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Analytic continuation in the case of non-regular dependency on a small parameter with an application to celestial mechanics

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Abstract

We consider a non-autonomous system of ordinary differential equations. Assume that the time dependence is periodic with a very high frequency $1/\varepsilon$, where ε is a small parameter and differentiability with respect to the parameter is lost when ε equals zero. We derive from Arenstorf's implicit function theorem a set of conditions to show the existence of periodic solutions. These conditions look formally like the standard analytic continuation method, namely, checking that a certain minor does not vanish. We apply this result to show the existence of a new class of periodic orbits of very large radii in the three-dimensional elliptic restricted three-body problem for arbitrary values of the masses of the primaries.

Keywords: Continuation method; Averaging; Periodic orbits; Symmetric orbits; Spatial restricted three body problem

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1. Introduction

In the study of non-integrable dynamical systems it is very difficult to obtain complete information on the behaviour of solutions for all values of time unless they are asymptotic, periodic or almost periodic. Hence the interest to show the existence of periodic solutions.

A classical tool to show the existence of periodic solutions in non-integrable systems is Poincare's analytic continuation method, which ultimately comes down to solving a system of equations with a small parameter by means of the implicit function theorem.

This method has been widely used in Celestial Mechanics for analytical studies. We call attention to the works of Poincare [13], Arenstorf [1], Hadjidemetriou [5], Meyer [10,7], Jefferys [8,9] and Guillaume [4].

Meyer and Howison show in [7] the existence of periodic solutions to the spatial circular restricted three-body problem, for all values of the mass parameter and large inclination. The solutions found are of very large radii compared to that of the primaries. When treating this as a perturbation problem the difficulty arises that the unperturbed orbits are not defined when the small parameter vanishes. This difficulty is overcome by using the implicit function theorem of Arenstorf [1]. We showed the existence of periodic solutions of this type in the case of elliptic motion of the primaries in [12]. As the elliptic case is non-autonomous, the period of the solution cannot be solved as a function of the small parameter, which results in making the problem more degenerate. In order to reduce the degeneracy we considered in that paper the very special case of primaries of equal mass.

In the present paper we get rid of this restriction and consider any possible value of the mass parameter. The fact that the masses are arbitrary precludes the use of a certain symmetry and adds to the degeneracy of the equations: one more equation is degenerate in this case.

To give some insight into the physics of the problem we recall that for moderate eccentricities of the primaries, the gravitational potential as seen from a body very far from the primaries resembles that of a slightly oblate planet. Circular orbits of a satellite undergo a slow rotation of their orbital plane around the z-axis, if the equatorial bulge is on the xy-plane. In astronomical terms this is known as *precession of the line of nodes*, the line of the nodes being the intersection of the orbital plane with the reference plane. Of course, this is not well defined for coplanar orbits. The velocity of precession depends on the inclination i of the orbit, being positive for $i < \pi/2$, zero (in a first approximation) for *polar* orbits, i.e. $i = \pi/2$, perpendicular to the equatorial plane, and negative if $i > \pi/2$. It seems then that periodic solutions of arbitrary inclination do not exist due to the fact that they tend to precess, so that they need a whole turn of the precession to fit into the original position. Of course, this is not a problem if the primaries move on circular orbits, because then all the positions of the primaries are equivalent (that is, the problem is invariant by rotations around the z-axis and we can think of a reduced phase space). This is not the case in the elliptic problem, where there is a privileged direction given by the major axes of the primaries and the infinitesimal body should come to exactly the same position in the inertial frame in order to have a periodic solution.

If the unperturbed orbit is a polar one, the precession induced by a small variation of the inclination can hopefully be used to compensate for any variation of the orbital plane due to the perturbation.

As precession is not defined for orbits coplanar with the primaries, the question of their existence is quite a different one. Another kind of variables and another transversality should be used in that case.

In the first two sections we deal with the analytic continuation in the case of non-regular dependency on the small parameter. We show that Arenstorf's theorem allows the problem to be solved by checking that a certain minor is non-vanishing, formally the same argument as in the classical method. The use of Arenstorf's theorem in this paper is slightly different from that in [7].

In the next sections we establish the existence of a discrete family of periodic solutions to the spatial restricted elliptic three-body problem. These solutions exist for all values of the mass ratio parameter and eccentricity of the primaries and are perturbations of circular solutions of the Kepler problem having very large radii on a plane perpendicular to that of the primaries. By the Kepler problem we mean the spatial central force problem with the inverse square law of attraction.

The small parameter ε is roughly the inverse of the distance to the primaries and it is introduced as a scale parameter. The perturbation problem is degenerate in the sense that for $\varepsilon = 0$ the solutions are not defined, their period tends to infinity and the minor relevant to the analytic continuation vanishes. These difficulties are overcome by first averaging on the fast variable (motion of the primaries) so that the problem is similar to the motion of a satellite around an oblate planet and then using an analytic continuation argument.

2. Analytic continuation with non-regular dependency on the small parameter

Poincare's method of analytic continuation reduces ultimately to solving a system of equations $f(x, \varepsilon) = 0$, with $f(0, 0) = 0$, for x as function of ε. If the system is analytic or differentiable enough and $f_x(0, 0) \neq 0$, then the implicit function theorem guarantees the existence of such a solution.

There are cases, however, where the function f is not differentiable with respect to ε, so the classical implicit function theorem does not apply. A result of Arenstorf can be used to show that differentiability with respect to ε can be dropped provided that the function f satisfies some mild regularity conditions. Arenstorf's fixed point theorem is as follows:

Theorem 1 (*Arenstorf*). *We assume X and P to be Banach spaces with elements x and p. Let g be a mapping from the product space $X \times P$ into X, given by $(x, \varepsilon) \to g(x, \varepsilon) \in X$, and defined for x in a ball $B = \{x \in X \text{ such that } \|x\| \leqslant \alpha, \alpha > 0\}$, and ε in a region V of P containing $\varepsilon = 0$, with $g(0, 0) = 0$.*

If, for every $\varepsilon \in V$, g is differentiable with respect to $x \in B$ and

$$\|g_x(x, \varepsilon)\| \leqslant \zeta \leqslant \tfrac{1}{2} \quad \text{on } B \times V$$

(*where* g_x *denotes the partial derivative of* g *with respect to* x, *and the norm of this linear operator from* X *into itself is the sup norm*) *and if*

$$\|g(0,\varepsilon)\| \leqslant \tfrac{1}{2} \quad \textit{on } V,$$

then there exists a function $x(\varepsilon)$ *with*

$$g(x(\varepsilon),\varepsilon) = x(\varepsilon), \quad x(\varepsilon) \in B \quad \textit{for } \varepsilon \in V, \quad x(0) = 0.$$

See [1] for more details.

By means of this theorem it can be seen that a sufficient condition for the existence of solution of $f(x,\varepsilon)=0$ in a neighbourhood of $x=0$, $\varepsilon=0$ is that the determinant of $f_x(0,0)$ does not vanish (formally the same condition as in the regular case) together with some regularity conditions, as stated in the following proposition.

Proposition 2. *Let* U *be an open domain in* $\mathbb{R}^n$, $I \subset \mathbb{R}$ *an open neighbourhood of the origin and* $f : U \times I \to \mathbb{R}^n$ *with* $f(0,0)=0$, *differentiable with respect to* $x \in U$, *and* $f_x(0,0)$ *non-singular. Assume that there exist* $c>0$, $k>0$ *such that for* $x \in U$, $\varepsilon \in I$

(1) $\|f_x(x,\varepsilon) - f_x(0,0)\| \leqslant c(\|x\| + \varepsilon)$,
(2) $\|f(0,\varepsilon)\| \leqslant k\varepsilon$.

Then there exists a function $x(\varepsilon) \in U$, *defined for* $\varepsilon \in I' \subset I$, *such that* $f(x(\varepsilon),\varepsilon)=0$ *and* $x(0)=0$.

Proof. Let

$$\alpha = \frac{k}{2c(mk+1/2)}, \quad \beta = \frac{1}{4mc(mk+1/2)},$$

where $m = \|f_x^{-1}(0,0)\|$.

We consider the function

$$g(x,\varepsilon) = x - f_x^{-1}(0,0) f(x,\varepsilon).$$

If $\|x\| \leqslant \alpha$ and $\varepsilon \leqslant \beta$ we have

$$\begin{aligned}\|g_x(x,\varepsilon)\| = \|Id - f_x^{-1}(0,0) f_x(x,\varepsilon)\| &\leqslant \|f_x^{-1}(0,0)\| \|f_x(x,\varepsilon) - f_x(0,0)\| \\ &\leqslant mc(\|x\| + \varepsilon) \leqslant mc(\alpha+\beta) \leqslant \tfrac{1}{2},\end{aligned}$$

where Id is the identity matrix.

On the other hand, the following inequality holds,

$$\|g(0,\varepsilon)\| = \|f_x^{-1}(0,0)f(0,\varepsilon)\| \leqslant mk\varepsilon \leqslant mk\beta \leqslant \tfrac{1}{2}\alpha.$$

Therefore, g satisfies the hypothesis of Arenstorf's fixed point theorem and there exists a neighbourhood of the origin $I' \subset I$ and a function $x(\varepsilon) \in U$ such that $f(x(\varepsilon),\varepsilon) = 0$ for ε in I'. $\square$

This result will be used to show the existence of periodic solutions in the three-dimensional elliptic restricted three-body problem, when the infinitesimal body is at great distance from the primaries and the perturbation can be seen as a fast periodic forcing.

3. Two lemmas on differential equations

We consider the differential equation

$$\dot{z} = \mathcal{F}(z,\varepsilon,t), \tag{1}$$

where $z \in \mathbb{R}^n$ and

$$\mathcal{F}(z,\varepsilon,t) = \mathcal{F}_0(z) + \varepsilon\mathcal{F}_1(z,\varepsilon,t) + \varepsilon^2\mathcal{F}_R(z,\varepsilon,t).$$

Let z_0 be initial conditions such that $z^{(0)}(t,z_0)$ is a solution of

$$\dot{z}^{(0)}(t,z_0) = \mathcal{F}_0(z^{(0)}) \tag{2}$$

which remains bounded and bounded away from the singularities of $\mathcal{F}$. Let $\mathcal{C} \subset \mathbb{R}^n$ be a compact neighbourhood of $z^{(0)}(t,z_0)$ without singularities. We assume that the functions $\mathcal{F}_0, \varepsilon\mathcal{F}_1, \varepsilon^2\mathcal{F}_R$ are continuous for $z \in \mathcal{C}$, $\varepsilon \in [0,\varepsilon_1]$, $t \in \mathbb{R}$. Furthermore, $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_R$ together with all their derivatives with respect to z are bounded on $\mathcal{C}$ by a constant C_1 independent of ε. In particular, $\mathcal{F}_0$ is Lipschitz with a constant C_2. In what follows the maximum norm $\|v\| = \max_i |v_i|$ for vectors $v \in \mathbb{R}^n$ and the usual norm of the supreme on the unit ball for linear operators will be used.

The next two lemmas show that the solution of Eq. (1) can be written as the solution of (2) plus terms which are of order ε, and the same is true about its partial derivatives with respect to the initial conditions.

Lemma 3. *For $\varepsilon \neq 0$ let $z(t,z_0,\varepsilon)$ be a solution of Eq.* (1) *with initial condition z_0 and let $z^{(1)}(t,z_0,\varepsilon)$ be the solution of*

$$\dot{z}^{(1)}(t,z_0,\varepsilon) = \mathcal{F}_1(z^{(0)},\varepsilon,t) + D\mathcal{F}_0(z^{(0)}(t,z_0))z^{(1)}(t,z_0,\varepsilon) \tag{3}$$

with initial condition $z^{(1)}(0, z_0, \varepsilon) = 0$, *where* D *is the matrix whose entries are the partial derivatives of* $\mathcal{F}$ *with respect to the* z *variables. Then we can write*

$$z(t, z_0, \varepsilon) = z^{(0)}(t, z_0) + \varepsilon z^{(1)}(t, z_0, \varepsilon) + z_R(t, z_0, \varepsilon)$$

and $z_R(t, z_0, \varepsilon)$ *is* $\mathcal{O}(\varepsilon^2)$ *in a finite interval of time.*

Proof. We first define $z_1(t, z_0, \varepsilon) = z(t, z_0, \varepsilon) - z^{(0)}(t, z_0)$ and we see that it is $\mathcal{O}(\varepsilon)$. Let $\mathcal{C}$ be a compact neighbourhood of $z^{(0)}(t, z_0)$. Then

$$\begin{aligned} \|z_1(t, z_0, \varepsilon)\| \leqslant & \int_0^t \|\mathcal{F}_0(z(\tau)) - \mathcal{F}_0(z^{(0)}(\tau))\| \\ & + \varepsilon \int_0^t \|\mathcal{F}_1(z(\tau), \varepsilon, \tau) + \varepsilon \mathcal{F}_R(z(\tau), \varepsilon, \tau)\| \, d\tau \\ \leqslant & \int_0^t C_2 \|z(\tau, z_0, \varepsilon) - z^{(0)}(\tau, z_0)\| + \varepsilon C_1 (1+\varepsilon) t \end{aligned}$$

and, applying Gronwall's inequality, we get

$$\|z_1(t, z_0, \varepsilon)\| \leqslant \varepsilon(1+\varepsilon)\frac{C_1}{C_2} \exp C_2 t - \varepsilon(1+\varepsilon)\frac{C_1}{C_2} \leqslant \varepsilon C_3$$

if $t \in [0, T_0]$. We see now that

$$z_R(t, z_0, \varepsilon) = z(t, z_0, \varepsilon) - z^{(0)}(t, z_0) - \varepsilon z^{(1)}(t, z_0, \varepsilon)$$

is $\mathcal{O}(\varepsilon^2)$. Let δ be such that the ball of radius δ and centre $z^{(0)}(t, z_0)$ is contained in $\mathcal{C}$ for all $t \in [0, T_0]$. By continuity with respect to the initial conditions and parameters, there exists ε_2 such that if $|z_0^* - z_0| < \varepsilon_2$ and $\varepsilon < \varepsilon_2$ then $z(t, z_0^*, \varepsilon)$ lies inside the ball of radius δ and centre $z^{(0)}(t, z_0)$ for all $t \in [0, T_0]$ (see [6]). Then we can write Taylor's formula,

$$\|\mathcal{F}_0(z(\tau), \tau) - \mathcal{F}_0(z^{(0)}(\tau), \tau) - D\mathcal{F}_0(z^{(0)}(\tau, z_0)) z_1(\tau, z_0, \varepsilon)\| \leqslant \varepsilon^2 C_4$$

and, since $\mathcal{F}_1(z, t, \varepsilon)$ and all its derivatives with respect to z are bounded by a constant independent of ε, we have

$$\|\mathcal{F}_1(z(\tau), \varepsilon, \tau) - \mathcal{F}_1(z^{(0)}(\tau), \varepsilon, \tau)\| \leqslant C_5 \|z_1(\tau, z_0, \varepsilon)\| \leqslant \varepsilon C_6.$$

Now we have

$$\begin{aligned}\|z_R(t,z_0,\varepsilon)\| &\leqslant \int_0^t \|\dot{z}(\tau,z_0,\varepsilon)-\dot{z}^{(0)}(\tau,z_0)-\varepsilon\dot{z}^{(1)}(\tau,z_0,\varepsilon)\|\,d\tau\\ &\leqslant \int_0^t \|\mathcal{F}_0(z(\tau),\tau)-\mathcal{F}_0(z^{(0)}(\tau),\tau)\\ &\quad -D\mathcal{F}_0(z^{(0)}(\tau,z_0))[z_1(\tau,z_0,\varepsilon)-z_R(\tau,z_0,\varepsilon)]\\ &\quad +\varepsilon[\mathcal{F}_1(z(\tau),\varepsilon,\tau)-\mathcal{F}_1(z^{(0)}(\tau),\varepsilon,\tau)]+\varepsilon^2\mathcal{F}_R(z(\tau),\varepsilon,\tau)\|\,d\tau,\end{aligned}$$

so we finally get

$$\|z_R(t,z_0,\varepsilon)\|\leqslant C_8\varepsilon^2 t+\int_0^t C_9\|z_R(t,z_0,\varepsilon)\|\,d\tau$$

and Gronwall's inequality gives

$$\|z_R(t,z_0,\varepsilon)\|\leqslant \frac{C_8\varepsilon^2}{C_9}\exp C_9 t-\frac{C_8\varepsilon^2}{C_9}$$

if $t\in[0,T_0]$. □

The next Lemma shows that similar bounds hold for the partials of z_R with respect to z.

Lemma 4. *Let* $z_R(t,z_0,\varepsilon)$ *be as in Lemma* 3. *Then*

$$D_{z_0}z_R(t,z_0,\varepsilon)=\mathcal{O}(\varepsilon^2)$$

for $t\in[0,T_0]$.

Proof. Let $z_1(t,z_0,\varepsilon)$ be as in Lemma 3. We first see that

$$\|D_{z_0}z_1(t,z_0,\varepsilon)\|\leqslant\varepsilon C_{10}.$$

We have

$$\|D_{z_0}z_1\|\leqslant\int_0^t \|D_{z_0}[\mathcal{F}(z(\tau,z_0,\varepsilon))-\mathcal{F}_0(z^{(0)}(\tau))]\|\,d\tau$$

and the integral in the right-hand side is bounded by $I_1(t)+I_2(t)$, where

$$I_1(t)=\int_0^t \|D_z\mathcal{F}(z)\|\|D_{z_0}z_1\|\,d\tau,$$

$$I_2(t) = \int_0^t \|D_z\mathcal{F}_0(z) - D_z\mathcal{F}_0(z^{(0)}) + \varepsilon D_z[\mathcal{F}_1 + \varepsilon\mathcal{F}_R](z)\| \|D_{z_0}z^{(0)}\| \, d\tau,$$

and then the inequality

$$\|D_{z_0}z_1\| \leqslant \int_0^t C_{11}\|D_{z_0}z_1\| \, d\tau + \varepsilon \int_0^t C_{12} \, d\tau$$

holds because all the functions involved, as well as their first and second order derivatives, are bounded on the compact $\mathcal{C}$. Gronwall's inequality readily gives

$$\|D_{z_0}z_1\| \leqslant \varepsilon \frac{C_{12}}{C_{11}} e^{C_{12}t} - \varepsilon \frac{C_{12}}{C_{11}}.$$

In order to see that $D_{z_0}z_R(t, z_0, \varepsilon)$ is $\mathcal{O}(\varepsilon^2)$, we write, as in Lemma 3,

$$\begin{aligned} \|D_{z_0}z_R(t, z_0, \varepsilon)\| \; \leqslant \; & \int_0^t \|D_{z_0}[\mathcal{F}(z(\tau, z_0, \varepsilon), \tau, \varepsilon) - \mathcal{F}_0(z^{(0)}(\tau, z_0)) \\ & -\varepsilon D_z\mathcal{F}_0(z^{(0)}(\tau, z_0))z^{(1)}(\tau, z_0, \varepsilon) - \varepsilon\mathcal{F}_1(z^{(0)}(\tau), \varepsilon, \tau)]\| \, d\tau. \end{aligned}$$

Let

$$\begin{aligned} M_1 \; = \; & D_z\mathcal{F}_0(z(t)) - D_z\mathcal{F}_0(z^{(0)}(t)) - D_{zz}\mathcal{F}_0(z^{(0)}(\tau, z_0))(z - z^0) \\ & +\varepsilon[D_z\mathcal{F}_1(z(t)) - D_z\mathcal{F}_1(z^{(0)}(t))] + \varepsilon^2 D_z\mathcal{F}_R(z(t)) + D_{zz}\mathcal{F}_0(z^{(0)})z_R, \end{aligned}$$

$$M_2 = D_z\mathcal{F}_0(z(t)) - D_z\mathcal{F}_0(z^{(0)}(t)) + \varepsilon D_z\mathcal{F}_1(z(t)) + \varepsilon^2 D_z\mathcal{F}_R(z(t)),$$

where $D_{zz}\mathcal{F}(p)h$ stands for $D_{zz}\mathcal{F}(p)(h, \cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $D_{zz}\mathcal{F}(p)$ is the second differential of $\mathcal{F}$. We then have

$$\begin{aligned} \|D_{z_0}z_R(t, z_0, \varepsilon)\| \; \leqslant \; & \int_0^t \|D_z\mathcal{F}_0(z^{(0)}(\tau, z_0))D_{z_0}z_1(\tau, z_0, \varepsilon)\| \, d\tau \\ & + \int_0^t (\|M_1\| \|D_{z_0}z^{(0)}(\tau, z_0)\| + \|M_2\| \|D_{z_0}z_R(t, z_0, \varepsilon)\|) \, d\tau, \end{aligned}$$

where $\|E\| = \sup \|Exy\|$, $\|x\| = \|y\| = 1$, $x, y \in \mathbb{R}^n$ is the norm of a bilinear continuous operator E mapping $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. As $D_z\mathcal{F}_R(z)$, $D_z\mathcal{F}_1(z)$ and $D_{zz}\mathcal{F}_0(z(t))$ are all bounded, we obtain

$$\|M_1\| \leqslant C_{12}\|z - z^{(0)}\|^2 + \varepsilon C_{13}\|z - z^{(0)}\| + \varepsilon^2 C_{14} \leqslant C_{15}\varepsilon^2$$

and

$$\|M_2\| \leqslant C_{16}\|z - z^{(0)}\| + \varepsilon C_{17} + \varepsilon^2 C_{18} \leqslant C_{19}\varepsilon.$$

From Lemma 3 we deduce

$$\|z - z^{(0)}\| \leqslant C_3\varepsilon,$$

so that

$$\|D_{z_0} z_R(t, z_0, \varepsilon)\| \leqslant C_{20}\varepsilon^2 t + \int_0^t C_{21} D_{z_0} z_R(\tau, z_0, \varepsilon)\, d\tau.$$

Then the Lemma follows from Gronwall's inequality. □

4. The elliptic three-dimensional restricted three-body problem

The elliptic restricted three-body problem describes the motion of a body of infinitesimal mass, m_3, in the gravitational field created by two bodies m_1 and m_2 called primaries. The primaries m_1 and m_2 are, respectively, of mass $1 - \mu$ and μ, with $\mu \in [0, 1)$, and are moving in elliptic orbits with eccentricity $\eta \in [0, 1)$ and semimajor axis μ and $1 - \mu$, around their centre of mass which remains fixed at the origin. The equations of motion are usually written in dimensionless coordinates, in such a way that the semimajor axis of each primary around the other is unity (see [14]). If the infinitesimal body is far away from the primaries, its motion must be close to a Keplerian motion although in the limit the orbit would be of infinite radius. Another system of units can be taken, in which the infinitesimal body is at distance unity from the origin and both primaries are very close to one another. Thus, the small parameter is the semimajor axis of the primaries and when it takes very small values the orbit of the infinitesimal body tends to a Keplerian circle of radius unity and the perturbation gives rise to a very fast periodic forcing, thus losing differentiability.

The equations of motion of the elliptic three-dimensional restricted three-body problem can be derived from the non-autonomous 2π-periodic Hamiltonian

$$\mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{1-\mu}{R_1} - \frac{\mu}{R_2}, \tag{4}$$

where $\mathbf{q} = (q_1, q_2, q_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$ are, respectively, the position and momentum of m_3 and R_1, R_2 are the distances from the infinitesimal body to the primaries

$$R_1^2 = (q_1 - \mu\rho\cos\varphi)^2 + (q_2 - \mu\rho\sin\varphi)^2 + q_3^2,$$

$$R_2^2 = (q_1 + (1-\mu)\rho\cos\varphi)^2 + (q_2 + (1-\mu)\rho\sin\varphi)^2 + q_3^2,$$

where $\rho(t)$ is the distance between the primaries and $\varphi(t)$ is the angular position of m_1 measured from the pericenter. The following expressions for $\rho(t)$ and $\varphi(t)$ can be found in [14]

$$\rho = \frac{(1-\eta^2)}{1+\eta\cos\varphi},$$

$$\frac{d\varphi}{dt} = \frac{(1+\eta\cos\varphi)^2}{(1-\eta^2)^{3/2}}.$$

It is easily seen that the equations of motion are invariant by the symmetry

$$S:(q_1,q_2,q_3,p_1,p_2,p_3,\varphi,t)\longrightarrow(q_1,-q_2,-q_3,-p_1,p_2,p_3,-\varphi,-t)$$

which can be used to show the existence of periodic solutions, in a way similar to that in [10,8,12], as stated in the following proposition.

Proposition 5. *Let $r(t)=(q_1(t),q_2(t),q_3(t),p_1(t),p_2(t),p_3(t),\varphi(t))$ be a solution of the equations of motion for m_3. If $(q_2(t),q_3(t),p_1(t),\varphi(t))$ are zero at $t=0$ and if $(q_2(t),q_3(t),p_1(t))$ is zero and $\varphi(t)=k\pi$ at $t=T/2$, then $r(t)$ is a periodic solution of period T.*

These periodic orbits are *symmetric periodic orbits* of the elliptic restricted three-body problem. Note that in order to have $\varphi(T/2)=k\pi$, we must take $T=2k\pi$.

As we intend to show the existence of symmetric periodic orbits close to infinity, we scale the variables by $\mathbf{q}=\varepsilon^{-2}\tilde{\mathbf{q}}$, $\mathbf{p}=\varepsilon\tilde{\mathbf{p}}$, $\widetilde{\mathcal{H}}=\varepsilon\mathcal{H}$.

Expanding $1/R_1$ and $1/R_2$ in terms of Legendre polynomials (as in [12]) and dropping tildes, we get the following expression for Hamiltonian $\mathcal{H}$.

$$\mathcal{H}(\mathbf{q},\mathbf{p},t,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\mathcal{H}_i^0(\mathbf{q},\mathbf{p},t), \tag{5}$$

where the non-zero terms are $\mathcal{H}_3^0(\mathbf{q},\mathbf{p},t)$ and $\mathcal{H}_{2i+1}^0(\mathbf{q},\mathbf{p},t)$, for $i\geqslant 3$. The functions $\mathcal{H}_3^0$ and $\mathcal{H}_7^0$ are given by

$$\mathcal{H}_3^0(\mathbf{q},\mathbf{p})=\frac{1}{2}|\mathbf{p}|^2-\frac{1}{|\mathbf{q}|}$$

$$\mathcal{H}_7^0(\mathbf{q},\mathbf{p},t)=-\mu(1-\mu)\rho^2\frac{1}{|\mathbf{q}|^3}\left(\frac{-1+3\cos^2 S}{2}\right),$$

with

$$\cos S=\frac{q_1\cos\varphi+q_2\sin\varphi}{|\mathbf{q}|}$$

Notice that $\mathcal{H}_3^0$ is the Hamiltonian of the Kepler problem and therefore Hamiltonian (5) can be seen as a small perturbation of $\mathcal{H}_3^0$. The functions $\mathcal{H}_7^0(\mathbf{q}, \mathbf{p}, t)$ and $\mathcal{H}_R^0(\mathbf{q}, \mathbf{p}, t, \varepsilon) = \sum_{i=9}^{\infty} \varepsilon^i \mathcal{H}_i^0(\mathbf{q}, \mathbf{p}, t)$ are bounded if $(1-\mu)\frac{\rho}{|\mathbf{q}|} \leqslant k$, for some $k < 1$.

5. Continuation of symmetric periodic solutions

In this section we show that circular solutions of the unperturbed Kepler problem can be continued to symmetric periodic solutions of the spatial elliptic restricted three-body problem for small values of ε. We introduce the Poincaré–Delaunay variables defined as

$$\begin{aligned} Q_1 &= l + g, \quad P_1 = L, \\ Q_2 &= -\sqrt{2(L-G)}\sin g, \quad P_2 = \sqrt{2(L-G)}\cos g, \\ Q_3 &= h, \quad P_3 = H, \end{aligned} \tag{6}$$

where $L = \sqrt{a}$, $H = G\cos i$, a is the semimajor axis of the infinitesimal mass, G its angular momentum, $e = \sqrt{1 - G^2/L^2}$ is the eccentricity of the infinitesimal body, i the inclination of the orbital plane to the q_1q_2 reference plane, l the mean anomaly, g the argument of the pericenter measured from the ascending node and h the longitude of the ascending node. These variables are defined on a neighbourhood of the circular Kepler orbits which occur at $Q_2 = 0$, $P_2 = 0$. If $P_3 = 0$ the orbit lies in a plane perpendicular to the q_1q_2 one, see [14] for more details.

The periodicity conditions given by Proposition 5 in Poincaré variables state that at time $t = 0$ we must have

$$Q_1 = 0 \bmod \pi, \quad Q_2 = 0, \quad Q_3 = 0 \bmod \pi \quad \text{and } \varphi = 0$$

and at time $t = T/2$

$$Q_1 = 0 \bmod \pi, \quad Q_2 = 0, \quad Q_3 = 0 \bmod \pi \quad \text{and } \varphi = k\pi.$$

The condition $Q_2 = 0$ implies either $g = 0 \bmod \pi$ or $L = G$, so that m_3 is on an elliptic orbit with its pericenter on the q_1 axis or on a circular orbit.

Applying the symplectic change of variables (6), Hamiltonian (5) becomes

$$\begin{aligned} \mathcal{H}(Q, P, t, \varepsilon) &= \sum_{i=0}^{\infty} \varepsilon^i \mathcal{H}_i^0(Q, P, t) \\ &= \varepsilon^3 \mathcal{H}_3^0(Q, P) + \varepsilon^7 \mathcal{H}_7^0(Q, P, t) + \varepsilon^9 \mathcal{H}_R^0(Q, P, t, \varepsilon), \end{aligned} \tag{7}$$

where

$$\mathcal{H}_3^0(Q, P) = -\frac{1}{2P_1^2},$$

$$\mathcal{H}_7^0(Q, P, t) = -\mu(1-\mu)\rho^2 \frac{1}{r^3}\left(\frac{-1}{2} + \frac{3}{2}\left(\cos\psi\cos(Q_3 - \varphi) - \frac{P_3}{G}\sin\psi\sin(Q_3-\varphi)\right)^2\right),$$

r is the distance from m_3 to the origin and ψ is the position of m_3 measured from the node.

The function $\mathcal{H}_7^0(Q, P, t)$ is 2π-periodic in t and it can be expanded as a Fourier series

$$\mathcal{H}_7^0(Q, P, t) = -\mu(1-\mu)\frac{1}{r^3}\sum_{k=-\infty}^{\infty} a_k(Q, P)\exp(ikt),$$

where

$$a_0(Q, P) = \frac{1}{8}(2+3\eta^2)\left(-2 + 3\cos^2\psi + \frac{3P_3^2\sin^2\psi}{G^2}\right) + \frac{15\eta^2}{8G^2}(-GP_3\sin 2Q_3\sin 2\psi + \cos 2Q_3(G^2\cos^2\psi - P_3^2\sin^2\psi)). \quad (8)$$

In this expression G is clearly a function of (Q, P), see (6), and so is ψ but a closed expression in terms of Q and P does not exist because it needs solving Kepler's equation. In the proof of Lemma 6 we will use an expansion as power series (see Appendix for details).

We will use the technique of the Lie transforms in order to simplify the Hamiltonian (7). As the Hamiltonian is non-autonomous, the new Hamiltonian $\bar{\mathcal{H}} = \sum_{i=0}^{\infty} \bar{\mathcal{H}}_0^i$ will be given $\bar{\mathcal{H}} = \mathcal{L}_W(\mathcal{H}) - \mathcal{L}_W(\partial W/\partial t)$ where $\mathcal{L}_W$ is the Lie transform generated by a function $W(P, Q, t)$ as defined in [11], Section VII, 2. The function W is given by a series expansion $W = \sum_{i=0}^{\infty} \varepsilon^i W_{i+1}$. We choose W_7 such that $\partial W_7/\partial t = \mathcal{H}_7^0(Q, P, t) + \mu(1-\mu)r^{-3}a_0(Q, P)$ and $W_i = 0$ for $i \neq 7$. In this way, $\bar{\mathcal{H}}_0^3 = \mathcal{H}_3^0$ and the periodic terms are removed from $\mathcal{H}_7^0$ and thrown into $\bar{\mathcal{H}}_R = \sum_{i=9}^{\infty} \varepsilon^i \bar{\mathcal{H}}_0^i$. Notice that we only rewrite the Hamiltonian in such a way that the term $\bar{\mathcal{H}}_7^0$ does not depend on t but all the terms in $\bar{\mathcal{H}}_R$ do depend on t.

$$\bar{\mathcal{H}}(Q, P, t, \varepsilon) = \varepsilon^3\bar{\mathcal{H}}_0^3(Q, P) + \varepsilon^7\bar{\mathcal{H}}_0^7(Q, P) + \varepsilon^9\bar{\mathcal{H}}_R(Q, P, t, \varepsilon).$$

If we change the scale of the time variable $t = \varepsilon^{-3}\tau$, the equations of motion for the infinitesimal mass are hamiltonian and the Hamiltonian $\mathcal{K}(Q, P, \tau, \varepsilon) = \frac{1}{\varepsilon^3}\bar{\mathcal{H}}(Q, P, \tau/\varepsilon^3, \varepsilon)$ is of the form

$$\mathcal{K}(Q, P, \tau, \varepsilon) = \mathcal{K}_0(Q, P) + \varepsilon^4\mathcal{K}_1(Q, P) + \varepsilon^6\mathcal{K}_R(Q, P, \tau, \varepsilon), \tag{9}$$

where $\mathcal{K}_0(Q, P) = -\frac{1}{2P_1^2}$, and $\mathcal{K}_1(Q, P)$ is given by

$$\mathcal{K}_1(Q, P) = -\mu(1-\mu)\frac{1}{r^3}a_0(Q, P). \tag{10}$$

The function $\mathcal{K}_R(Q, P, \tau, \varepsilon)$ is bounded by a constant independent of ε because the term τ/ε^3 appears only as the argument of circular functions. Note that $\varepsilon^5\mathcal{K}_R$ is continuous at $\varepsilon = 0$, but $\mathcal{K}_R$ is not so because the smaller gets ε the faster oscillate the terms $\cos\varphi$ and $\sin\varphi$. Note also that in the particular case of zero eccentricity of the primaries, the angle φ is given by $\varphi = \varepsilon^{-3}\tau$. This is the reason why expansions in power series in ε cannot be used but the results of Section 3 can be applied.

Let $z = (Q, P)$, then the equations of motion derived from the $2\pi\varepsilon^3$-periodic Hamiltonian (9) can be written as

$$\dot{z} = \mathcal{F}_0(z) + \varepsilon\mathcal{F}_1(z, \varepsilon) + \varepsilon^2\mathcal{F}_R(z, \varepsilon, \tau), \tag{11}$$

where

$$\begin{aligned} \mathcal{F}_0(z) &= (P_1^{-3}, 0, 0, 0, 0, 0), \\ \mathcal{F}_1(z, \varepsilon) &= \varepsilon^3\left(\frac{\partial\mathcal{K}_1}{\partial P_1}, \frac{\partial\mathcal{K}_1}{\partial P_2}, \frac{\partial\mathcal{K}_1}{\partial P_3}, -\frac{\partial\mathcal{K}_1}{\partial Q_1}, -\frac{\partial\mathcal{K}_1}{\partial Q_2}, -\frac{\partial\mathcal{K}_1}{\partial Q_3}\right). \end{aligned}$$

A solution of the Kepler problem with initial conditions $z_0^* = (Q_0^*, P_0^*) = (0, 0, 0, 1, 0, 0)$ and $\varphi^* = 0$ is

$$z^{(0)}(\tau, z_0^*) = (\tau, 0, 0, 1, 0, 0).$$

If $\varepsilon^3 = \frac{1}{k}$, then at time $\tau = T/2 = 2k\pi\varepsilon^3/2 = \pi$ we have $\varphi(\pi) = k\pi$ and we look for initial conditions in a neighbourhood of z_0^*, of form $z_0 = (0, 0, 0, P_1, P_2, P_3)$, in such a way that the solution $z(\tau, z_0, \varepsilon)$ of system (11), with $\varepsilon \neq 0$ small enough, is a symmetric periodic orbit.

From Lemma 3, we have that $z(\tau, z_0, \varepsilon) = z^{(0)}(\tau, z_0) + \varepsilon z^{(1)}(\tau, z_0, \varepsilon) + z_R(\tau, z_0, \varepsilon)$ where $z^{(0)}(\tau, z_0) = (P_1^{-3}\tau, 0, 0, P_1, P_2, P_3)$ and $z^{(1)}(\tau, z_0, \varepsilon)$ satisfies Eq. (3) and can be obtained through the formula

$$z^{(1)}(\tau, z_0, \varepsilon) = \mathcal{Z}(\tau, z_0)\int_0^\tau \mathcal{Z}^{-1}\mathcal{F}_1(z^{(0)}, \varepsilon, u)\, du, \tag{12}$$

where $\mathcal{Z}(\tau, z_0)$ is the matrix

$$\mathcal{Z}(\tau, z_0) = \left.\frac{\partial z^{(0)}(\tau, \xi)}{\partial \xi}\right|_{\xi = z_0}.$$

Then we have, correct to order ε^4

$$Q_1(\tau, z_0, \varepsilon) = P_1^{-3}\tau + O(\varepsilon^4),$$

and correct to order ε^6

$$Q_2(\tau, z_0, \varepsilon) = \varepsilon Q_2^{(1)}(\tau, z_0, \varepsilon) + O(\varepsilon^6),$$

$$Q_3(\tau, z_0, \varepsilon) = \varepsilon Q_3^{(1)}(\tau, z_0, \varepsilon) + O(\varepsilon^6),$$

where $Q_2^{(1)}(\tau, z_0, \varepsilon)$ and $Q_3^{(1)}(\tau, z_0, \varepsilon)$ are bounded on any fixed interval of time if $\varepsilon \neq 0$ is small enough and are given in the following lemma.

Lemma 6. *Let* $\delta P = (P_1 - 1, P_2, P_3) = (\Delta P_1, P_2, P_3)$. *Then*

$$Q_2^{(1)}(\pi, z_0, \varepsilon) = -\varepsilon^3 \left(\frac{3\pi}{8}\mu(1-\mu)(6\Delta P_1 - P_2 + 6(9\Delta P_1 + P_2)\eta^2) + O(\|\delta P\|^2) \right),$$

$$Q_3^{(1)}(\pi, z_0, \varepsilon) = -\varepsilon^3 \left(\frac{3\pi}{4}\mu(1-\mu)(1-\eta^2)P_3 + O(\|\delta P\|^2) \right).$$

Proof. From Eqs. (12) and (10) we have

$$\begin{aligned} Q_i^{(1)}(\pi, z_0, \varepsilon) &= \varepsilon^3 \int_0^\pi \left(\frac{\partial \mathcal{K}_1}{\partial P_i} \right)_{z^{(0)}(\tau, z_0)} d\tau \\ &= -\varepsilon^3 \mu(1-\mu) \int_0^\pi \left(\frac{\partial}{\partial P_i} \left(\frac{a_0(Q, P)}{r^3} \right) \right)_{z^{(0)}(\tau, z_0)} d\tau, \quad i = 2, 3. \end{aligned} \tag{13}$$

Now, $a_0(Q, P)$ is given explicitly in (8) as function of G, ψ, P and Q. Expressions of G and ψ as functions of P and Q must, of course, be substituted.

Note first that on $z^{(0)}(\tau, z_0)$ we have $Q_2 = Q_3 = 0$. As we are interested in a neighbourhood of the periodic orbit $z^{(0)}(\tau, z_0^*)$ we can use formulas (22) in order to expand $G, r,$ and ψ as power series in δP to order two.

$$\begin{aligned} G &= 1 + \Delta P_1 - \tfrac{1}{2} P_2^2, \\ r &= 1 + 2\Delta P_1 - \cos Q_1 P_2 + \Delta P_1^2 - \tfrac{3}{2}\cos Q_1 \Delta P_1 P_2 + \sin^2 Q_1 P_2^2 + O(|\delta P|^3), \end{aligned}$$

$$\psi = Q_1 + 2\sin Q_1 P_2 - \sin Q_1 \Delta P_1 P_2 + \tfrac{5}{4}\sin 2Q_1 P_2^2 + O(|\delta P|^3), \tag{14}$$

where $Q_1 = (1+\Delta P_1)^{-3}\tau = (1 - 3\Delta P_1 + 6\Delta P_1^2 + O(|\delta P|^3))\tau$.

These expansions are convergent in a small enough vicinity of $z^{(0)}(\tau, z_0^*)$ because all the functions involved are analytic in $(Q_2, Q_3, \Delta P_1, P_2, P_3)$.

Then we can substitute (14) in (13) and expand $a_0(Q,P)r^{-3}$ as power series in $(\Delta P_1, P_2, P_3)$ and coefficients periodic functions in (Q_2, Q_3). After a straight-forward computation the result follows. □

The conditions of symmetry which must be satisfied at time $\tau = T/2 = \pi$ for the existence of a periodic orbit are

$$Q_1(\pi, z_0, \varepsilon) = \pi,$$

$$Q_2(\pi, z_0, \varepsilon) = 0,$$

$$Q_3(\pi, z_0, \varepsilon) = 0.$$

Let us define $f = (f_1, f_2, f_3)$, where $f_1(\delta P, \varepsilon) = Q_1(\pi, z_0, \varepsilon) - \pi$, $f_i(\delta P, \varepsilon) = \varepsilon^{-4} Q_i(\pi, z_0, \varepsilon)$, $i = 2, 3$. Then

$$\begin{aligned}
f_1(\delta P, \varepsilon) &= (1+\Delta P_1)^{-3}\pi - \pi + O(\varepsilon^4),\\
f_2(\delta P, \varepsilon) &= -\frac{3\pi}{8}\mu(1-\mu)(6\Delta P_1 - P_2 + 6(9\Delta P_1 + P_2)\eta^2)\\
&\quad + O(\|\delta P\|^2) + O(\varepsilon),\\
f_3(\delta P, \varepsilon) &= -\frac{3\pi}{4}\mu(1-\mu)(1-\eta^2)P_3 + O(\|\delta P\|^2) + O(\varepsilon).
\end{aligned} \tag{15}$$

A sufficient condition to obtain symmetric periodic orbits of the elliptic problem is to find δP in such a way that $f(\delta P, \varepsilon) = 0$ for $\varepsilon \neq 0$. Strictly speaking, the function $f(\delta P, \varepsilon)$ is not defined for $\varepsilon = 0$, so we cannot use the standard implicit function theorem. We define $f(\delta P, 0)$ and its derivatives by making formally vanish the terms $O(\varepsilon)$ in (15). We shall see that the functions obtained meet the hypothesis of Arenstorf's theorem (see Section 2).

Theorem 7. *Consider the equations of motion for the spatial restricted three-body problem when the mass parameter $\mu \in (0,1)$, and the primaries move around each other on an elliptic orbit with semiaxis ε^2 and eccentricity $\eta \in J = [0, 1/\sqrt{6} - \lambda) \cup (1/\sqrt{6} + \lambda, 1)$, λ a small positive value. If $\varepsilon = k^{-1/3}$ for k a positive integer large enough, then there exist initial conditions for the infinitesimal body such that its motion is a symmetric periodic solution of period 2π, near a Keplerian circular orbit on a plane perpendicular to that of the primaries.*

Proof. We must see that the function $f(P,\varepsilon)$ satisfies the conditions stated in Proposition 2 that guarantee the existence of solutions of $f(\delta P,\varepsilon)=0$ in a neighbourhood of $(0,0)$. The system $f(\delta P,\varepsilon)=0$ has the solution $\delta P=0$ for $\varepsilon=0$.

Let $f_{\delta P}(\delta P,\varepsilon)$ be the Jacobian matrix of $f(\delta P,\varepsilon)$ with respect to δP. For $\varepsilon=0$ we have

$$f_{\delta P}(0,0)=\mu(1-\mu)\begin{pmatrix} -\frac{3\pi}{\mu(1-\mu)} & 0 & 0 \\ -\frac{9\pi}{4}(1+9\eta^2) & \frac{3\pi}{8}(1-6\eta^2) & 0 \\ 0 & 0 & -\frac{3\pi}{4}(1-\eta^2) \end{pmatrix}.$$

Then, if the eccentricity of the primaries $\eta\in J$, $f_{\delta P}(0,0)$ can be inverted and is bounded by a constant m. In order to prove condition (1) of Proposition 2, we write

$$\|f_{\delta P}(\delta P,\varepsilon)-f_{\delta P}(0,0)\|\leqslant\|f_{\delta P}(\delta P,\varepsilon)-f_{\delta P}(\delta P,0)\|+\|f_{\delta P}(\delta P,0)-f_{\delta P}(0,0)\|.$$

Now, the function $f(\delta P,0)$ being analytic, we have

$$\|f_{\delta P}(\delta P,0)-f_{\delta P}(0,0)\|\leqslant\sum_{i=1}^{3}\|f_{i,\delta P}(\delta P,0)-f_{i,\delta P}(0,0)\|\leqslant C_4\|\delta P\|,$$

and, on the other hand, in the inequality

$$\|f_{\delta P}(\delta P,\varepsilon)-f_{\delta P}(\delta P,0)\|\leqslant\sum_{i=1}^{3}\|f_{i,\delta P}(\delta P,\varepsilon)-f_{i,\delta P}(\delta P,0)\|,$$

the first term of the sum is bounded by $C_1\varepsilon^4$ and the second and third are less than $C_2\varepsilon$ because of (15). Then, for δP in a compact neighbourhood of the circular orbit (see Section 3),

$$\|f_{\delta P}(\delta P,\varepsilon)-f_{\delta P}(0,0)\|\leqslant C_5(\|\delta P\|+\varepsilon).$$

Condition (2) of Proposition 2 is a straightforward consequence of (15). □

This theorem yields a continuum of solutions of the system $f(\delta P,\varepsilon)=0$. In order to have a periodic solution of the elliptic problem, the above conditions must be satisfied simultaneously with $\varphi=k\pi$ (i.e. the primaries must be at either the pericenter or apocenter of their orbit). Thus, for each $\varepsilon=k^{-1/3}$, k a large positive integer, a periodic solution of the problem exists. Note that the solution $(\delta P,\varepsilon)$ is near $(0,0)$ which means that $P_3=G\cos i$ is near zero and $i\simeq\frac{\pi}{2}$.

Appendix. The neighbourhood of circular solutions in the Kepler problem

In this Section some convergent expansions relative to the transformation between polar coordinates and Poincare variables for the infinitesimal mass will be given. As use will be made of the classical orbital elements, we first recall some formulas that will be needed (see [3] for details). Let E be the eccentric anomaly, then we have the relation

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+3}}\tan\frac{v}{2}, \tag{16}$$

where v is the true anomaly. The mean anomaly l is related to the eccentric anomaly E through Kepler's equation

$$l = E - e\sin E. \tag{17}$$

Position and velocity can be calculated in terms of the orbital elements. The distance to the origin and the angle ψ are given by

$$\begin{aligned} r &= a(1 - e\cos E), \\ \psi &= v + g. \end{aligned} \tag{18}$$

If we denote by R and Ψ momenta conjugate to r and ψ respectively, we have

$$\begin{aligned} R &= a^{-1/2}(1 - e\cos E)^{-1} e\sin E, \\ \Psi &= G. \end{aligned} \tag{19}$$

Notice that ψ is well defined on circular orbits even though v and g are not themselves defined in this case.

Neither the classical orbital elements nor Delaunay elements are well defined on circular orbits, and the same is true for any of the anomalies. In contrast to that, the magnitudes $e\sin v$ and $e\cos v$ are well defined and depend smoothly on the variations of the initial conditions. The same can be said about $e\sin l$, $e\cos l$, $e\sin E$, $e\cos E$.

In passing from Poincare elements to polar coordinates, direct use of the angular variables l, v, E will be avoided, and pairs such as $(e\sin E, e\cos E)$ will be used instead. Notice that in each one of the pairs $(e\sin l, e\cos l)$, $(e\sin v, e\cos v)$, $(e\sin E, e\cos E)$, both variables can be expanded as a power series in the variables of any other pair. The differences $v - E$, $E - l$ and $v - l$ can be expanded in the same way as well. We will quote a few of these expansions that will prove useful in what follows. A standard reference for that subject is [2].

From Kepler's equation (17) the following expansions can be derived

$$e\sin E = e\sin l + e\sin l e\cos l + O_3(e\sin l, e\cos l),$$

$$e\cos E = e\cos l - (e\sin l)^2 + O_3(e\sin l, e\cos l). \tag{20}$$

The difference $v - E$ can be expanded from (16) as

$$v - E = e\sin E + \tfrac{1}{2}e\sin E e\cos E + O_3(e\sin E, e\cos E)$$

and the difference $E - l$ is just given by (17).

We also have

$$e\sin l = \sin Q_1 e\cos g - \cos Q_1 e\sin g,$$

$$e\cos l = \cos Q_1 e\cos g + \sin Q_1 e\sin g. \tag{21}$$

We look now for the formulas that change from Poincare variables to polar coordinates. From the definition of Q_2 and P_2 we have

$$e\sin g = -\frac{1}{\sqrt{2}}Q_2 P_1^{-1}\sqrt{2P_1 - \frac{1}{2}(Q_2^2 + P_2^2)}$$

$$e\cos g = \frac{1}{\sqrt{2}}P_2 P_1^{-1}\sqrt{2P_1 - \frac{1}{2}(Q_2^2 + P_2^2)},$$

and the right-hand sides can be expanded as power series in Q_2, P_1, P_2 near $Q_2 = 0, P_1 = 1, P_2 = 0$. Taking into account (21), we get $e\sin l$, $e\cos l$ as power series in the mentioned variables and coefficients trigonometric polynomials in Q_1 and from (20) we eventually find $e\sin E$ and $e\cos E$, again as series of the same type. From (18) and (19) we get the needed expansions for r and R. The expansion of ψ follows a similar reasoning using

$$\psi = Q_1 + v - E + E - l$$

and expanding $v - E$ and $E - l$ as before. The above procedures yield the series up to any order if care is taken to expand to the required order in each step. We quote the result for polar coordinates and momenta as functions of Poincare variables up to second order.

$$\begin{aligned} r \;=\; & 1 + 2\Delta P_1 + \Delta P_1^2 + \frac{Q_2^2}{2} - P_2\cos(Q_1) - \frac{3}{2}\Delta P_1 P_2\cos(Q_1) + \frac{1}{2}Q_2^2\cos(2Q_1) \\ & + Q_2\sin(Q_1) + \frac{3}{2}\Delta P_1 Q_2\sin(Q_1) + P_2^2\sin^2 Q_1 + Q_2 P_2\sin(2Q_1) \end{aligned}$$

$$\begin{aligned}
\psi \;=\; & Q_1 + 2Q_2\cos(Q_1) - \Delta P_1 Q_2\cos(Q_1) + \frac{5}{2}Q_2P_2\cos(2Q_1) \\
& +2P_2\sin(Q_1) - \Delta P_1 P_2\sin(Q_1) - \frac{5}{4}Q_2^2\sin(2Q_1) + \frac{5}{4}P_2^2\sin(2Q_1), \\
R \;=\; & Q_2\cos(Q_1) - \frac{3}{2}\Delta P_1 Q_2\cos(Q_1) + 2Q_2P_2\cos(2Q_1) + P_2\sin(Q_1) \\
& -\frac{3}{2}\Delta P_1 P_2\sin(Q_1) - Q_2^2\sin(2Q_1) + P_2^2\sin(2Q_1), \\
& \psi = 1 + \Delta P_1 - \frac{Q_2^2}{2} - \frac{P_2^2}{2},
\end{aligned} \tag{22}$$

where $\Delta P_1 = P_1 - 1$.

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