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Bifurcations of limit cycles in planar differential and piecewise differential systems

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Certifico que aquesta memòria ha estat realitzada per Leonardo Pereira Costa da Cruz sota la meva supervisió i que constitueix la seva tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona.

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Chapter 1

Introduction

The capacity of men to ask and answer is what pushes science. Taking into account that natural events depend continuously over time, it is clear that understanding natural phenomena of the past, present and future are big concerns of human beings. Mathematics, in turn, is without any doubt the basic language that describes this code behind natural events. Concretely, a field of mathematics that studies this is the so-called differential equations, which provides, for example, the rules for the evolution in time of a particle.

Since the 17th century, when I. Newton and G. Leibniz introduced the *differential calculus* more specifically in 1682, *differential equations* came to be a relevant and efficient tool to model in an abstract language what occurs in the real world, starting with the mechanical problems of bodies. Currently, the study of differential equations have become one of the fundamental pillars of the study of philosophy mathematics, because these equations have a big importance to the development of many areas of science, as well as engineering, biology, electronics, economy, etc.

When the derivation variable just plays an implicit role, the differential equation is said to be autonomous. The autonomous cases can be considered as dynamical systems and the time is taken as the derivation variable.

Ordinary differential equations of order n take the form

$$F\left(t, x, x', x'', \ldots, x^{(n)}\right) = 0, \tag{1.1}$$

where $x^{(n)}$ is the nth derivative of x with respect to t. The autonomous cases take place when F does not depend on t. If x is a vector instead of a real function, the equation (1.1) is called a differential system.

Only after approximately two hundred years of the statements of Newton and Leibniz, in the 19th century and more precisely around 1881, another brilliant mathematician appeared: H. Poincaré, who brought the splendor of the study of differential equations.

He realized that the qualitative properties of the solutions of a differential equation could be investigated, without such solutions having to be determined explicitly. Thus, instead of looking for the solution, he turned to a qualitative approach, using geometric and topological techniques. In his "Mémoire sur les courbes définies par une équation différentielle", he introduces these results which were a great breakthrough in the study of differential systems. Currently known as *Qualitative Theory of Differential Equations*.

Despite being applied to higher dimension fields, everything in his work was considered in equations in two variables, which means considering a system of the form (1.1) when the space where x is considered is $\mathbb{R}^2$ or any contained open subset, and we refer to it as a planar differential system.

Let $Z = (X, Y)$ be a first-order autonomous planar differential systems, defined by

$$\begin{cases} \dot{x} = X(x, y), \\ \dot{y} = Y(x, y), \end{cases} \tag{1.2}$$

where $x(t)$, $y(t)$, $X(x, y)$ and $Y(x, y)$ are real functions.

In order to understand this geometric point of view, let us consider the velocity field Z, which is the vector field whose components are X and Y, the functions in system (1.2). The solutions of the differential system are the trajectories of the vector field. It means that at any point the tangent vectors to the solution curves and the vector field are parallel. The trajectories are also known as the orbits of the vector field. The advantage of using orbits lies in the fact that if we change the time parametrization, they remain unchanged.

Among the contributions of Poincaré are the introduction of the concept of *phase portrait*, which is the sufficient information to determine the topological structure of the orbits of a differential equation. Moreover, he developed theoretical concepts such as *return map* or the *Annular Region Theorem*, which are fundamental for classifying orbits with particular behaviors. Some of them were characterized by Poincaré, as well as among others the equilibrium points and cycles.

Equilibrium points are the points where the vector field vanishes. They are also called singular, critical or fixed points. Cycles are the trajectories of the vector field that repeat themselves along time. Usually, they are also called closed or periodic orbits. Notice that equilibrium points are a particular type of cycles. For a point in a cycle after a finite time T, its orbit will be again on itself. For a fixed point, its orbit is on itself for every time t in $\mathbb{R}$.

The notion of *limit cycle* was also introduced in the first papers which dealt with qualitative theory. Essentially, a limit cycle, γ, is a periodic orbit such that at least one trajectory of the vector field, different from γ, approaches γ in positive or negative time. Usually, when the vector field is of class $\mathcal{C}^1$ an alternative definition is given. A closed orbit is named limit cycle if it is isolated from the other periodic orbits. This definition is, in general, more restrictive than the previous one, but both are equivalent in the analytic case.

In the not too distant future in 1926, B. van der Pol provided a differential equation that describes a non-conservative oscillator and used graphical methods to prove the existence of a periodic orbit. Already in 1929, A. Andronov established the relation between the limiting cycles of Poincaré.

Contemporaneous with Poincaré, and using his contributions, I. Bendixon presented in 1901 the well-known Poincaré–Bendixon Theorem. The result states that the solutions that really matter are called *singular or minimal sets* (critical points, periodic orbits and separatrix) defined a differential equation on a compact set has the property that the other solution goes to a singular solution. Consequently, the phase portrait is now determined by the set of singular solutions.

Thus, in a more specific way we can say that the qualitative theory aims to make the portrait of a differential equation, using the most significant solutions, that is, the minimal sets.

Interested in this new approach of the differential equations, the team led by A. Liapunov studied the behavior of solutions in a neighborhood of an equilibrium position, i.e. he founded the modern theory of stability of motion.

In the International Congress of Mathematics in 1900, D. Hilbert proposed 23 problems that in his opinion would motivate advances in mathematics during the 20th century. Among these there is the *16th Hilbert problem*, whose second part asking about the maximum number and the position of the limit cycles of a polynomial planar system in function of its degree, that is a system like (1.2) with X and Y being the polynomials of degree n. By convention, this number is called $H(n)$. As expected, these problems motivated many works and the 16th continues inspiring today. Recently it was considered one of the most relevant problems of the 21st century.

The first step in the direction of 16th Hilbert problem was given by H. Dulac in 1923. Currently called the finitude problem, his work goes in the direction of proving the finitude of the number of limit cycles in a polynomial vector field in the plane. This proof was considered valid for many years. It was not until the 1970s that Y. Ilyashenko did prove that it was false. So, some years later and independently, Y. Ilyashenko and J. Écalle provided a correct proof. Although the proof given by Dulac was wrong, the ideas given by him were very fruitful and generated results like the classical Dulac Theorem and its generalization: the Bendixon–Dulac Theorem.

Over the years many other works have been done in this direction of 16th Hilbert problem. But even the simplest case, $n = 2$, is still unsolved. N. Bautin (1952) states that $H(2) \geq 3$. Later, simultaneously, S. Shi (1979) and L. Chen and M. Wang found an example with $H(2) \geq 4$. For the next case, $n = 3$, J. Li and Q. Huan (1987) showed that $H(3) \geq 11$.

Given the big difficulty of the 16th Hilbert problem, mathematicians began to propose weaker versions of this problem. The more general version is the so-called Arnold–Hilbert problem, however it is still an open problem.

This problem says that: Let H, P, and Q be polynomials of degree n and R an integral factor. Given $\Gamma(h)$ a level curve $H(x, y) = h$ of the system

$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y}, \\ \dot{y} = \dfrac{\partial H}{\partial x}. \end{cases} \tag{1.3}$$

What is the number of zeroes of the integral

$$M(h) = \int_{\Gamma(h)} \frac{Pdx + Qdy}{R}? \tag{1.4}$$

The integral $M(h)$ is known as an *Abelian integral*, or in a broader context, as a *Melnikov's function*. The maximum number of simple zeros of $M(h)$ is also related to another problem, the highest multiplicity of a focus, which we call cyclicity (the maximum number of limit cycles that we get from an equilibrium point by a given polynomial perturbation).

The above problem appears when considering the polynomial perturbation of a Hamiltonian system,

$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y} + \varepsilon P, \\ \dot{y} = \dfrac{\partial H}{\partial x} + \varepsilon Q. \end{cases} \tag{1.5}$$

More generally the approach to this type of problem is through the *Averaging Theory* this method starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the mechanism. The first formalization of this procedure was given by Fatou in 1928. During the last decade many mathematicians have contributed to this problem. We highlight the works of A. Gasull, J. Libre, A. Varchenko, L. Gavrilov, E. Horozov, C. Li, D. Wang, H. Zoladek and others.

S. Smale also said that the computation of the Hilbert number can be notably difficult. So mathematicians must consider a special class of differential equations where it is proved that finitude is simple, but the upper bounds for $H(n)$ in this class remains unknown.

For example, considering the differential equations given by A. Liénard, published in a work in "Révue générale déletricité" in 1928, which has a formulation that has a strong relation with van der Pol oscillators.

He proposes that: Given a Liénard system

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -x, \end{cases} \tag{1.6}$$

where F is a real polynomial of degree n and satisfying $F(0) = 0$, which is the number $H(n)$ for this system? Although much more restricted, this problem also remains open.

Usually when we model a system in nature using differential equations they depend on some free parameters. The study of which phase portraits of a differential equation do not change it topology with small changes in the parameters is what we call *structural stability*. On the other hand, when the phase portrait changes we have a *bifurcation*. In the mid-1930s A. Andronov and L. Pontryagin were the pioneers in this subject. Soon after M. Peixoto, significantly extended the results given by the previous ones. Hence, the structural stability of a differential system is indicated by the stability of the phase portrait by varying the parameters.

The modeling of some systems in nature marked the history of the *applications* of the differential equations and boosted their development. Among others, three examples are worth mentioning.

Starting with the n-body problem, this question was motivated by the necessity to understand the movement of the sun, planets and stars. The first mathematical formulation was elaborated by Newton, who expressed the gravitational interactions in terms of differential equations.

Another landmark was the Lotka–Volterra equation proposed independently by the mathematician V. Volterra and the biophysicist A. J. Lotka in 1925. The first one found this model based on the work of U. d'Ancona, who developed its work analyzing the growth of the population of sharks and the decrease of the population of the other fishes in a sea of Italy, and the second studying the prey-predator relationship in a general way. Finally, the last example is the already mentioned modeling of van der Pol oscillators and Liénard equations.

Given a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $F(0) = 0$, such that the Jacobian matrix $JF(0)$ has pure imaginary eigenvalues. In these conditions the *center-focus* problem is based on finding conditions to the parameters to distinguish the equilibrium point between focus and center. From the return maps, Liapunov considers the importance of the terms of the series expansion of this application, which are the *Liapunov constants*. Thus, when all constants are identically zero we obtain the sufficient condition for a center. The problem however is extremely difficult due to the facts that the calculations of these quantities usually requires complex algebraic calculations and finding the points where all of them vanish is also very difficult.

The problem of *global stability*, both in points of equilibrium and in periodic orbits, has a special relevance in applications since it ensures the asymptotic tendency of any flow of a vector field that tends to a given singular solution or state of equilibrium. In 1960, L. Markus and H. Yamabe established the conjecture: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class $\mathcal{C}^1$ with $F(0) = 0$, such that for every $x \in \mathbb{R}^n$ the proper values of the Jacobian matrix $JF(x)$ have negative real part. Then, $x = 0$ is a globally asymptotically stable point. This conjecture was proved for $n = 2$ by C. Gutiérrez and R. Fessler in 1993. Years later in 1997 for $n = 3$, A. Gasull, A. Cima, F. Mañosa, A. van den Essen. and E. Hubbers gave a counterexample to the conjecture.

Considering the *local stability* of an equilibrium point, we can affirm that in general we have three types of singularities. The ones that are called hyperbolic, in which the Groebner–Hartman Theorem applies. The cases associated to the center-focus problem which are non-hyperbolic. And the rest of the points where usually the technique of blowing-up is used, which consists in exploding the critical point to others for which we can explore the local behavior of the orbits around them and so find the types of sectors of the original critical point. The works of F. Dumortier, R.Roussarie and J. Sotomayor from 1977 are fundamental to understand this technique.

Although the study of *piecewise systems* came from the first half of the last century, it was around 1950 that F. Filippov completely formalized this branch of qualitative theory by considering and defining the flow of the simultaneous systems on the manifold of separation.

We define a Σ-piecewise vector field as follows. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function. We denote the discontinuity boundary by $\Sigma = h^{-1}(0)$, being 0 a regular point and $\Sigma^{\pm} = \{\pm h > 0\}$. We consider the Σ-piecewise vector field by $Z^{\pm} = (X^{\pm}, Y^{\pm})$, where $X^{\pm}$, $Y^{\pm} : \mathbb{R}^n \to \mathbb{R}^n$ are real functions and the vector field $Z^{\pm}$ is defined on Σ following Filippov's terminology.

In last years, emerged a big interest for the study of piecewise systemshas emerged, due to the fact that many real phenomena can be modeled with this class of systems. In particular in electrical and mechanical systems, in control theory, and even genetic networks, among others.

Motivated by the importance of piecewise systems, to extending the tools and classic problems of Qualitative Theory in analytical systems to this kind of systems, became a relevant and interesting question. A large set of classical theorems are not satisfied by the piecewise systems. Among others, we can cite the Existence and Uniqueness Theorem or the Hartman–Grobman Theorem.

For piecewise systems, similarly as for analytical differential systems, a *limit cycle* is an isolated periodic orbit. On Σ, Filippov defined the regions of escaping, sliding, crossing and tangency points. Consequently, a limit cycle here can have points in those regions. Periodic orbits that contain only crossing points are the ones closest to that given by analytical systems.

For piecewise systems, the classical *Poincaré–Bendixson theorems* are not satisfied due to the fact that there exist different minimals sets. However recently in 2018 C. A. Buzzi, T. Carvalho, and R. D. Euzébio presented an extension of this theorem for piecewise systems. Over some extra hypothesis they proved a larger number of minimal sets by adding the pseudo-cycle, the pseudo-graph and the singular tangencies.

Assuming that $Z^{\pm}$ are polynomials, we can consider an *extension of the Hilbert's 16th problem* to piecewise systems restricting, if necessary, the studied family.

That is, for $n = 1$. Considering the hypothesis that Σ is a straight line and the system is continuous ($Z^{+}(x, y) = Z^{-}(x, y)$ with $(x, y) \in \Sigma$), in 1998, E. Freire, E. Ponce, F. Rodrigo and F. Torres proved that $H(1) = 1$.

For the non-continuous but linear case and also when Σ is a straight line we have that $H(1) \geq 3$. This number was firstly detected numerically by Huan and Yang. Later, it was analytically proved by Llibre and Ponce. Other authors have also proved the same. For analytical systems one of the most used tools to find limit cycles is the Average Theory, which in general way applies in a very similar way to piecewise systems.

For piecewise systems, the study of the hyperbolicity of minimal sets (equilibrium points, limit cycles,...) is even more complicated. First, the classification of equilibrium points in Σ, because among the hyperbolic, non-hyperbolic and the other cases, the combination of two or more equilibria is needed to study. Furthermore, in piecewise systems the so-called tagency points also fulfill this role.

The study of hyperbolicity of limit cycles and other minimal sets also depends on the behavior of the systems on the manifold separation, Σ. It is worth quoting Llibre for his contributions in this subject and also M. A. Teixeira. Teixeira is known for the T-singularity that shows that the behavior of singularities given by tangencies has a really interesting behavior and very different from the analytic cases.

Due to the difficulties on the computations of the Liapunov constants on piecewise systems, the center-focus problem for these type of systems has not been completely developed, as well as many other results and problems that we can extend to piecewise systems but were not developed due to the fact that they have a very high degree of difficulty. This thesis is our contribution to this recent and interesting field.

The work has been developed in collaboration with J. Torregrosa, and it is structured in an introduction as the first chapter and then four chapters where the results and proofs are developed. The main techniques in each chapter are different and so they are written in an independent form. As it is explained in the title, the main results are concerning to limit cycles in differential and piecewise differential systems in the plane. Basically, almost all the studied vector fields are polynomial or piecewise polynomial.

In Chapter 2 we use the averaging technique to study the simultaneous bifurcation of isolate periodic orbits in a polynomial cubic planar system that has two period annuli. Although the unperturbed system is analytic and also its perturbation is considered inside the piecewise class of two zones separated by a straight line, the x-axis. We consider two different type of problems. First we study the number of limit cycles up to first order polynomial perturbation of degree n. More concretely, we prove that the inner and outer Abelian integrals are rational functions and we provide an upper bound for the number of simple zeros. Second, for a cubic perturbation, we can improve the general result by obtaining the maximum number of these periodic solutions, always up to first order perturbation. This maximum number is 9 and 8, for the inner and outer regions, respectively. Finally, the simultaneous bifurcation problem is also considered. Then, 12 limit cycles exist and they appear in three different types of configurations: (9,3), (6,6) and (4,8). We remark that, in the non-piecewise scenario, only 5 limit cycles were found. The results on this chapter are already published in [CT18b].

The aim of Chapter 3 is to provide the best lower bound known up to now for the Hilbert number in the quadratic polynomial piecewise vector fields defined in two zones separated by a straight line. We prove that, in this class, at least 16 limit cycles appear. We study, using first and second order averaging method, the perturbation of all isochronous quadratic polynomial vector fields having a birational linearization. For the first order we provide some upper bounds that are reached. For the second order, we study the Taylor developments near the origin to provide the 16 limit cycles. The result is proved by doing a careful study of the intersection of second order varieties. The Poincaré–Miranda theorem and some computer assisted proofs have been necessary to complete the proof. This work has been developed together with D. Novaes and it is submitted to be published, see [CNT18].

The Bendixson–Dulac Theorem provides a criterion to find upper bounds to the number of limit cycles in analytic differential systems. In Chapter 4 we extend this classical result to some classes of piecewise differential systems. We apply it to three different Liénard piecewise differential systems. In all cases, the systems present regions in the parameter space with no limit cycles and others having at most one. The results are submitted to be published, see [CT18a].

In Chapter 5 we study the family of quartic linear-like time reversible polynomial systems having a nondegenerate center at the origin. This family is defined by quartic vectors fields having only degree one in one variable. In this class, we classify all systems having two extra non-degenerate centers out of the symmetry line. There are only two configuration types, when the three centers are aligned or when they are located at the vertex of an isosceles triangle. Next, we are interested in the simultaneous bifurcation of small limit cycles in these systems. This work is not finished yet. We have studied all the limit cycles appearing with first order Taylor developments. We have obtained five different configurations of limit cycles surrounding the three centers. The highest number has been 13 in two configurations. We are still working in the high order analysis to improve our results. The results has been done in collaborations with V. Romanovsky and can be found in [CRT18].