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Finite-time scaling in local bifurcations

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Finite-size scaling is a key tool in statistical physics, used to infer critical behavior in finite systems. Here we have made use of the analogous concept of finite-time scaling to describe the bifurcation diagram at finite times in discrete (deterministic) dynamical systems. We analytically derive finite-time scaling laws for two ubiquitous transitions given by the transcritical and the saddle-node bifurcation, obtaining exact expressions for the critical exponents and scaling functions. One of the scaling laws, corresponding to the distance of the dynamical variable to the attractor, turns out to be universal, in the sense that it holds for both bifurcations, yielding the same exponents and scaling function. Remarkably, the resulting scaling behavior in the transcritical bifurcation is precisely the same as the one in the (stochastic) Galton-Watson process. Our work establishes a new connection between thermodynamic phase transitions and bifurcations in low-dimensional dynamical systems, and opens new avenues to identify the nature of dynamical shifts in systems for which only short time series are available.

Bifurcations separate qualitatively different dynamics in dynamical systems as one or more parameters are changed. Bifurcations have been mathematically characterized in elastic-plastic materials[1], electronic circuits[2], or in open quantum systems[3]. Also, bifurcations have been theoretically described in population dynamics[4–6], in socioecological systems[7,8], as well as in fixation of alleles in population genetics and computer virus propagation, to name a few examples[9,10]. More importantly, bifurcations have been identified experimentally in physical[11–14], chemical[15,16], and biological systems[17,18]. The simplest cases of local bifurcations, such as the transcritical and the saddle-node bifurcations, only involve changes in the stability and existence of fixed points.

Although, strictly speaking, attractors (such as stable fixed points) are only reached in the infinite-time limit, some studies near local bifurcations have focused on the dependence of the characteristic time needed to approach the attractor as a function of the distance of the bifurcation parameter to the bifurcation point. For example, for the transcritical bifurcation it is known that the transient time, τ, diverges as a power law[19], as $\tau \sim |\mu - \mu_c|^{-1}$, with μ and μ_c being the bifurcation parameter and the bifurcation point, respectively, while for the saddle-node bifurcation[20] this time goes as $\tau \sim |\mu - \mu_c|^{-1/2}$ (see[12] for an experimental evidence of this power law in an electronic circuit).

Thermodynamic phase transitions[21,22], where an order parameter suddenly changes its behavior as a response to small changes in one or several control parameters, can be considered as bifurcations[23]. Three important peculiarities of thermodynamic phase transitions within this picture are that the order parameter has to be equal to zero in one of the phases or regimes, that the bifurcation does not arise (in principle) from a simple low-dimensional dynamical system but from the cooperative effects of many-body interactions, and that at thermodynamic equilibrium there is no (macroscopic) dynamics at all. Non-equilibrium phase transitions[24,25] are also bifurcations and share these characteristics, except the last one. Particular interest has been paid to second-order phase transitions, where the sudden change of the order parameter is nevertheless continuous and associated to the existence of a critical point.

A key ingredient of second-order phase transitions is finite-size scaling[26,27], which describes how the sharpness of the transition emerges in the thermodynamic (infinite-system) limit. For instance, if m is magnetization (order parameter), T temperature (control parameter), and ℓ system's size, then for zero applied field and close to the critical point, the equation of state can be approximated as a finite-size scaling law,

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