions $f_{1j}(\mathbf{x})$, $j = 1, \dots, 4$ do not depend on α and so the Jacobian of the function $f_1(\mathbf{x})$ at any of the solutions of $f_1(\mathbf{x}) = (f_{11}(\mathbf{x}), f_{12}(\mathbf{x}), f_{13}(\mathbf{x}), f_{14}(\mathbf{x})) = 0$ will be zero, so we cannot go to second order averaging theory. Hence in this case we also do not get any information on the periodic solutions. This completes the proof of statements (ii) and (iii).

Finally we compute the solutions of the system $f_1(\mathbf{x}) = 0$ when $k_1 = k_2 = k_3$. Since the pairs (k_1, k_3) and (k_1, k_3) are coprime, we have $k_1 = k_2 = k_3 = 1$. Notice that we are not interested in the solutions with $R = 0$ because at these solutions the function F_{14} is not defined. Moreover we consider $b \neq 0$ otherwise $f_{11}(\mathbf{x}) = 0$ and $f_{12}(\mathbf{x})$ is not identically zero and we cannot go to the second order.

Now we compute the solutions of system $f_1(\mathbf{x}) = 0$ with $R \neq 0$ and $b \neq 0$. From $f_{11}(\mathbf{x}) = 0$ we get either $\beta = j\pi$ for $j = 0, 1$, or $r = 0$. By substituting $\beta = j\pi$ into $f_{12}(\mathbf{x}) = 0$ we get the equation $f_{12}(\mathbf{x}) = \frac{1}{4}(-1)^j R(\cos(2\alpha) - b) = 0$, whose solutions $\alpha = (-1)^k \arccos(b)/2 + \ell\pi$ with $k, \ell = 0, 1$ are defined when $b \in [-1, 0) \cup (0, 1]$. We substitute

α into $f_{14}(\mathbf{x})=0$ and we obtain $R=\sqrt{2h/3}$. Finally substituting these values into $f_{13}(\mathbf{x})=0$ we get equation $f_{13}(\mathbf{x})=(-1)^{j+k+1}\left(4h-3r^2\right)\sqrt{1-b^2}/12=0$, which provides solutions for r whenever $b\neq\pm1$. In short we have the following solution

$$(r_{(j,k,\ell)},\alpha_{(j,k,\ell)},R_{(j,k,\ell)},\beta_{(j,k,\ell)})=\left(2\sqrt{h/3},(-1)^k\arccos(b)/2+\ell\pi,\sqrt{2h/3},j\pi\right),$$

for $b\in(-1,0)\cup(0,1)$ and $j,k,l=0,1$.

Now we substitute $r=0$ into $f_{13}(\mathbf{x})=0$ and we get either $R=\sqrt{2h}$, or $\beta=2\alpha-j\pi$ for $j=0,1$. First we substitute $r=0$ and $R=\sqrt{2h}$ into the remaining equations $f_{12}(\mathbf{x})=0$, $f_{14}(\mathbf{x})=0$ and we get the solution

$$(r^*_{(j,\ell)},\alpha^*_{(j,\ell)},R^*_{(j,\ell)},\beta^*_{(j,\ell)})=(0,\ell\pi/2,\sqrt{2h},\pi/2+j\pi),$$

with $j,\ell=0,1$ which is defined for all $b\neq0$.

We substitute $r=0$ and $\beta=2\alpha-j\pi$ into $f_{12}(\mathbf{x})=0$ and we get the equation $f_{12}(\mathbf{x})=-\frac{1}{4}(-1)^jR(b\cos(2\alpha)-1)$, whose solution $\alpha=(-1)^k\arccos(1/b)/2+\ell\pi$ with $k,\ell=0,1$ is defined for all $b\in(-\infty,-1]\cup[1,+\infty)$. Finally we compute the value of R from equation $f_{14}(\mathbf{x})=0$ and we get the solution

$$(\overline{r}_{(j,k,\ell)},\overline{\alpha}_{(j,k,\ell)},\overline{R}_{(j,k,\ell)},\overline{\beta}_{(j,k,\ell)})=(0,(-1)^k\arccos(1/b)/2+\ell\pi,\sqrt{2h/3},(-1)^k\arccos(1/b)-j\pi),$$

defined for $b\in(-\infty,-1]\cup[1,+\infty)$ and $j,k,l=0,1$.

Now we compute the Jacobian matrix of f_1

$$\mathcal{J}=\quad D_{\mathbf{x}}f_1=\begin{pmatrix}\dfrac{\partial f_{11}}{\partial r}&\dfrac{\partial f_{11}}{\partial\alpha}&\dfrac{\partial f_{11}}{\partial R}&\dfrac{\partial f_{11}}{\partial\beta}\\ \dfrac{\partial f_{12}}{\partial r}&\dfrac{\partial f_{12}}{\partial\alpha}&\dfrac{\partial f_{12}}{\partial R}&\dfrac{\partial f_{12}}{\partial\beta}\\ \dfrac{\partial f_{13}}{\partial r}&\dfrac{\partial f_{13}}{\partial\alpha}&\dfrac{\partial f_{13}}{\partial R}&\dfrac{\partial f_{13}}{\partial\beta}\\ \dfrac{\partial f_{14}}{\partial r}&\dfrac{\partial f_{14}}{\partial\alpha}&\dfrac{\partial f_{14}}{\partial R}&\dfrac{\partial f_{14}}{\partial\beta}\end{pmatrix}. \tag{17}$$

The determinant of $\mathcal{J}$ evaluated at the solutions $(r_{(j,k,\ell)},\alpha_{(j,k,\ell)},R_{(j,k,\ell)},\beta_{(j,k,\ell)})$ is equal to $h^2(b^2-1)b^2/12$. The determinant of $\mathcal{J}$ evaluated at the solutions $(r^*_{(j,\ell)},\alpha^*_{(j,\ell)},R^*_{(j,\ell)},\beta^*_{(j,\ell)})$ is $b^2h^2/8$. Finally the determinant $\mathcal{J}$ evaluated at the solutions $(\overline{r}_{(j,\ell)},\overline{\alpha}_{(j,k,\ell)},\overline{R}_{(j,k,\ell)},\overline{\beta}_{(j,k,\ell)})$ is $-h^2(b^2-1)/24$. Hence the averaging theory can be applied for the two first solutions and it can be applied for the third solution when $b\in(-\infty,-1)\cup(1,+\infty)$.

It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) for $b\in(-1,0)\cup(0,1)$ has eight π–periodic solutions, $\varphi^{(j,k,\ell)}(\theta,\varepsilon)=(r^{(j,k,\ell)}(\theta,\varepsilon),\ \alpha^{(j,k,\ell)}(\theta,\varepsilon),\ R^{(j,k,\ell)}(\theta,\varepsilon),\ \beta^{(j,k,\ell)}(\theta,\varepsilon))$ for $j,k,\ell=0,1$ such that $\varphi^{(j,k,\ell)}(0,\varepsilon)$ tend to $(r_{(j,k,\ell)},\alpha_{(j,k,\ell)},R_{(j,k,\ell)},\beta_{(j,k,\ell)})$ when $\varepsilon\to0$.

For any given $h>0$ and for $|\varepsilon|$ sufficiently small, it follows also from Theorem 4 that system (16) for $b\neq0$ has four π–periodic solutions $\varphi^{*(j,\ell)}(\theta,\varepsilon)=(r^{*(j,\ell)}(\theta,\varepsilon),\ \alpha^{*(j,\ell)}(\theta,\varepsilon),\ R^{*(j,\ell)}(\theta,\varepsilon),\ \beta^{*(j,\ell)}(\theta,\varepsilon))$ for $j,\ell=0,1$ such that $\varphi^{*(j,\ell)}(0,\varepsilon)$ tend to $(r^*_{(j,\ell)},\alpha^*_{(j,\ell)},R^*_{(j,\ell)},\beta^*_{(j,\ell)})$ when $\varepsilon\to0$.

Again it follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) for $b\in(-\infty,-1)\cup(1,+\infty)$ has eight π–periodic solutions, $\overline{\varphi}^{(j,k,\ell)}(\theta,\varepsilon)=(\overline{r}^{(j,k,\ell)}(\theta,\varepsilon),\ \overline{\alpha}^{(j,k,\ell)}(\theta,\varepsilon),\ \overline{R}^{(j,k,\ell)}(\theta,\varepsilon),\ \overline{\beta}^{(j,k,\ell)}(\theta,\varepsilon))$ for $j,k,\ell=0,1$ such that $\overline{\varphi}^{(j,k,\ell)}(0,\varepsilon)$ tend to $(\overline{r}_{(j,k,\ell)},\overline{\alpha}_{(j,k,\ell)},\overline{R}_{(j,k,\ell)},\overline{\beta}_{(j,k,\ell)})$ when $\varepsilon\to0$.

The eigenvalues of the matrix $\mathcal{J}$ evaluated at $(r_{(j,k,\ell)},\alpha_{(j,k,\ell)},R_{(j,k,\ell)},\beta_{(j,k,\ell)})$ for $j,k,\ell=0,1$ are

$$\pm(-1)^kb\sqrt{\frac{h}{2}}i,\quad\pm(-1)^{k+j}\sqrt{\frac{h(1-b^2)}{6}}.$$

In this case $b\in(-1,0)\cup(0,1)$, so two of them are complex and the other two are real with opposite sign. Since the eigenvalues of the matrix (17) evaluated at the solutions $(r_{(j,k,\ell)},\alpha_{(j,k,\ell)},R_{(j,k,\ell)},\beta_{(j,k,\ell)})$ provide the linear stability of the fixed point corresponding to the Poincaré map defined in a neighborhood of the periodic solution associated to $(r_{(j,k,\ell)},\alpha_{(j,k,\ell)},R_{(j,k,\ell)},\beta_{(j,k,\ell)})$ (see for instance the proof of Theorem 11.6 of [20]), the eight periodic $\varphi^{(j,k,\ell)}(\theta,\varepsilon)$ solutions are linearly unstable.

The eigenvalues of the matrix $\mathcal{J}$ evaluated at $(r^*_{(j,\ell)}, \alpha^*_{(j,\ell)}, R^*_{(j,\ell)}, \beta^*_{(j,\ell)})$ for $j, \ell = 0, 1$ are

$$\pm(-1)^{j+\ell}\sqrt{\frac{h}{2}}, \quad -(-1)^j b\sqrt{\frac{h}{8}}, \quad (-1)^j b\sqrt{\frac{h}{2}}.$$

They are all real: two positive and two negative. Arguing as above we have that for all $b \in \mathbb{R}\setminus\{0\}$, the four periodic solutions $\varphi^{*(j,\ell)}(\theta,\varepsilon)$ are linearly unstable.

The eigenvalues of the matrix $\mathcal{J}$ evaluated at $(\overline{r}_{(j,k,\ell)}, \overline{\alpha}_{(j,k,\ell)}, \overline{R}_{(j,k,\ell)}, \overline{\beta}_{(j,k,\ell)})$ for $j, k, \ell = 0, 1$

$$\pm(-1)^j\sqrt{\frac{h}{2}}i, \quad (-1)^{j+k}\mathrm{sign}(b)\sqrt{\frac{h(b^2-1)}{6}}, \quad (-1)^{j+k+1}\mathrm{sign}(b)\sqrt{\frac{h(b^2-1)}{24}},$$

where sign is the sign function. In this case $b \in (-\infty,-1)\cup(1,+\infty)$, so two of them are complex and the other two are real with opposite sign. So the eight periodic solutions $\overline{\varphi}^{(j,k,\ell)}(\theta,\varepsilon)$ will be linearly unstable.

Now we shall go back through the changes of variables in order to see how the π–periodic solutions $\varphi^{(j,k,\ell)}(\theta,\varepsilon)$, $\varphi^{*(j,\ell)}(\theta,\varepsilon)$ and $\overline{\varphi}^{(j,k,\ell)}(\theta,\varepsilon)$ with $j,k,\ell = 0,1$ of the differential system (16) are written in the original variables (x, y, z, p_x, p_y, p_z). Here we shall do the computations for $\varphi^{(j,k,\ell)}(\theta,\varepsilon)$, the other two cases can be done in a similar way.

By substituting $\varphi^{(j,k,\ell)}(\theta,\varepsilon)$ into (15) we get $\Gamma^{(j,k,\ell)}(\theta,\varepsilon)$. Then $\psi^{(j,k,\ell)} = (\varphi^{(j,k,\ell)}(\theta,\varepsilon), \Gamma^{(j,k,\ell)}(\theta,\varepsilon))$ is a π–periodic solution for the differential system (14). This solution provides the 2π–periodic solution for the differential system (13)

$$\begin{aligned}\Psi^{(j,k,\ell)}(t,\varepsilon) &= (\varphi(\theta^{(j,k,\ell)}(t,\varepsilon),\varepsilon), \theta^{(j,k,\ell)}(t,\varepsilon))\\ &= (r_{(j,k,\ell)} + O(\varepsilon), \alpha_{(j,k,\ell)} + O(\varepsilon), R_{(j,k,\ell)} + O(\varepsilon), \beta_{(j,k,\ell)} + O(\varepsilon),\\ &\quad 2h - (r_{(j,k,\ell)})^2 - (R_{(j,k,\ell)})^2 + O(\varepsilon), -t + O(\varepsilon)).\end{aligned}$$

Now we introduce the variable

$$\rho^{(j,k,\ell)}(\theta^{(j,k,\ell)}(t,\varepsilon),\varepsilon) = \sqrt{\Gamma^{(j,k,\ell)}(\theta^{(j,k,\ell)}(t,\varepsilon),\varepsilon)}.$$

Going back to the change of variables (11) we get the 2π–periodic solutions of system (9) given in Theorem 1. We note that the terms of order zero of the solutions of system (9) are the same for the solutions with $\ell = 0,1$ and $k = 0,1$. So in principle for $|\varepsilon|$ sufficiently small we can only guarantee that we have two different solutions of system (9) for any of the solutions $\Psi^{(j,k,\ell)}(t,\varepsilon)$, i.e. the ones for $j = 0,1$. This completes the proof of statement (a) of Theorem 1.

4. Proof of Theorem 2

We write again

$$\omega_2 = \frac{k_2}{k_1} \quad \omega_3 = \frac{2k_3}{k_1}$$

and we recall that we are under the assumptions $k_1 \neq k_3$, $k_2 \neq k_3$ and also $(k_1-k_2)^2 + (k_1-2k_3)^2 \neq 0$.

Since we proved in Theorem 1 that $f_{1i}(\mathbf{x}) = 0$ for $i = 1,\ldots,4$, we shall apply the averaging of second order. We compute for our system the integral $\int_0^s F_1(t,z)dt$ of (8), and after tedious computations we compute the integrals of (8). The values of $f_2(\mathbf{x}) = (f_{21}(\mathbf{x}), f_{22}(\mathbf{x}), f_{23}(\mathbf{x}), f_{24}(\mathbf{x}))$ that we obtain depend on the values of k_1, k_2, k_3. So we need to consider different cases.

a) If $k_1 \neq 2k_3$, $k_1 \neq k_2$, $k_1-k_2-2k_3 \neq 0$, $k_1+k_2-2k_3 \neq 0$, $k_2 \neq 2k_3$, $k_1 \neq 4k_3$, $k_1-k_2+2k_3 \neq 0$, and $k_2 \neq 4k_3$, then $f_{21}(\mathbf{x}) = 0$, $f_{23}(\mathbf{x}) = 0$ but $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are functions depending on $a, b, h, k_1, k_2, k_3, r, R$. We prove that there are no integer values of k_1, k_2 and k_3 and no real values of a, b such that $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. Therefore we cannot go to third order in the averaging theory and so we do not get information on the periodic solutions in these cases.

b) If $k_1 = k_2$ we must distinguish the following cases $k_1 = k_3$, $k_1 = 2k_3$ (both are not possible by assumptions), $k_1 = 4k_3$, and $k_1 \neq 2k_3, k_3, 4k_3$. When either $k_1 = 4k_3$ or $k_1 \neq 2k_3, k_3, 4k_3$ we have that $f_{23}(\mathbf{x}) = 0$, but we cannot find integer values of k_2, k_3 and real values of a, b such that $f_{21}(\mathbf{x}), f_{22}(\mathbf{x}), f_{24}(\mathbf{x})$ be identically zero, we cannot go to third order in the averaging theory and so we do not get information on the periodic solutions in this case.

c) If $k_1 = 2k_3$, then we must distinguish the following cases $k_2 = k_3$, $k_2 = 2k_3$ (both are not possible by assumptions), $k_2 = 4k_3$, and $k_2 \neq k_3, 2k_3, 4k_3$. In the last two cases we have that the functions $f_{2j}(\mathbf{x})$ for $j = 1, \ldots, 4$ do not depend on α and so the Jacobian of the function $f_2(\mathbf{x})$ at any of the solutions of

$$f_2(\mathbf{x}) = (f_{21}(\mathbf{x}), f_{22}(\mathbf{x}), f_{23}(\mathbf{x}), f_{24}(\mathbf{x})) = 0$$

will be zero, so we cannot go to third order in the averaging theory, and so we do not get information on the periodic solutions in this case.

d) If $k_2 = 2k_3$ then either $k_1 = k_3$ or $k_1 = 2k_3$ (both are not possible by assumptions), or $k_1 = 4k_3$, or $k_1 \neq k_3, 2k_3, 4k_3$. In the last two cases we have that $f_{21}(\mathbf{x}) = 0$ but there are no integer values of k_1, k_3 and no real values of a, b such that $f_{22}(\mathbf{x})$, $f_{23}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. Hence we cannot go to third order in the averaging theory and so we do not get information on the periodic solutions in this case.

e) If $k_1 = 4k_3$ then either $k_2 = k_3$ (which is not possible by assumptions), or $k_2 = 4k_3$ (this implies $k_1 = k_2$, and it was studied in case b), or $k_2 = 2k_3$ (which was studied in case d), or $k_2 = 6k_3$, or $k_2 \neq k_3, 2k_3, 4k_3, 6k_3$. In the last two cases we get that $f_{21}(\mathbf{x}) = f_{23}(\mathbf{x}) = 0$, but there are no integer values of k_2, k_3 and real values of a, b such that $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. Again we cannot go to third order in the averaging theory and we do not get information on the periodic solutions in this case.

f) If $k_2 = 4k_3$ then either $k_1 = k_3$ (which is not possible by assumptions), or $k_1 = 2k_3$ (which was studied in case c), or $k_1 = 4k_3$ (that was studied in case e), or $k_1 = 6k_3$, or $k_1 \neq k_3, 2k_3, 4k_3, 6k_3$. In the last two cases we have that $f_{21}(\mathbf{x}) = f_{23}(\mathbf{x}) = 0$, but there are not integer values of k_1, k_3 and real values of a, b so that $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. Hence we cannot go to third order in the averaging theory and we do not get information on the periodic solutions in this case.

g) If $k_2 = 2k_3 - k_1$. In this case either $k_1 = 3k_3$ or $k_1 = 2k_3$, or $k_1 = 4k_3$, or $k_1 = 6k_3$ (all of them are not possible because then $k_2 \leq 0$), or $k_1 = k_3$ (which is not possible by assumptions), or $k_1 \neq k_1, 2k_3, 3k_3, 4k_3, 6k_3$. In the last case we get that $f_{21}(\mathbf{x}) = f_{23}(\mathbf{x}) = 0$, but there are not integer values of k_1, k_3 and real values of a, b so that $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. We cannot go to third order in the averaging theory.

h) If $k_2 = k_1 - 2k_3$. In this case either $k_1 = k_3$ (which is not possible by assumptions), or $k_1 = 3k_3$ (which implies $k_1 = k_2$ and it was studied in case b), or $k_1 = 2k_3$ (which is not possible because then $k_2 = 0$), or $k_1 = 4k_3$ (that was studied in case d), or $k_1 = 6k_3$ (that was studied in case f), or $k_1 \neq k_1, 2k_3, 3k_3, 4k_3, 6k_3$. In the last case we get that $f_{21}(\mathbf{x}) = f_{23}(\mathbf{x}) = 0$, but there are not integer values of k_1, k_3 and real values of a, b so that $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. We cannot go to third order in the averaging theory.

j) If $k_2 = k_1 + 2k_3$. In this case either $k_1 = k_3$ (which is not possible by assumptions), or $k_1 = 2k_3$ (that was studied in case c), or $k_1 = 4k_3$ (that was studied in case e), or $k_1 \neq k_1, 2k_3, 4k_3$. In the last case we get that $f_{21}(\mathbf{x}) = f_{23}(\mathbf{x}) = 0$, but there are not integer values of k_1, k_3 and real values of a, b so that $f_{22}(\mathbf{x})$ and $f_{24}(\mathbf{x})$ are identically zero. We cannot go to third order in the averaging theory.

This completes the proof of the theorem.

5. Proof of Theorem 4

The case $\omega_1 = \omega_2 = \omega_3$ corresponds to $k_1 = k_2 = 2k_3$. We apply the averaging of second order. By computing for our system the integral $\int_0^s F_1(t, z)dt$ of

(8), and after tedious computations we compute (8), and we get $f_2(\mathbf{x})$ and we get

$$\begin{aligned}
f_{21}(\mathbf{x}) &= \frac{1}{6}\pi b k_3 r\big(\sin(2\alpha)(r^2+R^2-2h)-3R^2(a-2b)\sin(2\beta)\big),\\
f_{21}(\mathbf{x}) &= \frac{1}{6}\pi k_3(18abR^2-18aR^2+5b^2r^2+4b^2R^2+12bh-12br^2-6bR^2\\
&\quad -10h+5r^2+R^2)-\frac{1}{2}\pi b k_3 R^2(a-2b)\cos(2\beta)+\frac{1}{2}\pi(a-2)k_3R^2\\
&\quad \cos(2(\alpha-\beta))-\frac{1}{6}\pi b k_3\cos(2\alpha)(2h-2r^2-R^2),\\
f_{23}(\mathbf{x}) &= \frac{1}{2}\pi k_3 R\big(br^2(a-2b)\sin(2\beta)+(2-a)\sin(2(\alpha-\beta))(2h-r^2-R^2)\big),\\
f_{24}(\mathbf{x}) &= -\frac{1}{6}\pi k_3(45a^2R^2+18abr^2-18abR^2+36ah-18ar^2-18aR^2-b^2r^2\\
&\quad -4b^2R^2-12bh+6br^2+6bR^2+8h-4r^2-4R^2)+\frac{1}{2}\pi b k_3(a-2b)\\
&\quad \cos(2\beta)(r-R)(r+R)+\frac{1}{2}\pi(a-2)k_3\cos(2(\alpha-\beta))(2h-r^2-R^2)\\
&\quad -\frac{1}{6}\pi b k_3\cos(2\alpha)(2h-r^2-R^2).
\end{aligned}$$

It follows from $f_{21}(\mathbf{x})=0$ that there are three cases to consider: $b=0$, $r=0$ and $\kappa=\sin(2\alpha)(r^2+R^2-2h)-3R^2(a-2b)\sin(2\beta)=0$. We study them separately.

Case 1: $b=0$. In this case the functions $f_{2j}(\mathbf{x})$ for $j=2,3,4$ only depend on the variables $r,R,\alpha-\beta$. Therefore the Jacobian of the function $f_2(\mathbf{x})$ at any of the solutions of

$$f_2(\mathbf{x})=(f_{21}(\mathbf{x}),f_{22}(\mathbf{x}),f_{23}(\mathbf{x}),f_{24}(\mathbf{x}))=0$$

will be zero, so we cannot go to third order in the averaging theory, and so we do not get information on the periodic solutions in this case.

Case 2: $b\neq 0$ and $r=0$. In this case, from $f_{23}(\mathbf{x})=0$ we get three subcases: $a=2$, $R=\sqrt{2h}$ or $2(\alpha-\beta)=k\pi$ with $k\in\{0,1\}$. The case $a=2$ does not provide any solution because if $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ then $f_{22}(\mathbf{x})-f_{24}(\mathbf{x})=35k_3\pi(2h+3R^2)/6=0$ which is not possible. So $a\neq 2$.

Case 2.1. If $b\neq 0$, $r=0$, $a\neq 2$, and $R=\sqrt{2h}$, then from $f_{24}(\mathbf{x})=0$ we get that either $a=2b$, or if $a\neq 2b$ then $\beta=\pm\frac{1}{2}A_{a,b}+j\pi$ with $j\in\{0,1\}$ where $A_{a,b}$ is given in (4). In the first case, i.e. when $a=2b$ then $f_{24}(\mathbf{x})=-140b^2hk_3\pi/3$ which is not zero, so this case is not possible. Hence $a\neq 2b$ and

$$\beta=\beta_{j_1,j}=\frac{(-1)^{j_1}}{2}A_{a,b}+j\pi,\quad j_1,j\in\{0,1\}.$$

Imposing this value of $\beta_{j_1,j}$ in $f_{22}(\mathbf{x})=0$ we get that

$$\alpha=\alpha_{j_1,j_2,\ell}=\frac{(-1)^{j_2}}{2}A_{a,1}+j_3\pi+\beta_{j_1,j}=\frac{(-1)^{j_2}}{2}A_{a,1}+\frac{(-1)^{j_1}}{2}A_{a,b}+\ell\pi,$$

with $j_1,j_2,\ell\in\{0,1\}$. In order that $\beta_{j_1,j}$ and $\alpha_{j_1,j_2,\ell}$ are well-defined we must have

$$-1\le\frac{4b^2+18ab-45a^2}{3(a-2b)b}\le 1\quad\text{and}\quad -1\le\frac{-45a^2+18a+4}{3(a-2)}\le 1. \tag{18}$$

The second condition in (18) is equivalent to

$$\frac{(3a-2)(3a+1)}{3(a-2)}\ge 0\quad\text{and}\quad\frac{(3a-1)(15a-2)}{3(a-2)}\le 0. \tag{19}$$

This implies that $a\in[-1/3,2/15]\cup[1/3,2/3]$. Going back to (19) we see that this is indeed the set of values for which (19) holds. Moreover the first condition in (18) is equivalent to

$$\frac{(3a-2b)(3a+b)}{3(a-2b)b}\ge 0\quad\text{and}\quad\frac{(15a-2b)(3a-b)}{3(a-2b)b}\le 0,$$

This yields that the set of parameters $(a,b)\in\mathbb{R}^2$ for which there exist $\alpha_{j_1,j_2,\ell}$ and $\beta_{j_1,j}$ is the set $(a,b)\in\mathbb{R}^2$ such that

$$\Big\{a\in[-1/3,2/15]\cup[1/3,2/3],\ \frac{(3a-2b)(3a+b)}{3(a-2b)b}\geq 0,\ \frac{(15a-2b)(3a-b)}{3(a-2b)b}\leq 0\Big\}.$$

The determinant of the Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(0,\alpha_{j_1,j_2,\ell},\sqrt{2h},\beta_{j_1,j})$ is

$$\frac{200}{81}h^4k_3^4\pi^4(3a-2)(3a-1)(3a+1)(15a-2)(3a-2b)(15a-2b)(3a-b)(3a+b).$$

Since this determinant must be different from zero, we have that the set of parameters $(a,b)\in\mathbb{R}^2$ for which the second averaging order provides a solution is S_1 given in (5).

Moreover the eigenvalues are

$$\begin{aligned}
\lambda_{1,2} &= \pm\frac{2(-1)^{j_2}hk_3\pi}{3}\sqrt{-5(3a-2)(3a-1)(3a+1)(15a-2)},\\
\lambda_3 &= \frac{(-1)^{j_1}hk_3\pi\,\mathrm{sign}\,(b(a-2b))}{3}\sqrt{-5(3a-2b)(15a-2b)(3a-b)(3a+b)},\\
\lambda_4 &= -2\lambda_3,
\end{aligned}$$

where $\mathrm{sign}(x)$ denotes the sign of x. By the definition of S_1 we have that $\lambda_{1,2}\in\mathbb{R}$ with different sign and λ_3,λ_4 are also real with different sign.

It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) has eight linearly unstable $k_1\pi$-periodic solutions

$$\varphi_{j_1,j_2,j,\ell}(\theta,\varepsilon)=(r_{j_1,j_2,j,\ell}(\theta,\varepsilon),\alpha_{j_1,j_2,j,\ell}(\theta,\varepsilon),R_{j_1,j_2,j,\ell}(\theta,\varepsilon),\beta_{j_1,j_2,j,\ell}(\theta,\varepsilon)),$$

that tend to $(0,\alpha_{j_1,j_2,\ell},\sqrt{2h},\beta_{j_1,j})$ when $\varepsilon\to 0$.

Now we go back through the changes of variables in (11). Substituting $\varphi_{j_1,j_2,j,\ell}(\theta,\varepsilon)$ in (15) we get $\Gamma_{j_1,j_2,j,\ell}(\theta,\varepsilon)$ and so $\rho_{j_1,j_2,j,\ell}(\theta,\varepsilon)$. Therefore $\varphi_{j_1,j_2,j,\ell}(\theta,\varepsilon)$ is a 2π-periodic solution for the differential system (13). This solution provides the $2\pi/\omega_1$-periodic solutions for the differential system (9) given in the statement of the theorem. Note that the terms of order zero of the solutions in (9) are the same for the solutions $(\varphi_{j_1,j_2,j,\ell}(\theta,\varepsilon),\rho_{j_1,j_2,j}(\theta,\varepsilon))$ with $j_2=1$ and $j_2=0$. Moreover the terms of order zero of the solution with $j=1$ correspond to the terms of order zero of the solution with $j=0$ taking as initial angle $A_{a,b}/2+\pi/(2\omega_1)$ instead of $A_{a,b}/2$. So in principle for $|\varepsilon|$ sufficiently small we can only guarantee that we have one solution of system (9) which is the one given in the statement of the theorem.

Case 2.2. If $b\neq 0$, $r=0$, $2(\alpha-\beta)=0$ then $\alpha=\beta$. Moreover setting $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ we get that either $a=1$, or $a\neq 1$, $b=1/3$, or $a\neq 1$, $b\neq 1/3$, $R=\sqrt{\frac{2h}{3(1-a)}}$ with $a<1$ and $\cos(2\beta)=2(5-3b-b^2)/(b(3b-1))$. The first case, i.e. $a=1$ does not provide any solution because if $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ then $f_{22}(\mathbf{x})-f_{24}(\mathbf{x})=20k_3\pi h/3=0$, which is not possible. So $a\neq 1$. If $b=1/3$ then $f_{22}(\mathbf{x})+f_{24}(\mathbf{x})=5k_3\pi(3a+1)(2h-3R^2-3aR^2)/6$, so either $a=-1/3$ or $R=\sqrt{\frac{2h}{3(1-a)}}$ with $a<1$. When $a=-1/3$ then $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})$ are not identically zero and depend on R and β. Therefore the Jacobian matrix of $f_2(\mathbf{x})$ evaluated at any solution of $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})$ will be zero and we cannot go to third order in averaging theory. When $R=\sqrt{\frac{2h}{3(1-a)}}$ then we cannot find values of a,b and k_3 such that $f_{22}(\mathbf{x})=0$. So this case is not possible which implies $a\neq 1$ and $b\neq 1/3$. Therefore $R=\sqrt{\frac{2h}{3(1-a)}}$ with $a<1$ and $\beta=\beta_{j_1,k}=\frac{(-1)^{j_1}}{2}B_b+k\pi$ with $j_1,k\in\{0,1\}$ that is well-defined if and only if

$$-1\leq\frac{2(b^2+3b-5)}{b(1-3b)}\leq 1.$$

This condition is equivalent to

$$\frac{(b-1)(2+b)}{b(1-3b)}\leq 0\quad\text{and}\quad\frac{(b-5)(b-2)}{b(1-3b)}\leq 0. \tag{20}$$

Moreover

$$\rho_1=\sqrt{2h-r_1^2-R_1^2}=\sqrt{\frac{2(3a-2)h}{3(a-1)}}$$

must be well defined, so we must have

$$\frac{3a-2}{a-1} \geqslant 0$$

This yields that $a \leqslant 2/3$ and $b \in (-\infty,-2] \cup [1,2] \cup [5,+\infty)$. Imposing that it satisfies (20) we get that indeed this is the set of parameters for which $\beta_{j_1,k}$ is well-defined, i.e. for which (20) holds. In this case the solution is $(0, \beta_{j_1,k}, \sqrt{\frac{2h}{3(1-a)}}, \beta_{j_1,k})$. The Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(0, \beta_{j_1,k}, \sqrt{\frac{2h}{3(1-a)}}, \beta_{j_1,k})$ is

$$\frac{800h^4k_3^4\pi^4}{729(a-1)^3}((a-2)(3a-2)(1+3a)(b-5)(b-2)(b-1)(b+2).$$

Therefore the set of parameters $(a,b) \in \mathbb{R}^2$ for which the second averaging order provides a solution is $a \in (-\infty,-1/3) \cup (-1/3,2/3)$ and $b \in (-\infty,-2) \cup (1,2) \cup (5,+\infty)$. The eigenvalues are

$$\begin{aligned}
\lambda_{1,2} &= \pm\frac{2hk_3\pi}{3\sqrt{1-a}}\sqrt{5(a-2)(3a-2)(1+3a)}i,\\
\lambda_3 &= (-1)^{j_1+1}\frac{2k_3h\pi\mathrm{sign}(b(3b-1))}{9(a-1)}\sqrt{5(b-5)(b-2)(b-1)(b+2)},\\
\lambda_4 &= -2\lambda_3
\end{aligned}$$

Note that λ_3 and λ_4 are real eigenvalues with different sign. It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) has four linearly unstable $k_1\pi$-periodic solutions

$$\varphi_{j_1,\ell,k}(\theta,\varepsilon) = (r_{j_1,\ell,k}(\theta,\varepsilon), \alpha_{j_1,\ell,k}(\theta,\varepsilon), R_{j_1,\ell,k}(\theta,\varepsilon), \beta_{j_1,\ell,k}(\theta,\varepsilon)),$$

that tend to $(0, \alpha_{j_1,\ell}, \sqrt{\frac{2h}{3(1-a)}}, \beta_{j_1,k})$ when $\varepsilon \to 0$.

Now we go back through the changes of variables in (11). Substituting $\varphi_{j_1,\ell,k}(\theta,\varepsilon)$ in (15) we get $\Gamma_{j_1,\ell,k}(\theta,\varepsilon)$ and so $\rho_{j_1,\ell,k}(\theta,\varepsilon)$. Therefore $(\varphi_{j_1,\ell,k}(\theta,\varepsilon), \rho_{j_1,\ell,k}(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13). Moreover the terms of order zero of the solution with $k=1$ correspond to the terms of order zero of the solution with $k=0$ taking as initial angle $B_b/2+\pi/(2\omega_1)$ instead of $B_b/2$. So in principle for $|\varepsilon|$ sufficiently small we can only guarantee that we have two different solutions of system (9), which are the ones given in the statement of the theorem.

Case 2.3. If $b \neq 0$, $r=0$, $2(\alpha-\beta)=\pi$ then $\alpha=\beta+\pi/2$. Moreover setting $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ we get that either $a=3/5$, or $a \neq 3/5$, $b=(18a-1)/21$, or $a \neq 3/5$, $b \neq (18a-1)/21$ and $R=\sqrt{\frac{14h}{3(3-5a)}}$ with $a<3/5$ and $\cos(2\beta) = 2(1-18a+3b+9ab+7b^2)/(b(-21b+18a-1))$. The first case, i.e. $a=3/5$ does not provide any solution because if $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ then $f_{22}(\mathbf{x}) - f_{24}(\mathbf{x}) = 28hk_3\pi/15 = 0$ which is not possible. So $a \neq 3/5$. In a similar manner if $a \neq 3/5$ and $b=(18a-1)/21$ then if $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ we get $f_{22}(\mathbf{x}) - f_{24}(\mathbf{x}) = (3a-1)k_3\pi(14h-(9-15a)R^2)/6 = 0$, so either $a=1/3$, or $R=\sqrt{\frac{14h}{3(3-5a)}}$ with $a<3/5$. When $a=1/3$ then $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x})$ depend on R,β and are not identically zero. Therefore the Jacobian matrix of $f_2(\mathbf{x})$ evaluated at any solution of $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ will be zero, so we cannot go to third order in the averaging theory, and we do not get information on the periodic solutions in this case. When $R=\sqrt{\frac{14h}{3(3-5a)}}$ with $a<3/5$, then $f_{22}(\mathbf{x}) + f_{24}(\mathbf{x}) = 40(9a-11)(1-18a)hk_3\pi/(567(5a-3)) = 0$. Note that $a \neq 11/9$ because $a<3/5$ and $a \neq 1/18$, otherwise $b=0$ which is not possible. So this case is not possible which implies that $a \neq 3/5$ and $b \neq (18a-1)/21$. Therefore $R=\sqrt{\frac{14h}{3(3-5a)}}$ with $a<3/5$ and $\beta=\beta_{j_1,j}=\frac{(-1)^{j_1}}{2}C_{a,b}+j\pi$ with $j,j_1 \in \{0,1\}$ and $C_{a,b}$ was introduced in (4). In order that $\beta_{j_1,j}$ it is well defined we must have

$$-1 \leq \frac{2(1-18a+3b+9ab+7b^2)}{b(-21b+18a-1)} \leq 1.$$

This condition is equivalent to

$$\frac{-2+36a-7b-35b^2}{b(-21b+18a-1)} \geq 0 \quad \text{and} \quad \frac{(b-1)(-2+36a+7b)}{b(-21b+18a-1)} \geq 0.$$

Moreover

$$\rho_1 = \sqrt{2h-r_1^2-R_1^2} = \sqrt{\frac{2(15a-2)h}{3(5a-3)}}$$

must be well defined, so we must have

$$\frac{15a-2}{5a-3} \geqslant 0.$$

This yields that the set of parameters $(a,b) \in \mathbb{R}^2$ for which there exist solutions is: $a \leqslant 2/15$ and b such that for these values of a it is satisfied $\frac{-2+36a-7b-35b^2}{b(-21b+18a-1)} \geq 0$ and $\frac{(b-1)(-2+36a+7b)}{b(-21b+18a-1)} \geq 0$. In this case the solution is $(0, \alpha_{j_1,j}, \sqrt{\frac{14h}{3(3-5a)}}, \beta_{j_1,j})$ where $\alpha_{j_1,\ell} = \beta_{j_1,\ell}+\pi/2$. The Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(0, \alpha_{j_1,j}, \sqrt{\frac{14h}{3(3-5a)}}, \beta_{j_1,j})$ is

$$\frac{224h^4k_3^4\pi^4}{729(3-5a)^3}((a-2)(3a-1)(15a-2)(b-1)(-2+36a-7b)(-2+36a-7b-35b^2)$$

Therefore the set of parameters $(a,b) \in \mathbb{R}^2$ for which the second averaging order provides a solution is S_3 (see (5)). The eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= \pm\frac{2hk_3\pi}{3}\sqrt{\frac{7(a-2)(3a-1)(15a-2)}{3-5a}}, \\ \lambda_3 &= (-1)^{j_1+1}\frac{2hk_3\pi\mathrm{sign}(b(1-18a+21b))}{9(5a-3)} \\ &\quad \sqrt{(b-1)(-2+36a-7b)(-2+36a-7b-35b^2)}, \\ \lambda_4 &= -2\lambda_3. \end{aligned}$$

Notice that the eigenvalues λ_3 and λ_4 are real with different sign. It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) has four linearly unstable $k_1\pi$-periodic solutions

$$\varphi_{j_1,j}(\theta,\varepsilon) = (r_{j_1,j}(\theta,\varepsilon), \alpha_{j_1,j}(\theta,\varepsilon), R_{j_1,j}(\theta,\varepsilon), \beta_{j_1,j}(\theta,\varepsilon)),$$

that tend to $(0, \alpha_{j_1,j}, \sqrt{\frac{14h}{3(3-5a)}}, \beta_{j_1,j})$ when $\varepsilon \to 0$.

Now we go back through the changes of variables in (11). Substituting $\varphi_{j_1,j}(\theta,\varepsilon)$ in (15) and taking the square-root we get $\rho_{j_1,j}(\theta,\varepsilon)$. Therefore $(\varphi_{j_1,j}(\theta,\varepsilon), \rho_{j_1,j}(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13). Moreover the terms of order zero of the solution with $j=1$ correspond to the terms of order zero of the solution with $j=0$ taking as initial angle $C_{a,b}/2+\pi/(2\omega_1)$ instead of $C_{a,b}/2$. So in principle for $|\varepsilon|$ sufficiently small we can only guarantee that we have two different solutions of system (9), which are the ones given in the statement of the theorem.

From now one we can consider that $rb \neq 0$. Moreover we distinguish between the cases $\sin(2\alpha)=0$ and $\sin(2\alpha) \neq 0$.

Case 3: *$rb \neq 0$ and* $\sin(2\alpha)=0$. Hence $\alpha = j\pi/2$ with $j \in \{0,1\}$ and $f_{21}(\mathbf{x}) = -bk_3\pi r(a-2b)R^2\sin(2\beta)/2$. Setting $f_{21}(\mathbf{x})=0$ we have two possible cases: either $\sin(2\beta)=0$, or $\sin(2\beta)\neq 0$ and $a=2b$. We separate the study in these two cases.

Case 3.1 Assume $rb \neq 0$, $\sin(2\alpha)=0$, and $\sin(2\beta)=0$. In this case we write $\beta = k\pi/2$ with $k \in \{0,1\}$. We consider the different values of j and k separately.

Assume first that $j=k=0$, i.e. $\alpha=\beta=0$. In this case $f_{22}(\mathbf{x}) = 5k_3\pi(b-1)(2h+(b-1)r^2+(2b+3a+1)R^2)/6$. If $b=1$ then $f_{24}(\mathbf{x}) = 5k_3\pi(-2(1+3a)h+3(1+2a-3a^2)R^2)/6=0$. Any solution of $f_2(\mathbf{x})=0$ does not depend on r and so the Jacobian will be zero. This case is not possible. If $b\neq 1$ then solving $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ we get that either $D_{1,a,b}=0$, or $D_{1,a,b}\neq 0$ and

$$r = r_0 = \sqrt{\frac{-(1+3a)(2+b)h}{(b-1)D_{1,a,b}}}, \quad R = R_0 = \sqrt{\frac{-bh}{D_{1,a,b}}}.$$

If $D_{1,a,b}=0$, i.e. $a = -\frac{b^2+b+1}{3(b+1)}$ then solving $f_{22}(\mathbf{x})=0$ we get $r = \sqrt{(2(b+1)h+b(b+2)R^2)/(b^2-1)}$ but then $f_{24}(\mathbf{x}) = 10bhk_3\pi/3 \neq 0$, so this case is not possible. We note that if $b=-1$ then $D_{1,a,b}\neq 0$. If $D_{1,a,b}\neq 0$, then the set of parameters $(a,b)\in\mathbb{R}^2$ for which there exists solution must satisfy

$$\frac{(1+3a)(2+b)h}{(b-1)D_{1,a,b}} \leqslant 0, \quad \frac{bh}{D_{1,a,b}} < 0, \tag{21}$$

and the solution is $(r_0,0,R_0,0)$. Moreover

$$\rho_0=\sqrt{2h-r_0^2-R_0^2}=\sqrt{b(3a+b)(1+2b)h/((b-1)D_{1,a,b})},$$

must be well defined, so we must have

$$\frac{b(3a+b)(1+2b)}{(b-1)D_{1,a,b}}\geqslant 0. \tag{22}$$

In addition the Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(r_0,0,R_0,0)$ must be different from zero, i.e,

$$\frac{-1400(1+3a)b^4(2+b)(3a+b)(1+2b)(a-b+ab)h^4k_3^4\pi^4}{27D_{1,a,b}^3}\neq 0. \tag{23}$$

All these conditions (equations (21), (23) and (22)) define the domain S_4. The eigenvalues of the Jacobian matrix at $(r_0,0,R_0,0)$ are

$$\lambda_{1,2,3,4}=\pm\frac{\sqrt{5}\pi k_3h|b|\sqrt{T_1\pm\sqrt{T_2}}}{3\sqrt{2}D_{1,a,b}},$$

where

$$T_1=-(3a+1)(3a+b)(-9a^2-27ab-27a+16b^2+49b+16),$$

and

$$\begin{aligned}T_2=&(1+3a)(3a+b)(2112a+3744a^2-6453a^3+5427a^4+4617a^5\\&+729a^6-1088b+6624ab+19575a^2b-28053a^3b+8181a^4b\\&+4617a^5b-3136b^2+6573ab^2+33957a^2b^2-28053a^3b^2+5427a^4b^2\\&-3135b^3+6573ab^3+19575a^2b^3-6453a^3b^3-3136b^4+6624ab^4\\&+3744a^2b^4-1088b^5+2112ab^5).\end{aligned}$$

Clearly the set of solutions of equation $T_1^2-T_2=0$ contains the sets of solutions of the two equations $T_1\pm\sqrt{T_2}=0$, moreover

$$T_1^2-T_2=-672(3a+1)(b+2)(3a+b)(1+2b)(a-b+ab)D_{1,a,b}.$$

Therefore the solutions of equation $T_1\pm\sqrt{T_2}=0$ belong to the boundary of the domain S_4. Computing the sign of $T_1\pm\sqrt{T_2}$ at a point in each connected component of S_4 we get that in all domain S_4 the eigenvalues $\lambda_{1,2}$ are complex while $\lambda_{3,4}$ are real with different signs. It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) has one unstable $k_1\pi$-periodic solution

$$\varphi_0(\theta,\varepsilon)=(r_0(\theta,\varepsilon),\alpha_0(\theta,\varepsilon),R_0(\theta,\varepsilon),\beta_0(\theta,\varepsilon)),$$

that tend to $(r_0,0,R_0,0)$ when $\varepsilon\to 0$. Now we go back through the changes of variables in (11). Substituting $\varphi_0(\theta,\varepsilon)$ in (15) we get $\Gamma_0(\theta,\varepsilon)$ and so $\rho_0(\theta,\varepsilon)$, where $\rho_0(\theta,0)=\rho_0$. Then in the domain S_4, $(\varphi_0(\theta,\varepsilon),\rho_0(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13) and for $|\varepsilon|$ sufficiently small, we have the solution of system (9) in the statement of the theorem.

Now we consider the case $j=0$, $k=1$, i.e. $\alpha=0$, $\beta=\pi/2$. In this case $f_{22}(\mathbf{x})=(b-1)(10h+5(b-1)r^2-(7-21a+2b)R^2)k_3\pi/6$. If $b=1$ then $f_{24}(\mathbf{x})=(2(7-21a)h+3(-3+14a-15a^2)R^2)hk_3\pi/6=0$. Any solution of $f_2(\mathbf{x})=0$ does not depend on r and so the Jacobian will be zero. This case is not possible. If $b\neq 1$ then solving $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ we get that either $D_{2,a,b}=0$, or $D_{2,a,b}\neq 0$ and

$$r=r_1=\sqrt{\frac{-(-1+3a)(-2+36a-7b)h}{(b-1)D_{2,a,b}}},\quad R=R_1=\sqrt{\frac{5bh}{D_{2,a,b}}}.$$

If $D_{2,a,b}=0$, then $b=(-7+21a\pm\sqrt{15}\sqrt{3-14a+15a^2})/2$. When $b=(-7+21a-\sqrt{15}\sqrt{3-14a+15a^2})/2$ we compute r from equation $f_{22}(\mathbf{x})=0$ and we substitute it into $f_{24}(\mathbf{x})$ and we get $f_{24}(\mathbf{x})=(7-21a+\sqrt{15}\sqrt{(3a-1)(5a-2)})hk_3\pi/3=0$ which implies $a=1/3$. So when $a=1/3$ we obtain a solution in function of R. When $b=(-7+21a+\sqrt{15}\sqrt{3-14a+15a^2})/2$ we compute r from equation $f_{22}(\mathbf{x})=0$ and we substitute it into $f_{24}(\mathbf{x})$ we get $f_{24}(\mathbf{x})=-(-7+21a+\sqrt{15}\sqrt{(3a-1)(5a-2)})hk_3\pi/3=0$ which implies either $a=1/3$ or $a=1/18$. So when $a=1/3$ and $a=1/18$ we obtain solutions that are functions of R. Therefore the Jacobian matrix of $f_2(\mathbf{x})$ evaluated at the solutions with $b=(-7+21a\pm\sqrt{15}\sqrt{3-14a+15a^2})/2$ will be zero and we cannot go to third order in the averaging theory. If $D_{2,a,b} \neq 0$, then the set of parameters $(a, b) \in \mathbb{R}^2$ for which there exists solution must satisfy

$$\frac{(-1+3a)(-2+36a-7b)}{(b-1)D_{2,a,b}} \leqslant 0, \qquad \frac{5b}{D_{2,a,b}} > 0, \tag{24}$$

and the solution is $(r_1, 0, R_1, \pi/2)$. Moreover

$$\rho_1 = \sqrt{2h - r_1^2 - R_1^2} = \sqrt{\frac{b(3a-b)(-7+36a-2b)h}{(b-1)D_{2,a,b}}}$$

must be well defined, so we must have

$$\frac{b(3a-b)(-7+36a-2b)}{(b-1)D_{2,a,b}} \geqslant 0. \tag{25}$$

In addition the Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(r_1, 0, R_1, \pi/2)$ must be different from zero, i.e,

$$-\frac{40}{27} r_1^2 \rho_1^2 R_1^2 b^2 (b-1)^2 (a - 18a^2 + 5b + ab) h k_3^4 \pi^4. \tag{26}$$

Taking into account the conditions in (24), (26) and (25) we get the domain S_5. The eigenvalues of the Jacobian matrix at $(r_1, 0, R_1, \pi/2)$ are

$$\lambda_{1,2,3,4} = \pm \frac{\sqrt{5}\pi k_3 h |b| \sqrt{T_3 \pm \sqrt{T_4}}}{3\sqrt{2} D_{2,a,b}},$$

where

$$T_3 = 49(3a-1)(3a-b)(117a^2 - 63ab - 63a + 8b^2 + 11b + 8)$$

and

$$\begin{aligned} T_4 = (3a-1)(3a-b)(&416737953a^6 - 501142545a^5 b - 501142545a^5 \\ &+ 252000315a^4 b^2 + 491523309a^4 b + 252000315a^4 - 65772675a^3 b^3 \\ &- 193668867a^3 b^2 - 193668867a^3 b - 65772675a^3 + 8727120a^2 b^4 \\ &+ 38785311a^2 b^3 + 54305037a^2 b^2 + 38785311a^2 b + 8727120a^2 \\ &- 462336ab^5 - 3868128ab^4 - 6942549ab^3 - 6942549ab^2 - 3868128ab \\ &- 462336a + 146944b^5 + 350096b^4 + 406329b^3 + 350096b^2 + 146944b). \end{aligned}$$

Now

$$T_3^2 - T_4 = -96(3a-1)(36a-7b-2)(36a-2b-7)(3a-b)\left(18a^2 - ab - a - 5b\right) D_{2,a,b},$$

so all the solutions of equation $T_3 \pm \sqrt{T_4} = 0$ belong to the boundary of the domain S_5. Computing the sign of $T_3 \pm \sqrt{T_4}$ at a point in each connected component of S_5 we get that for different values of $(a, b) \in S_5$ the eigenvalues $\lambda_{1,2}, \lambda_{3,4}$ are all complex, or two complex and two real (with different signs). It follows from Theorem 4 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (16) has one $k_1\pi$-periodic solution

$$\varphi_1(\theta, \varepsilon) = (r_1(\theta, \varepsilon), \alpha_1(\theta, \varepsilon), R_1(\theta, \varepsilon), \beta_1(\theta, \varepsilon)),$$

that tend to $(r_1, 0, R_1, \pi/2)$ when $\varepsilon \to 0$. Now we go back through the changes of variables in (11). Substituting $\varphi(\theta, \varepsilon)$ in (15) we get $\Gamma_1(\theta, \varepsilon)$ and so $\rho_1(\theta, \varepsilon)$ where $\rho_1(\theta, 0) = \rho_1$. Then in S_6 $(\varphi_1(\theta, \varepsilon), \rho_1(\theta, \varepsilon))$ is a 2π-periodic solution for the differential system (13) and for $|\varepsilon|$ sufficiently small, we have the solution of system (9) in the statement of the theorem.

Now consider the case $j = 1$, $k = 0$, i.e. $\alpha = \pi/2$ and $\beta = 0$. In this case solving $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ we get that either $D_{3,a,b} = 0$, or $D_{3,a,b} \neq 0$ and

$$r = r_2 = \sqrt{\frac{(3a-1)(-2+36a-7b-35b^2)h}{D_{3,a,b}}},$$

$$R = R_2 = \sqrt{\frac{b(7-36a+32b-35b^2)h}{D_{3,a,b}}}.$$

If $D_{3,a,b} = 0$, then proceeding as in the previous case we get that there is no solution. Hence $D_{3,a,b} \neq 0$ and the set of parameters $(a, b) \in \mathbb{R}^2$ for which there exists solution must satisfy

$$\frac{(3a-1)(-2+36a-7b-35b^2)}{D_{3,a,b}} \geqslant 0, \quad \frac{b(7-36a+32b-35b^2)}{D_{3,a,b}} > 0, \tag{27}$$

and the solution is $(r_2, \pi/2, R_2, 0)$. Note that

$$\rho_2 = \sqrt{2h - r_2^2 - R_2^2} = \sqrt{5}\sqrt{b(3a+b)(2b-1)(5b-1)h/D_{3,a,b}}$$

must be well defined, so we must have that

$$\frac{b(3a+b)(2b-1)(5b-1)h}{D_{3,a,b}} \geqslant 0. \tag{28}$$

The Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(r_2, \pi/2, R_2, 0)$ is

$$-\frac{40}{27} r_2^2 \rho_2^2 R_2^2 (b-1) b^2 D_{4,a,b} h k_3^4 \pi^4. \tag{29}$$

It must be different from zero. Taking into account the conditions in (25), (27) and (29) we get the domain S_6. The eigenvalues of the Jacobian matrix are

$$\lambda_{1,2,3,4} = \pm \frac{\sqrt{5}\pi k_3 h |b| \sqrt{T_5 \pm \sqrt{T_6}}}{3\sqrt{2} D_{3,a,b}},$$

where

$$\begin{aligned} T_5 = (3a-1)(3a+b)(&11664a^4 - 16200a^3b^2 - 36288a^3b - 4536a^3 - 26775a^2b^4 \\ &+ 33300a^2b^3 + 98442a^2b^2 - 20484a^2b + 4653a^2 + 33075ab^5 + 19125ab^4 \\ &- 108450ab^3 - 702ab^2 + 13455ab - 2583a - 23800b^6 + 21525b^5 \\ &- 15948b^4 + 33538b^3 - 10092b^2 - 855b + 368), \end{aligned}$$

$$
\begin{aligned}
T_6 = {}& (3a-1)(3a+b)(1224440064a^{10} - 3401222400a^9b^2 - 7210591488a^9b \\
& - 1360488960a^9 - 3259504800a^8b^4 + 16439241600a^8b^3 + 32436324288a^8b^2 \\
& + 748268928a^8b + 1479531744a^8 + 7807590000a^7b^6 + 13636382400a^7b^5 \\
& - 37311094800a^7b^4 - 78014803968a^7b^3 - 7042105008a^7b^2 + 2397231936a^7b \\
& - 1309575600a^7 + 6452105625a^6b^8 - 23091075000a^6b^7 - 67336782300a^6b^6 \\
& + 70531770600a^6b^5 + 121271458014a^6b^4 + 7283485656a^6b^3 + 1400627700a^6b^2 \\
& - 2954683656a^6b + 911349873a^6 - 13789794375a^5b^9 + 6832704375a^5b^8 \\
& + 112731369300a^5b^7 + 9815863500a^5b^6 - 182653848018a^5b^5 - 14817951918a^5b^4 \\
& + 17277847236a^5b^3 - 4867747812a^5b^2 + 2409516801a^5b - 491368113a^5 \\
& + 16002511875a^4b^{10} - 2807713125a^4b^9 - 33725324025a^4b^8 - 82827238500a^4b^7 \\
& + 23736107646a^4b^6 + 148014385290a^4b^5 - 56156297058a^4b^4 - 839571156a^4b^3 \\
& + 3036868767a^4b^2 - 1228397805a^4b + 192591675a^4 - 3888793125a^3b^{11} \\
& - 15653806875a^3b^{10} + 24647153175a^3b^9 - 23433071175a^3b^8 + 107442882054a^3b^7 \\
& - 74209627926a^3b^6 - 38992058154a^3b^5 + 33659294202a^3b^4 - 4412798433a^3b^3 \\
& - 638016399a^3b^2 + 354837267a^3b - 49317363a^3 - 1830150000a^2b^{12} \\
& - 3442359375a^2b^{11} + 40547641725a^2b^{10} - 58132343475a^2b^9 + 43818769476a^2b^8 \\
& - 59563412694a^2b^7 + 43152985926a^2b^6 - 1509382998a^2b^5 - 7374977532a^2b^4 \\
& + 1593243909a^2b^3 + 18855621a^2b^2 - 51671511a^2b + 7005600a^2 + 611520000ab^{13} \\
& + 2038260000ab^{12} - 4237211475ab^{11} - 9607597725ab^{10} + 20409022452ab^9 \\
& - 15003156852ab^8 + 12463368246ab^7 - 8118301878ab^6 + 1377168804ab^5 \\
& + 676821084ab^4 - 176089371ab^3 - 470133ab^2 + 3544224ab - 407616a \\
& + 580160000b^{13} - 3084270000b^{12} + 5231017575b^{11} - 2611352200b^{10} \\
& - 1359776484b^9 + 2134756440b^8 - 1389500574b^7 + 569072808b^6 \\
& - 56961396b^5 - 43497928b^4 + 7278159b^3 + 767232b^2 - 123328b).
\end{aligned}
$$

Now

$$T_5^2 - T_6 = 96(3a-1)(b-1)(2b-1)(5b-1)(3a+b)\left(36a - 35b^2 - 7b - 2\right)\left(36a + 35b^2 - 32b - 7\right) D_{4,a,b}D_{3,a,b},$$

so all the solutions of equation $T_5 \pm \sqrt{T_6} = 0$ belong to the boundary of the domain S_6. Computing the sign of $T_5 \pm \sqrt{T_6}$ at a point in each connected component of S_6 it can be shown that for different values of $(a,b) \in S_6$ we have that the eigenvalues $\lambda_{1,2}, \lambda_{3,4}$ are all complex, two complex and two real (with different signs), or all real (also with different signs). It follows from Theorem 4 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (16) has one $k_1\pi$-periodic solution

$$\varphi_2(\theta,\varepsilon) = (r_2(\theta,\varepsilon), \alpha_2(\theta,\varepsilon), R_2(\theta,\varepsilon), \beta_2(\theta,\varepsilon)),$$

that tend to $(r_2, 0, R_2, 0)$ when $\varepsilon \to 0$. Now we go back through the changes of variables in (11). Substituting $\varphi_2(\theta,\varepsilon)$ in (15) we get $\Gamma_2(\theta,\varepsilon)$ and so $\rho_2(\theta,\varepsilon)$ with $\rho_2(\theta,0) = \rho_2$. Then in this domain $(\varphi_2(\theta,\varepsilon), \rho_2(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13) and for $|\varepsilon|$ sufficiently small, we have the solution of system (9) in the statement of the theorem.

Finally we study the case $j = k = 1$, i.e. $\alpha = \beta = \pi/2$. In this case solving $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ we get that either $D_{5,a,b} = 0$, or $D_{5,a,b} \neq 0$ and

$$r = r_3 = \sqrt{\frac{5(1+3a)(b-2)(b-5)h}{D_{5,a,b}}},$$

$$R = R_3 = \sqrt{\frac{b(-35 - 4(-8+9a)b + 7b^2)h}{D_{5,a,b}}}.$$

If $D_{5,a,b} = 0$, then proceeding as in the previous cases we get that there is no solution. Hence $D_{5,a,b} \neq 0$ and the set of parameters $(a, b) \in \mathbb{R}^2$ for which there exists solution must satisfy

$$\frac{(1+3a)(b-2)(b-5)}{D_{5,a,b}} \geqslant 0, \quad \frac{b(-35-4(-8+9a)b+7b^2)}{D_{5,a,b}} > 0, \tag{30}$$

and the solution is $(r_3, \pi/2, R_3, \pi/2)$. Moreover

$$\rho_3 = \sqrt{2h - r_3^2 - R_3^2} = \sqrt{b(3a-b)(-35-7b+36ab-2b^2)h/D_{5,a,b}},$$

must be well defined, so we must have that

$$\frac{b(3a-b)(-35-7b+36ab-2b^2)}{D_{5,a,b}} \geqslant 0. \tag{31}$$

The Jacobian matrix of $f_2(\mathbf{x})$ evaluated at $(r_3, \pi/2, R_3, \pi/2)$ is

$$-\frac{40}{27} r_3^2 \rho_3^2 R_3^2 (b-1) b^2 D_{6,a,b} h k_3^4 \pi^4. \tag{32}$$

It must be different from zero. Taking into account the conditions in (30), (32) and (31) we get the domain S_7. The eigenvalues are

$$\lambda = \pm \frac{\sqrt{5}\pi k_3 h |b| \sqrt{T_7 \pm \sqrt{T_8}}}{2\sqrt{3} D_{5,a,b}},$$

where

$$\begin{aligned} T_7 = & -(1+3a)(-3a+b)(-23800+33075a-26775a^2+21525b+19125ab \\ & +33300a^2b-16200a^3b-15948b^2-108450ab^2+98442a^2b^2-36288a^3b^2 \\ & +11664a^4b^2+33538b^3-702ab^3-20484a^2b^3-4536a^3b^3-10092b^4 \\ & +13455ab^4+4653a^2b^4-855b^5-2583ab^5+368b^6), \end{aligned}$$

$$\begin{aligned}T_8 = {}&(3a+1)(3a-b)(1224440064a^{10}b^4 - 1360488960a^9b^5 - 7210591488a^9b^4\\
&- 3401222400a^9b^3 + 1479531744a^8b^6 + 748268928a^8b^5 + 32436324288a^8b^4\\
&+ 16439241600a^8b^3 - 3259504800a^8b^2 - 1309575600a^7b^7 + 2397231936a^7b^6\\
&- 7042105008a^7b^5 - 78014803968a^7b^4 - 37311094800a^7b^3 + 13636382400a^7b^2\\
&+ 7807590000a^7b + 911349873a^6b^8 - 2954683656a^6b^7 + 1400627700a^6b^6\\
&+ 7283485656a^6b^5 + 121271458014a^6b^4 + 70531770600a^6b^3 - 67336782300a^6b^2\\
&- 23091075000a^6b + 6452105625a^6 - 491368113a^5b^9 + 2409516801a^5b^8\\
&- 4867747812a^5b^7 + 17277847236a^5b^6 - 14817951918a^5b^5 - 182653848018a^5b^4\\
&+ 9815863500a^5b^3 + 112731369300a^5b^2 + 6832704375a^5b - 13789794375a^5\\
&+ 192591675a^4b^{10} - 1228397805a^4b^9 + 3036868767a^4b^8 - 839571156a^4b^7\\
&- 56156297058a^4b^6 + 148014385290a^4b^5 + 23736107646a^4b^4 - 82827238500a^4b^3\\
&- 33725324025a^4b^2 - 2807713125a^4b + 16002511875a^4 - 49317363a^3b^{11}\\
&+ 354837267a^3b^{10} - 638016399a^3b^9 - 4412798433a^3b^8 + 33659294202a^3b^7\\
&- 38992058154a^3b^6 - 74209627926a^3b^5 + 107442882054a^3b^4 - 23433071175a^3b^3\\
&+ 24647153175a^3b^2 - 15653806875a^3b - 3888793125a^3 + 7005600a^2b^{12}\\
&- 51671511a^2b^{11} + 18855621a^2b^{10} + 1593243909a^2b^9 - 7374977532a^2b^8\\
&- 1509382998a^2b^7 + 43152985926a^2b^6 - 59563412694a^2b^5 + 43818769476a^2b^4\\
&- 58132343475a^2b^3 + 40547641725a^2b^2 - 3442359375a^2b - 1830150000a^2\\
&- 407616ab^{13} + 3544224ab^{12} - 470133ab^{11} - 176089371ab^{10} + 676821084ab^9\\
&+ 1377168804ab^8 - 8118301878ab^7 + 12463368246ab^6 - 15003156852ab^5\\
&+ 20409022452ab^4 - 9607597725ab^3 - 4237211475ab^2 + 2038260000ab\\
&+ 611520000a - 123328b^{13} + 767232b^{12} + 7278159b^{11} - 43497928b^{10}\\
&- 56961396b^9 + 569072808b^8 - 1389500574b^7 + 2134756440b^6 - 1359776484b^5\\
&- 2611352200b^4 + 5231017575b^3 - 3084270000b^2 + 580160000b).\end{aligned}$$

Now

$$T_7^2 - T_8 = -96(3a+1)(b-5)(b-2)(b-1)(3a-b)\left(36ab - 7b^2 - 32b + 35\right)\left(36ab - 2b^2 - 7b - 35\right)D_{6,a,b}D_{5,a,b},$$

so all the solutions of equation $T_5 \pm \sqrt{T_6} = 0$ belong to the boundary of the domain S_7. Computing the sign of $T_5 \pm \sqrt{T_6}$ at a point in each connected component of S_7 it can be shown that for different values of $(a,b) \in S_7$ we have that the eigenvalues $\lambda_{1,2}, \lambda_{3,4}$ are all complex, two complex and two real (with different signs), or all real (also with different signs). It follows from Theorem 4 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (16) has one $k_1\pi$-periodic solution

$$\varphi_3(\theta,\varepsilon) = (r_3(\theta,\varepsilon), \alpha_3(\theta,\varepsilon), R_3(\theta,\varepsilon), \beta_3(\theta,\varepsilon)),$$

that tend to $(r_3, 0, R_3, 0)$ when $\varepsilon \to 0$. Now we go back through the changes of variables in (11). Substituting $\varphi_2(\theta,\varepsilon)$ in (15) we get $\Gamma_3(\theta,\varepsilon)$ and so $\rho_3(\theta,\varepsilon)$, where $\rho_3(\theta,0) = \rho_3$. Then in this domain $(\varphi_3(\theta,\varepsilon), \rho_3(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13) and for $|\varepsilon|$ sufficiently small, we have the solution of system (9) in the statement of the theorem.

Case 3.2 Here $rb \neq 0$, $\sin(2\alpha) = 0$, $\sin(2\beta) \neq 0$ and $a = 2b$. In this case $f_{23}(\mathbf{x}) = -(-1)^j k_3\pi R(1-b)(2h - r^2 - R^2)\sin(2\beta)$. Setting $f_{23}(\mathbf{x}) = 0$ we get that either $b = 1$, or $b \neq 1$ and $2h - r^2 - R^2 = 0$. When $b = 1$ solving $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ we obtain $r = \sqrt{\frac{2h}{3}}$ and $R = \sqrt{\frac{2h}{3}}i$ when $j = 0$ and $r = \sqrt{\frac{70h}{53}}$ and $R = \sqrt{\frac{34h}{53}}i$ when $j = 1$ which is not possible. So $b \neq 1$. If $r = \sqrt{2h - R^2}$ then solving $f_{22}(\mathbf{x}) = f_{24}(\mathbf{x}) = 0$ we obtain $R = \sqrt{\frac{2h}{3}}i$ which is not possible.

Case 4: $rb\sin(2\alpha)\neq 0$. In this case we consider that either $a=2b$ and $r=\sqrt{2h-R^2}$, or $a\neq 2b$, $\sin(2\beta)=0$ and $r=\sqrt{2h-R^2}$, or $a\neq 2b$, $\sin(2\beta)\neq 0$, $r\neq\sqrt{2h-R^2}$ and $\kappa_1 := b\sin(2\alpha)+3(2-a)\sin(2(\alpha-\beta))+3b(2b-a)\sin(2\beta)=0$, or $a\neq 2b$, $\sin(2\beta)\neq 0$, $r\neq\sqrt{2h-R^2}$, and $\kappa_1\neq 0$.

Case 4.1 Assume $rb\sin(2\alpha)\neq 0$, $a=2b$ and $r=\sqrt{2h-R^2}$. Solving $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ we get $R=\sqrt{\frac{2h}{3}}i$ which is not possible.

Case 4.2 Assume $rb\sin(2\alpha)\neq 0$, $a\neq 2b$, $\sin(2\beta)=0$ and $r=\sqrt{2h-R^2}$. So $\beta=k\pi/2$ with $k\in\{0,1\}$.

First we study the case $k=0$. If $k=0$ and $b=a$ then $f_{24}(\mathbf{x})=-20a^2hk_3\pi/3$ which is different from zero if $a\neq 0$. When $a=0$, $b=0$ which is not possible.

If $k=0$, $b\neq a$ and $b=3$ then $f_{24}(\mathbf{x})=-15(a+1)k_3\pi(2h+(a-3)R^2)/2$ so either $R=\sqrt{2h/(3-a)}$ (recall that $a\neq 3$) or $a=-1$. If $a=-1$ then $f_{24}(\mathbf{x})=0$ and $f_{22}(\mathbf{x})$ is a function of R and α which is not identically zero, therefore we cannot go to averaging of third order. If $R=\sqrt{2h/(3-a)}$, then $f_{22}=140hk_3\pi/(9-3a)\neq 0$.

If $k=0$, $b\neq a$ and $b\neq 3$, then solving $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ we get

$$R=\sqrt{\frac{2bh}{3(b-a)}},\quad \cos(2\alpha)=-\frac{2(-1-3b+5b^2)}{b-3}.$$

In order that R is well defined we must have $b(b-a)>0$ and in order that α is well defined we must have

$$-1\leq-\frac{2(-1-3b+5b^2)}{b-3}\leq 1.$$

The condition is equivalent to

$$\frac{(b-1)(1+2b)}{b-3}\geq 0\quad\text{and}\quad\frac{(2b-1)(5b-1)}{b-3}\leq 0.$$

So $b\in[-1/2,1/5]\cup[1/2,1]$ and we have that

$$\alpha_{j_1,j}=\frac{(-1)^{j_1}}{2}\arccos\Big(-\frac{2(-1-3b+5b^2)}{b-3}\Big)+l\pi,\quad j_1,j\in\{0,1\}$$

is well-defined. On the other hand $r=\sqrt{2h-R^2}=\sqrt{2(3a-2b)h/(3(a-b))}$ which is well defined when $(3a-2b)/(a-b)\geqslant 0$. The Jacobian matrix of $f_2(\mathbf{x})$ evaluated at the solution $\Big(\sqrt{\frac{2h(3a-2b)}{3a-3b}},\alpha_{j_1,j},\sqrt{\frac{2bh}{3(b-a)}},0\Big)$ is

$$1600\frac{(-1)^j h^4k_3^4\pi^4}{729(a-b)^3}(2b-a)(b-1)(2b-1)(1+2b)(5b-1)(3a-2b)(3a+b).$$

Therefore the set of parameters $(a,b)\in\mathbb{R}^2$ for which the second averaging order provides a solution is S_8 (see (5)). The eigenvalues are

$$\lambda_{1,2}=\pm\frac{(-1)^{j_1}4b\,\mathrm{sign}(b-3)hk_3\pi}{9(a-b)}\sqrt{-5(b-1)(2b-1)(1+2b)(5b-1)},$$
$$\lambda_{3,4}=\pm\frac{2bhk_3\pi\sqrt{5(a-2b)(3a-2b)(a-b)(3a+b)}}{3(a-b)}.$$

Analyzing these eigenvalues in each connected component of S_8 it can be shown that for different values of $(a,b)\in S_8$ we have that the eigenvalues $\lambda_{1,2,3,4}$ are all real or two complex and two real (with different signs). It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) has four unstable $k_1\pi$-periodic solutions

$$\varphi_{j_1,j}(\theta,\varepsilon)=(r_{j_1,j}(\theta,\varepsilon),\alpha_{j_1,j}(\theta,\varepsilon),R_{j_1,j}(\theta,\varepsilon),\beta_{j_1,j}(\theta,\varepsilon)),$$

that tend to $\Big(\sqrt{\frac{2h(3a-2b)}{3a-3b}},\alpha_{j_1,j},\sqrt{\frac{2bh}{3(b-a)}},0\Big)$ when $\varepsilon\to 0$.

Now we go back through the changes of variables in (11). Substituting $\varphi_{j_1,j}(\theta,\varepsilon)$ in (15) and taking the square-root we get $\rho_{j_1,j}(\theta,\varepsilon)$. Therefore $(\varphi_{j_1,j}(\theta,\varepsilon),\rho_{j_1,j}(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13). Moreover the terms of order zero of the solution with $j\in\{0,1\}$ and $j_1\in\{0,1\}$ are the same, so we can only guarantee that we have the solution of system (9) given in the statement of the theorem.

Now we study the case $k=1$. If $k=1$ and $b=5a/3$, then $f_{24}(\mathbf{x})=-140a^2hk_3\pi/27$ which is diferent from zero if $a\neq 0$. When $a=0$, $b=0$ which is not possible. If $k=1$, $b\neq 5a/3$ and $b=3(6a-7)$, then $f_{24} = 21k3\pi(5a-7)(-14h+12ah+9R^2-7aR^2)/2$. Thus either $a=7/5$, or $a=9/7$, or $a\neq 9/7$ and $R=\sqrt{2(6a-7)h/(7a-9)}$. When $a=7/5$ then $f_{24}(\mathbf{x})=0$ and $f_{22}(\mathbf{x})$ is a function of R and α which is not identically zero, therefore we cannot go to averaging of third order. When $a=9/7$ then $f_{24}(\mathbf{x})=-60hk_3\pi/7\neq 0$. When $a\neq 9/7$ and $R=\sqrt{2(6a-7)h/(7a-9)}$ then $f_{22}(\mathbf{x})=20(-7+6a)(-11+9a)hk3\pi/(3(-9+7a))$ which is zero when $a=7/6$ and $a=11/9$ but in these last two cases the solutions depend on α, so they are not possible.

If $k=1$, $b\neq 5a/3$ and $b\neq 3(6a-7)$, then then solving $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ we get

$$R=\sqrt{\frac{14bh}{3(3b-5a)}},\quad \cos(2\alpha)=\frac{2(-7-9a-3b+18ab-b^2)}{-21+18a-b}.$$

In order that R is well defined we must have $b(3b-5a)>0$ and in order that α is well defined we must have

$$-1\leq\frac{2(-7-9a-3b+18ab-b^2)}{-21+18a-b}\leq 1.$$

The condition is equivalent to

$$\frac{(b-1)(-7+36a-2b)}{-21+18a-b}\leq 0\quad\text{and}\quad\frac{-35-7b+36ab-2b^2}{-21+18a-b}\geq 0. \tag{33}$$

So when condition (33) holds then $\alpha=\alpha_{j_1,j}$ with

$$\alpha_{j_1,j}=\frac{(-1)^{j_1}}{2}\arccos\Big(-\frac{2(-7-9a-3b+18ab-b^2)}{-21+18a-b}\Big)+j\pi,\quad j,j_1\in\{0,1\}.$$

If, additionally, $(15a-2b)/(5a-3b)\geqslant 0$, then we have that the solution exists and it is

$$\Big(\sqrt{\frac{2h(15a-2b)}{3(5a-3b)}},\alpha_{j_1,j},\sqrt{\frac{14bh}{3(3b-5a)}},\frac{\pi}{2}\Big).$$

The Jacobian matrix of $f_2(\mathbf{x})$ evaluated at the above solution is

$$\frac{(-1)^{j+1}448h^4k_3^4\pi^4}{729(3b-5a)^3}(b^4(b-1)(b-3a)(7-36a+2b)(2b-15a)(2b-a)(35+7b-36ab+2b^2).$$

Therefore the set of parameters $(a,b)\in\mathbb{R}^2$ for which the second averaging order provides a solution is S_9 (see (5)). The eigenvalues are

$$\lambda_{1,2}=\pm\frac{4b(-1)^{j_1}\operatorname{sign}(21-18a+b)hk_3\pi}{9(5a-3b)}\sqrt{-(b-1)(-7+36a-2b)(-35-7b+36ab-2b^2)},$$
$$\lambda_{3,4}=i\pm\frac{2bhk_3\pi\sqrt{7(5a-3b)(a-2b)(15a-2b)(3a-b)}}{15a-9b}.$$

Analyzing these eigenvalues in each connected component of S_9 it can be shown that in all S_9 they are two complex and two real (with different signs). It follows from Theorem 4 that for any given $h>0$ and for $|\varepsilon|$ sufficiently small, system (16) has four unstable $k_1\pi$-periodic solutions

$$\varphi_{j_1,j}(\theta,\varepsilon)=(r_{j_1,j}(\theta,\varepsilon),\alpha_{j_1,j}(\theta,\varepsilon),R_{j_1,j}(\theta,\varepsilon),\beta_{j_1,j}(\theta,\varepsilon)),$$

that tend to $\Big(\sqrt{\frac{2h(15a-2b)}{3(5a-3b)}},\alpha_{j_1,j},\sqrt{\frac{14bh}{3(3b-5a)}},\frac{\pi}{2}\Big)$ when $\varepsilon\to 0$.

Now we go back through the changes of variables in (11). Substituting $\varphi_{j_1,j}(\theta,\varepsilon)$ in (15) and taking the square-root we get $\rho_{j_1,j}(\theta,\varepsilon)$. Therefore $(\varphi_{j_1,j}(\theta,\varepsilon),\rho_{j_1,j}(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13). Moreover the terms of order zero of the solution with $j\in\{0,1\}$ and $j_1\in\{0,1\}$ are the same, so we can only guarantee that we have the solution of system (9) given in the statement of the theorem.

Case 4.3 Assume $rb\sin(2\alpha)\neq 0$, $a\neq 2b$, $\sin(2\beta)\neq 0$, $r\neq\sqrt{2h-R^2}$ and $\kappa_1=0$. Solving $\kappa_1=0$ in $\sin(2\alpha)$ we get $\sin(2\alpha)=\frac{3((a-2)\sin(2(\alpha-\beta))-b(2b-a)\sin(2\beta))}{b}$. Then solving $f_{21}(\mathbf{x})=0$ in $\sin(2(\alpha-\beta))$ we obtain that either $a=2$, or $a\neq 2$ and $r=\sqrt{2h-R^2}$ (which is not possible by assumptions), or $a\neq 2$, $r\neq\sqrt{2h-R^2}$ and $\sin(2(\alpha-\beta))=\frac{b(2b-a)(2h-r^2)\sin(2\beta)}{(a-2)(2h-r^2-R^2)}$. If $a=2$ then from $f_{21}(\mathbf{x})=0$ we get that $(b-1)(2h-r^2)\sin(2\beta)=0$. Since $\sin(2\beta)\neq 0$ and $a\neq 2b$ we only have the case $r=\sqrt{2h}$. But then $f_{23}(\mathbf{x})=-2(b-1)bhk_3\pi\sin(2\beta)$ which is never zero, so this case is not possible. If $a\neq 2$, $r\neq\sqrt{2h-R^2}$ and $\sin(2(\alpha-\beta))=\frac{b(2b-a)(2h-r^2)\sin(2\beta)}{(a-2)(2h-r^2-R^2)}$ then $f_{23}(\mathbf{x})=-b(2b-a)hk_3h\pi R\sin(2\beta)$ which is also never possible. Hence, this case is not possible.

Case 4.4 Assume $rb\sin(2\alpha)\neq 0$, $a\neq 2b$, $\sin(2\beta)\neq 0$, $r\neq\sqrt{2h-R^2}$ and $\kappa_1\neq 0$. Solving $f_{21}(\mathbf{x})=f_{23}(\mathbf{x})=0$ in (r,R) we get

$$r=\sqrt{\frac{-6(-2+a)h\sin(2(\alpha-\beta))}{\kappa_1}} \tag{34}$$

and

$$R=\sqrt{\frac{2bh\sin(2\alpha)}{\kappa_1}}. \tag{35}$$

Then after removing the denominator κ_1 (which we know is different from zero) equation $f_{22}(\mathbf{x})=0$ is equivalent to

$$\widetilde{f}_{22}(\mathbf{x})=(b-1)((2+9a+2b)\sin(2\alpha)+3(6-3a+b)\sin(2(\alpha-\beta))+3(1-3a+6b)\sin(2\beta)=0 \tag{36}$$

and equation $f_{24}(\mathbf{x})=0$ is equivalent to

$$\begin{aligned}\widetilde{f}_{24}(\mathbf{x})=\,&b(-27a^2+2(-9+b)b+9a(1+2b))\sin(2\alpha)+3(9a^2+b(1+b)\\&-a(18+b))\sin(2(\alpha-\beta))+3(9a^2+a(2-21b)+2b(-2+3b))\sin(2\beta)=0.\end{aligned}$$

Case 4.4.1 Assume $b=1$. In this case $f_{22}(\mathbf{x})=0$ and $\widetilde{f}_{24}(\mathbf{x})=((-16+27a-27a^2)\sin(2\alpha)+3(2-19a+9a^2)(\sin(2(\alpha-\beta))+\sin(2\beta))$. Note that the solution of $\widetilde{f}_{24}(\mathbf{x})=0$ does not determine both variables α and β and so the Jacobian matrix of $f_2(\mathbf{x})$ at any solution of $f_{2j}(\mathbf{x})=0$ for $j=1,2,3,4$ will be zero.

Case 4.4.2 Assume $b\neq 1$, $b=\frac{3a-1}{6}$, $a=-1/3$. Setting $\widetilde{f}_{22}(\mathbf{x})=\widetilde{f}_{24}(\mathbf{x})=0$ we get

$$\frac{20}{9}(\sin(2\alpha)-12\sin(2(\alpha-\beta)))=0 \quad\text{and}\quad \frac{20}{9}(\sin(2\alpha)+9\sin(2(\alpha-\beta)))=0,$$

which in particular implies that $\sin(2\alpha)=0$ which is not possible.

Case 4.4.3 Assume $b\neq 1$, $b=\frac{3a-1}{6}$, $a=-1/6$. Setting $f_{22}(\mathbf{x})=f_{24}(\mathbf{x})=0$ given in (36) we get

$$\sin(2(\alpha-\beta))=0 \quad\text{and}\quad 4\sin(2\beta)=-10\sin(2\alpha),$$

which implies that $\sin(2\alpha)=0$ which is not possible.

Case 4.4.4 Assume $b\neq 1$, $b=\frac{3a-1}{6}$, $a\neq -1/3$, $a\neq -1/6$. Setting $\widetilde{f}_{22}(\mathbf{x})=\widetilde{f}_{24}(\mathbf{x})=0$ and solving in $\sin(2\beta)$ and $\sin(2(\alpha-\beta))$ we get

$$\begin{aligned}\sin(2\beta)&=\frac{26+5a+96a^2-63a^3}{3(1+3a)(-7+3a)}\sin(2\alpha),\\ \sin(2(\alpha-\beta))&=\frac{2(1+6a)}{3(3a-7)}\sin(2\alpha).\end{aligned} \tag{37}$$

We note that if $a=7/3$ then $b=1$ which is not possible by assumptions, so $a\neq 7/3$. Substituting (37) into equation

$$\sin(2(\alpha-\beta))=\cos(2\beta)\sin(2\alpha)-\cos(2\alpha)\sin(2\beta),$$

and using that $\sin(2\alpha)\neq 0$ we get

$$\cos(2\beta)=\frac{2+18a+36a^2+(26+5a+96a^2-63a^3)\cos(2\alpha)}{3(-7+3a)(1+3a)}. \tag{38}$$

Then using (38) and (37) together with $\sin^2(2\beta)+\cos^2(2\beta)=1$ we get

$$\cos(2\alpha)=\frac{239-1936a+3703a^2+1896a^3+9153a^4-12096a^5+3969a^6}{4(1+3a)(1+6a)(-26-5a-96a^2+63a^3)}, \tag{39}$$

and

$$\cos(2\beta)=\frac{33-283a+274a^2+18a^3+945a^4-1323a^5}{12(1+3a)^2(1+6a)}. \tag{40}$$

In order that $\cos(2\alpha)$ be well defined we need that the denominator in (39) be different from zero and that $-1 \leqslant \cos(2\alpha) \leqslant 1$. The first conditions is accomplished when $a \neq 1/63(32 + (89405 - 252\sqrt{103209})^{1/3} + (89405 + 252\sqrt{103209})^{1/3})$ and the second one when

$$\frac{7(3a-7)^2\left(3a^2+2a+1\right)\left(21a^2-4a+1\right)}{4(3a+1)(6a+1)\left(63a^3-96a^2-5a-26\right)} \leqslant 0, \quad \text{and} \quad \frac{(21a-1)\left(3a^2-4a+3\right)\left(63a^3-33a^2-41a-45\right)}{4(3a+1)(6a+1)\left(63a^3-96a^2-5a-26\right)} \geqslant 0.$$

The denominator of $\cos(2\beta)$ is different from zero by assumptions, then in order that $\cos(2\beta)$ be well defined we need $-1 \leqslant \cos(2\beta) \leqslant 1$ which is equivalent to

$$\frac{7(21a-1)\left(3a^2-4a+3\right)\left(3a^2+2a+1\right)}{12(3a+1)^2(6a+1)} \geqslant 0 \quad \text{and} \quad \frac{\left(21a^2-4a+1\right)\left(63a^3-33a^2-41a-45\right)}{12(3a+1)^2(6a+1)} \leqslant 0.$$

Substituting (37) into (34) and (35) we get $r = \sqrt{\overline{r}}$ and $R = \sqrt{\overline{R}}$ with $\overline{r}$ and $\overline{R}$ given in (4). Then

$$\rho = \sqrt{2h - r^2 - R^2} = \sqrt{\overline{\rho}} = \sqrt{\frac{2(3a-1)\overline{\rho}_1 h}{(3a-7)\overline{r}_2}},$$

with $\overline{r}_2$ and $\overline{\rho}_1$ as in (4). Analyzing the intervals where $-1 \leqslant \cos(2\alpha) \leqslant 1$, $-1 \leqslant \cos(2\beta) \leqslant 1$, $\overline{r} \geqslant 0$, $\overline{R} > 0$ and $\overline{\rho} \geqslant 0$ we get $1/21 \leqslant a < 1/3$. From (39) and (40) we get

$$\alpha = \overline{\alpha}_{j_1,\ell_1} = \frac{(-1)^{j_1}}{2}\arccos(C_A) + \ell_1\pi$$

with C_A as in (4), $j_1, \ell_1 \in \{0,1\}$, and

$$\beta = \overline{\beta}_{k_1,\ell_2} = \frac{(-1)^{k_1}}{2}\arccos(C_B) + \ell_2\pi$$

with C_B as in (4) and $k_1, \ell_2 \in \{0,1\}$. Imposing that this solution is indeed a solution of the system $f_2(\mathbf{x}) = 0$ for $1/21 \leqslant a < 1/3$ (recall that $\sin(2\alpha) = (-1)^{j_1}\sqrt{1-\cos^2(2\alpha)}$ and $\sin(2\beta) = (-1)^{k_1}\sqrt{1-\cos^2(2\beta)}$) we obtain that in fact if $j_1 = 0$ then $k_1 = 1$ and if $j_1 = 1$ then $k_1 = 0$. So

$$\overline{\alpha}_{j_1,\ell_1} = \frac{(-1)^{j_1}}{2}\arccos(C_A) + \ell_1\pi, \quad \overline{\beta}_{j_1+1,\ell_2}, \quad j_1, \ell_1, \ell_2 \in \{0,1\},$$

with the convention that $\overline{\beta}_{2,\ell_2} = \overline{\beta}_{0,\ell_2}$.

We compute the Jacobian matrix of $f_2(\mathbf{x})$ on the solution $(\sqrt{\overline{r}}, \overline{\alpha}_{j_1,\ell_1}, \sqrt{\overline{R}}, \overline{\beta}_{j_1+1,\ell_2})$ and we obtain

$$\frac{175\pi^4(a-2)(3a-1)^4(21a-1)\left(3a^2-4a+3\right)\left(3a^2+2a+1\right)\left(21a^2-4a+1\right)\left(63a^3-33a^2-41a-45\right)h^4k_3^4}{729\overline{r_2}^3}$$

which is different from zero in the domain S_{10} (see (5)). We compute the eigenvalues of the Jacobian matrix of $f_2(\mathbf{x})$ on the solution (we do not write them explicitly because of the length of their expressions) and we see that for all the different values of $(a,b) \in S_{10}$ we get two complex conjugate eigenvalues and two real eigenvalues with different sign.

It follows from Theorem 4 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (16) has eight unstable $k_1\pi$-periodic solutions

$$\overline{\varphi}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon) = (\overline{r}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon), \overline{\alpha}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon), \overline{R}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon), \overline{\beta}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon)),$$

that tend to $\left(\sqrt{\overline{r}}, \overline{\alpha}_{j_1,\ell_1}, \sqrt{\overline{R}}, \overline{\beta}_{j_1+1,\ell_2}\right)$ when $\varepsilon \to 0$.

Now we go back through the changes of variables in (11). Substituting $\overline{\varphi}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon)$ in (15) and taking the square-root we get $\overline{\rho}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon)$. Therefore $(\overline{\varphi}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon), \overline{\rho}_{j_1,\ell_1,\ell_2}(\theta,\varepsilon))$ is a 2π-periodic solution for the differential system (13).

Case 4.4.5 Assume $b \neq 1$, $b \neq \frac{3a-1}{6}$, and $b = \frac{3(a^2-2a)}{6a-5}$. Setting $\widetilde{f}_{22}(\mathbf{x}) = 0$ given in (36) and solving in $\sin(2\beta)$ we have

$$\sin(2\beta) = \frac{\left(12a^2-9a-2\right)\sin(2\alpha) - 9\left(a^2-3a+2\right)\sin(2(\alpha-\beta))}{9a+3}, \tag{41}$$

which is well-defined because if $a = -1/3$ then $b = -1/3$ and this contradicts the assumption $b \neq \frac{3a-1}{6}$. Hence substituting this value of $\sin(2\beta)$ in $\widetilde{f}_{24}(\mathbf{x}) = 0$, if $a = 5/6$ then $f_{24}(\mathbf{x}) = 0$ has no solution, so we can assume that $a \neq 5/6$. Now taking into account that $b = \frac{3(a^2-2a)}{6a-5}$, and using (42) we get

$$\widetilde{f}_{24}(\mathbf{x}) = \frac{105(a-2)a^2\left(3a^2-3a+1\right)\sin(2\alpha)}{(5-6a)^2}.$$

Setting $\widetilde{f}_{24}(\mathbf{x}) = 0$ using that $\sin(2\alpha) \neq 0$ we get that either $a = 0$, or $a = 2$ (which is not possible because $b \neq 0$, or $a = (3 \pm i\sqrt{3})/6$ (which is not possible). If $a = 0$ then solution of $f_{24}(\mathbf{x}) = 0$ does not determine both variables α and β and so the Jacobian matrix of $f_2(\mathbf{x})$ at any solution of $f_{2j}(\mathbf{x}) = 0$ for $j = 1, 2, 3, 4$ will be zero. So this case is not possible.

Case 4.4.6 Assume $b \neq 1$, $b \neq \frac{3a-1}{6}$, $b \neq \frac{3(a^2-2a)}{6a-5}$. Since $b \neq \frac{3a-1}{6}$, solving $f_{22}(\mathbf{x}) = 0$ with respect to $\sin(2\beta)$ we get

$$\sin(2\beta) = \frac{3(-3a+b+6)\sin(2(\alpha-\beta)) + (9a+2b+2)\sin(2\alpha)}{9a-18b-3}. \tag{42}$$

Hence substituting this value of $\sin(2\beta)$ in $f_{24}(\mathbf{x}) = 0$ and solving in $\sin(2(\alpha-\beta))$ we get

$$\sin(2(\alpha-\beta)) = \frac{(9a^2(2+b) + 2b(1+11b) - a(1+40b+18b^2))\sin(2\alpha)}{3(-1+b)(-3a^2-5b+6a(1+b))}, \tag{43}$$

which is well defined because $b \neq 3(a^2-2a)/(6a-5)$. Then from (42) and (43) we can express $\sin(2\beta)$ in terms of $\sin(2\alpha)$

$$\sin(2\beta) = \frac{\left(-9a^2(2b+1) + a\left(b^2+40b+18\right) - 2b(b+11)\right)\sin(2\alpha)}{3(b-1)\left(-3a^2+6a(b+1)-5b\right)}. \tag{44}$$

Using $\sin(2(\alpha-\beta)) = \cos(2\beta)\sin(2\alpha) - \cos(2\alpha)\sin(2\beta)$ together with (42), (43) and that $\sin(2\alpha) \neq 0$ we get

$$\cos(2\beta) = \frac{\left(-9a^2(2b+1) + a\left(b^2+40b+18\right) - 2b(b+11)\right)\cos(2\alpha) + 9a^2(b+2) - a\left(18b^2+40b+1\right) + 2b(11b+1)}{3(b-1)\left(-3a^2+6a(b+1)-5b\right)}. \tag{45}$$

From (45) and (44) imposing that $\cos^2(2\beta) + \sin^2(2\beta) = 1$ we obtain $\cos(2\alpha) = c_a$ and using again (45) we get $\cos(2\beta) = c_b$ with c_a and c_b given in (4). In order that $\cos(2\alpha)$ and $\cos(2\beta)$ be well-defined we must have that $c_{2A1}c_{2A2} \neq 0$ with c_{2A1}, and c_{2A2} given in (4). Note that $c_{2B} \neq 0$ because $b \neq 3(a^2-2a)/(6a-5)$. On the other hand, $-1 \leqslant c_a \leqslant 1$ which is equivalent to

$$\frac{\left(18a^2-ab-a-5b\right)(ab+a-b)}{c_{2A1}c_{2A2}} \leqslant 0 \quad \text{and} \quad \frac{D_{7,a,b}D_{8,a,b}}{c_{2A1}c_{2A2}} \geqslant 0, \tag{46}$$

and $-1 \leqslant c_b \leqslant 1$ which is equivalent to

$$\frac{(ab+a-b)D_{7,a,b}}{c_{2B}c_{2A2}} \geqslant 0 \quad \text{and} \quad \frac{\left(18a^2-ab-a-5b\right)D_{8,a,b,}}{c_{2B}c_{2A2}} \leqslant 0, \tag{47}$$

where $D_{7,a,b}$, $D_{8,a,b}$ and c_{2B} given in (4). When conditions (46) and (47) are satisfied, we have

$$\alpha = \tilde{\alpha}_{j_1,\ell_1} = \frac{(-1)^{j_1}}{2}\arccos(c_a) + \ell_1\pi$$

with c_a as in (4), $j_1, \ell_1 \in \{0, 1\}$, and

$$\beta = \tilde{\beta}_{k_1,\ell_2} = \frac{(-1)^{k_1}}{2}\arccos(c_b) + \ell_2\pi$$

with c_b as in (4) and $k_1, \ell_2 \in \{0, 1\}$.

Substituting (43) and (44) into (34) and (35) we get $r = \sqrt{\widetilde{r}}$ and $R = \sqrt{\widetilde{R}}$ with $\widetilde{r}$ and $\widetilde{R}$ given in (4). Moreover

$$\rho = \sqrt{2h - r^2 - R^2} = \sqrt{\widetilde{\rho}} = \sqrt{\frac{2bh(2b-a)\tilde{\rho}_1}{(b-1)\widetilde{r}_2}},$$

with $\widetilde{r}_2$ and $\tilde{\rho}_1$ given in (4) should be well defined. Let $\widetilde{S}_{10}$ the domain $(a, b) \in \mathbb{R}^2$ where $\widetilde{r} \geqslant 0$, $\widetilde{R} > 0$ and $\widetilde{\rho} \geqslant 0$ and conditions (46), (47) are satisfied. Imposing the solution $(\sqrt{\widetilde{r}}, \widetilde{\alpha}_{j_1,l_1}, \sqrt{\widetilde{R}}, \widetilde{\beta}_{k_1,l_1})$, is indeed a solution of

$f_{22}(\mathbf{x}) = 0$ in the domain $\widetilde{S}_{10}$ (recall that $\sin(2\alpha) = \pm\sqrt{1-\cos^2(2\alpha)}$ and $\sin(2\beta) = \pm\sqrt{1-\cos^2(2\beta)}$) we obtain that in fact if $j_1 = 0$ then $k_1 = 1$ and if $j_1 = 1$ then $k_1 = 0$. So

$$\tilde{\alpha}_{j_1,\ell_1} = \frac{(-1)^{j_1}}{2}\arccos(c_a) + \ell_1\pi, \quad \tilde{\beta}_{j_1+1,\ell_2}, \quad j_1, \ell_1, \ell_2 \in \{0, 1\},$$

with the convention that $\tilde{\beta}_{2,\ell_2} = \tilde{\beta}_{0,\ell_2}$.

The Jacobian matrix of $f_2(\mathbf{x})$ on the solution $(\sqrt{\tilde{r}}, \tilde{\alpha}_{j_1,\ell_1}, \sqrt{\tilde{R}}, \tilde{\beta}_{j_1+1,\ell_2})$ is

$$-\frac{2800\pi^4 k_3^4 h^4}{81\tilde{r}_2^3}(a-2)b^4(2b-a)\left(18a^2 - ab - a - 5b\right)(ab + a - b)D_{7,a,b}D_{8,a,b},$$

and we obtain that it is different from zero in the region S_{11} as in (5). We then compute the eigenvalues of the Jacobian matrix evaluated at the solution on the values in the domain S_{11} (we do not write them explicitly because of the length of their expressions). Depending on the values of $(a, b) \in S_{11}$ we get either two pairs of complex conjugate eigenvalues or two complex conjugate eigenvalues and two real eigenvalues with different sign.

It follows from Theorem 4 that for any given $h > 0$ and for $|\varepsilon|$ sufficiently small, system (16) has eight unstable $k_1\pi$-periodic solutions

$$\tilde{\varphi}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon) = (\tilde{r}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon), \tilde{\alpha}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon), \tilde{R}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon), \tilde{\beta}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon)),$$

that tend to $\left(\sqrt{\tilde{r}}, \tilde{\alpha}_{j_1,\ell_1}, \sqrt{\tilde{R}}, \tilde{\beta}_{j_1+1,\ell_2}\right)$ when $\varepsilon \to 0$.

Now we go back through the changes of variables in (11). Substituting $\tilde{\varphi}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon)$ in (15) and taking the square-root we get $\tilde{\rho}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon)$. Therefore $(\tilde{\varphi}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon), \tilde{\rho}_{j_1,\ell_1,\ell_2}(\theta, \varepsilon))$ is a 2π-periodic solution for the differential system (13).

In order to conclude this section we must study the cases in which condition $c_{2A1}c_{2A2} \neq 0$ does not hold. Solving $c_{2A1}c_{2A2} = 0$ in a we obtain

$$a_{1,2} = \frac{18 + 40b + b^2 \pm \sqrt{324 + 648b20b^2 - 64b^3 + b^4}}{18 + 36b},$$
$$a_{3,4} = \frac{1 + 40b + 18b^2 \pm \sqrt{1 - 64b - 20b^2 + 648b^3 + 324b^4}}{36 + 18b},$$

which are not defined when $b = -2$ or $b = -1/2$.

If $b = -2$ then either $a = -12$, or a is complex. Hence $b = -2$ and $a = -12$. In this case equations (43) and (44) become

$$\sin(2\beta) = \frac{22\sin(2\alpha)}{15}, \quad \sin(2(\alpha - \beta)) = 0,$$

which yields that either $\sin(2\alpha) = 0$, or $\sin(2\beta) = 0$. Both cases are not possible.

If $b = -1/2$ then either $a = 6$, or a is complex. Hence $b = -1/2$ and $a = 6$. In this case equations (43) and (44) become

$$\sin(2\beta) = 0, \quad \sin(2(\alpha - \beta)) = \frac{22}{15}\sin(2\alpha),$$

which is not possible because $\sin(2\beta) \neq 0$. So $b \neq -2$ and $b \neq -1/2$.

If $a = a_2$ then imposing the condition $\cos^2(2\beta) + \sin^2(2\beta) = 1$ using equations (45) and (44) and solving in b we get that either $b = -4$, $b = -1/2$, $b = 0$ or $b = 16/5$. Since the case $b \neq 0$ and $b \neq -1/2$ was studied before, we only need to consider the cases $a = 4/3, b = -4$ and $a = 16/15, b = 16/5$. In the first case equations (43) and (44) become

$$\sin(2\beta) = 0, \quad \sin(2(\alpha - \beta)) = \sin(2\alpha),$$

which is not possible because $\sin(2\beta) \neq 0$. In the second case equations (43) and (44) become

$$\sin(2\beta) = 0, \quad \sin(2(\alpha - \beta)) = -\sin(2\alpha),$$

which is also not possible because $\sin(2\beta) \neq 0$.

If $a = a_1$ imposing the condition $\cos^2(2\beta) + \sin^2(2\beta) = 1$ using equations (45) and (44) and solving in b we get that $b = -1/2$ which is not possible.

If $a = a_4$ imposing the condition $\cos^2(2\beta) + \sin^2(2\beta) = 1$ using equations (45) and (44) and solving in b we get that $b = -2$, $b = -1/4$, $b = 0$ or $b = 5/16$. The case $b = -2$ was studied before and is not possible and the case

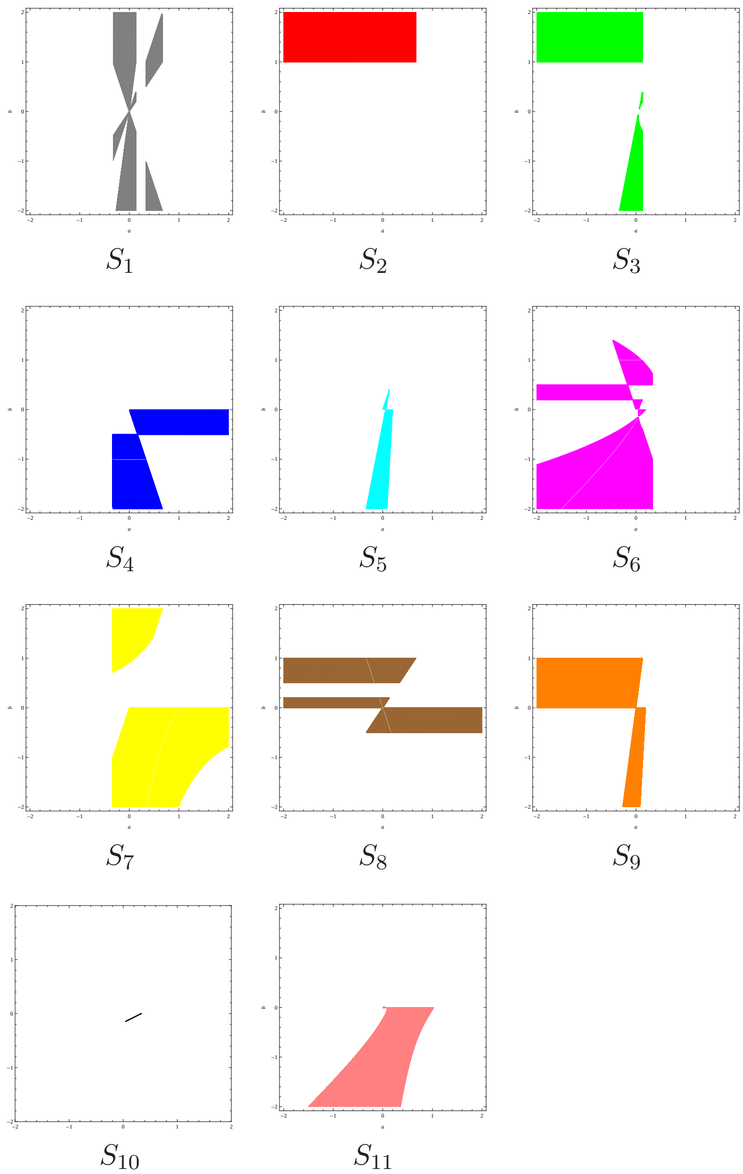


FIGURE 1. The plot of the regions S_i.

$b = 0$ is not possible. We only need to consider the cases $a = -1/3, b = -1/4$ and $a = 1/3, b = 5/16$. In the first case equations (43) and (44) become

$$\sin(2\beta) = \sin(2\alpha), \quad \sin(2(\alpha - \beta)) = 0,$$

which is not possible because $\sin(2\alpha) \neq 0$. In the second case equations (43) and (44) become

$$\sin(2\beta) = -\sin(2\alpha), \quad \sin(2(\alpha - \beta)) = 0,$$

which is also not possible because $\sin(2\alpha) \neq 0$.

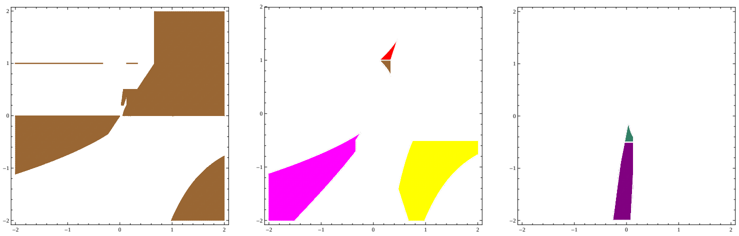
Finally if $a = a_3$ imposing the condition $\cos^2(2\beta) + \sin^2(2\beta) = 1$ using equations (45) and (44) and solving in b we get that $b = -2$, $b = 0$ which are both not possible.

In short if condition (37) holds we have no solutions. This concludes the proof of the theorem.

6. APPENDIX

In Figure 1 we plot the sets S_i for $i = 1, \ldots, 11$. We have chosen the region $[-2, 2] \times [-2, 2]$ to show the shape of these sets. This region is not the whole domain, is just appropriate for clearness.

The intersection of the regions S_i is a tedious set, here to illustrate how complex these regions can be we show the regions in the parameters (a, b) in the following cases: when $\cap_{i=1}^{11} S_i = \emptyset$, when only one condition S_i is satisfied, and when 8 different conditions S_i are satisfied simultaneously. There are also regions where two, three,



a) b) c)

Figure 2. Examples of the intersection of the regions S_i. a) the case $\cap_{i=1}^{11} S_i = \emptyset$. b) the case where only one condition S_i is satisfied. The top of the upper region corresponds to S_2, the bottom of the upper region to S_8, the left hand side region to S_6 and the right hand side region to S_7. c) the case where 8 different conditions S_i are satisfied simultaneously. The upper region corresponds to $S_1 \cap S_3 \cap S_5 \cap S_6 \cap S_7 \cap S_8 \cap S_9 \cap S_{11}$ and the lower one to $S_1 \cap S_3 \cap S_4 \cap S_5 \cap S_6 \cap S_7 \cap S_9 \cap S_{11}$.

four, five, six, and seven different S_i are satisfied simultaneously but due to the big number of possibilities, we do not gain any insight in including them.

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[1] Departament de Tecnologies Digitals i de la Informació, Escola Politècnica Superior, Universitat de Vic - Universitat Central de Catalunya (UVic-UCC), C. de la Laura, 13, 08500 Vic, Catalonia, Spain

E-mail address: `montserrat.corbera@uvic.cat`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

[3] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal

E-mail address: `cvalls@math.ist.utl.pt`