Study of a class of skew-products defined on the cylinder

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Chapter 1

Introduction

In 1984 Grebogi, Ott, Pelikan and Yorke [16] studied the following system defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}$

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = 2\sigma \tanh(x_n)\cos(2\pi\theta_n), \end{cases}$$

where ω is an irrational number and σ is greater than 1. They found two invariant curves; one of them is always the circle $x \equiv 0$ and the other one intersects $x \equiv 0$ in a dense and negligible set of points. This is where the strangeness of the manifold came from. Moreover, they assert that the closure of this second invariant curve is an attractor, and that the Lyapunov exponents on it are non-positive; consequently it is non-chaotic. Therefore they called it *Strange Nonchaotic Attractor* (briefly SNA). Since then many mathematicians have studied two dimensional skew-products, with the objective of finding more attractors of this kind (see, [3, 15, 18, 17] and [20]). In fact these objects had already appeared (for instance in [19, 24, 25] and [33]) and even baptised with other names, as *absorbing Cantor sets* (see [9] and [10]). These objects happen also to appear in the "real life", i.e. in physical experiments (see [14] and [29]).

When we started to work on this subject we realised that there was no consensus on the definition of SNA and, unfortunately, one can find much more empirical and rude numerical studies about SNAs than rigorous proofs of their existence. Waving hands all the definitions are the same; they all agree in the following: the attractor, whatever it is, is invariant; the strangeness makes reference to the smoothness of the attractor, and the chaoticity is studied in terms of Lyapunov exponents. All the slightly different definitions we have found where used to study SNAs for two-dimensional systems, but for higher dimensions the only definition considered is the one proposed by Grebogi et al. in [16].

The third chapter of this memoir is devoted to the concept of SNA. We propose a new definition of SNA which encompasses all the definitions we have found in the literature, is independent of the dimension of the system and is satisfied by the two paradigmatic examples of this field: Grebogi et al. model [16] and Keller's model [22]. Moreover, we show the pros and cons of the definition we propose. At the end of this third chapter we prove that the two paradigmatic examples are, according to our definition, SNAs; and we provide one-dimensional examples of this kind of attractors: the absorbing Cantor sets and the *solenoidal attractors*.

By the other hand, all the examples we have found are quasiperiodically forced skew-products, i.e. the basis is an irrational rotation, and most of them are two-dimensional and defined on the positive semi-cylinder. In this memoir, we are only interested in systems of the form:

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = g(\theta_n) f(x_n) \end{cases}$$

and defined on the cylinder, $\mathbb{S}^1 \times \mathbb{R}$, or in part of it: $\mathbb{S}^1 \times [0, \infty)$, $\mathbb{S}^1 \times [0, 1]$ or $\mathbb{S}^1 \times [-1, 1]$. We have only found a few analytical proofs of existence of SNAs for systems defined on the whole cylinder $\mathbb{S}^1 \times \mathbb{R}$, and all of them are for concrete functions f and g (see, for example, [7]). As far as we know, the general results of existence of this kind of attractors are only for the cases where f is monotonous and the system is defined on $\mathbb{S}^1 \times [0, \infty)$ (see [17] and [22]) or f is an unimodal map, in which case the system is defined on $\mathbb{S}^1 \times [0, 1]$ (see [3] and [8]). In both cases f maps zero into zero, therefore the circle $x \equiv 0$ is invariant. Under some conditions, the authors in the above works prove the existence of another invariant manifold given by the graph of a map. They rigorously prove that the closure of the graph of this map, whenever it exists, is strange and, in some sense, non-chaotic.

What happens when we extend the map f to the negative numbers? In Chapter 4 we extend Keller's [22] and Haro's [17] results to the case when f is monotonous and defined on the whole real line $\mathbb{R}$, and in Chapter 5 we extend Alsedà and Misiurewic's result [3] to a bimodal map defined on $\mathcal{I} = [-1, 1]$. In both cases we suppose $f(0) = 0$ so that the circle is still invariant. We prove the existence of a bivaluated correspondence such that the closure of its graph is an attractor, but a priori, it does not satisfy the definition of SNA we propose in Chapter 3. This is because we only control the values of the Lyapunov exponents on the graph of the correspondence, therefore we cannot assure the nonchaoticity as we require. Moreover, the attractor we find may be either a minimal attractor, (roughly speaking, there exists a dense orbit) or the graph of the above correspondence can be split into exactly two disjoint and invariant parts which are different from zero almost everywhere. We provide examples of both situations. If the map g vanishes at some point, then both components of the correspondence vanish at this point, and the circle $x \equiv 0$ belongs to the attractor. Moreover, we prove that both components of the correspondence are different from zero in a set of full Lebesgue measure, consequently they are discontinuous at almost every point, and in particular strange. This implies that if there are two attractors they have the same regularity; if one of them is strange, the other also.

As we said before, all the systems where the existence of SNAs have been studied are skew-products whose basis is an irrational rotation. What changes if we exchange the irrational rotation of the basis by another circle map R? In order to answer this question we consider systems of the form:

$$\begin{cases} \theta_{n+1} = R(\theta_n), \\ x_{n+1} = g(\theta_n) f(x_n), \end{cases}$$

where R is a degree one circle map. First we consider the case where R is a continuous degree-one map without periodic points, then it is uniquely ergodic (see [4]).

The forth and fifth chapters of this memoir are devoted to this first case. In fact, we change both the basis of the system and the map f. One of the main differences between this case and the quasiperiodically forced one is that the invariant correspondence (or map) may not be defined on $\mathbb{S}^1$, not even on a set of full Lebesgue measure but only on a set of full measure with respect the R-ergodic measure. The main consequence of this fact is that the closure of the graph of this correspondence (or map) may not be an attractor, although it attracts all points in almost every fibre ("vertical line") except those lying on the circle. The strangeness in this situation is more complicated to handle because it has to be controlled the relationship between the vanishing set of g and where the correspondence (or map) is defined. The circle $x \equiv 0$ belongs to the closure of the invariant graph but may not intersect the graph.

In Chapter 6, we finally treat the case where R is a $\mathscr{C}^1$ degree one map such that the rotation interval of any of its lifts is non-degenerate. We prove the existence of a bivaluated correspondence which is positively invariant. The closure of this graph is in fact the union of uncountably many different attracting sets, which coexists simultaneously. Each of these attracting sets corresponds to the ones obtained in Chapter 4, when the system is monotonous on the fibres, or in Chapter 5, when it is bimodal, with a different rotation number.

We note that when R has no periodic points, the graph we find is both positively and negatively invariant, whereas in the general situation we can only assure positive invariance. Concerning the Lyapunov exponents, in the particular cases treated in Chapters 4 and 5 we know for almost every point in the invariant graph the value of the two Lyapunov exponents, whereas in the general case, for the points on the graph, we only know the value of one of them.

To make this memoir more self-contained we introduce some notations and concepts from Dynamical Systems, Ergodic Theory, in particular Lyapunov exponents, circle maps and some general properties of two-dimensional skew-products. Chapter 2 is devoted to all these preliminary results.

Finally in the last chapter of this memoir we discuss some future perspectives. They are mainly related with the analysis SNAs in the systems studied in the three previous chapters.