Master Thesis

MASTER IN ADVANCED MATHEMATICS

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

The parameterization method for invariant curves associated to parabolic points

Clara Cufí Cabré

Advisor: Dr. Ernest Fontich Julià
Performed at: Departament de Matemàtiques i Informàtica

Barcelona, 7th December 2018

Acknowledgments

I would like to gratefully thank my advisor, professor Ernest Fontich, for his patience and dedication to this project, his inestimable advice and finally for opening me the door of mathematical research.

Abstract

In the first part of this work we present the parameterization method for invariant manifolds and we apply it to prove the existence of stable invariant curves of planar maps associated to a fixed point with an eigenvalue λ such that $0 < |\lambda| < 1$. We study both the case in which the map is analytic and the case in which it is differentiable. In the second part, we apply the parameterization method to obtain the existence of a stable analytic curve associated to a nilpotent parabolic fixed point of an analytic map. The main result of this master thesis is the existence of such a stable curve. Finally, we perform a numerical simulation in order to estimate the growth of the coefficients of a parameterization of this curve.

Contents

1 Introduction

In this master thesis we deal with invariant manifolds of discrete dynamical systems. A dynamical system, $(\mathcal{M}, \mathbb{T}, \Phi)$, is defined as the action Φ of a group $\mathbb{T}$, which represents the time, on an abstract space $\mathcal{M}$, of the form

$$\begin{aligned} \Phi : \mathbb{T} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (t,\, x) &\longmapsto \Phi(t, x), \end{aligned}$$

such that for all $x \in \mathcal{M}$ and all $t, s \in \mathbb{T}$ we have

$$\Phi(0,\, x) = x \quad \text{and} \quad \Phi(t+s,\, x) = \Phi(s,\, \Phi(t,\, x)).$$

Given a set $\mathcal{M}$ and a map $f : \mathcal{M} \to \mathcal{M}$, a discrete dynamical system, $(\mathcal{M},\, \mathbb{Z},\, \Phi)$, is usually defined by the action

$$\Phi(n,\, x) := f^n(x), \quad n \in \mathbb{Z}, \quad x \in \mathcal{M};$$

that is, a discrete dynamical system is the action of iterating a map in a certain set, which we denote as the *phase space*.

Given a dynamical system induced by a map $f : \mathcal{M} \to \mathcal{M}$, we define the *orbit* of a point $x \in \mathcal{M}$ as the set of its iterates, namely $\{f^n(x) \,|\, n \in \mathbb{Z}\}$. A point $x \in \mathcal{M}$ is called a *fixed point* if $f^n(x) = x$, for all $n \in \mathbb{Z}$, ant thus, if $f(x) = x$.

Let $(\mathcal{M},\, \mathbb{Z}, \Phi)$ be a discrete dynamical system. We say that $\mathcal{N} \subset \mathcal{M}$ is an *invariant set* if

$$\Phi\,(n,\, x) \in \mathcal{N}, \quad \forall\, n \in \mathbb{Z}, \quad \forall\, x \in \mathcal{N}.$$

In the case of a dynamical system induced by a map f defined in a topological space, if such an invariant set is a manifold, we call it an *invariant manifold.* In this work we are interested in invariant one-dimensional manifolds of two-dimensional maps.

We say that two dynamical systems, $(X,\, \mathbb{Z}, \Phi)$ and $(Y,\, \mathbb{Z}, \Psi)$, defined on differentiable manifolds, are $C^r -$ *conjugate* if there exists a map $h : X \to Y$ of class C^r such that

$$h\,(\Phi\,(n,\, x)) = \Psi\,(n,\, h\,(x)), \quad \forall\, n \in \mathbb{Z}, \quad \forall\, x \in X.$$

Let us consider a discrete dynamical system given by a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$, and let $x_0 \in U$ be a fixed point of F. We say that x_0 is *hyperbolic* if none of the eigenvalues of $DF(x_0)$ has modulus equal to 1. In the other hand, if all the eigenvalues of $DF(x_0)$ have modulus equal to 1, we say that the fixed point x_0 is *parabolic.*

Hartman's theorem (see [9]) establishes that given a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 with an hyperbolic fixed point $x_0 \in U$ and such that $DF(x_0)$ is invertible, there exists a neighborhood of x_0 where the map F is conjugated to the linear map given by $DF(x_0)$. In this master thesis we study the behavior of the dynamics near parabolic fixed points of planar maps, where Hartman's theorem does not apply. In particular, we are interested in the existence of invariant curves associated to this class of fixed points.

In Section 2 we present the parameterization method for invariant manifolds of dynamical systems, which is a recently appeared tool to study several types of invariant manifolds based on looking for them as solutions of functional equations. An introductory exposition of this method can be found in [7].

In Section 3 we apply the parameterization method to planar maps with fixed points. We consider a map $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ with a fixed point $x_0 \in U$ and such that $DF(x_0)$ has an eigenvalue λ with $0 < |\lambda| < 1$, and we prove the existence of a stable invariant curve starting at x_0 and being tangent to the eigenvector associated to λ. We obtain this result for both the case when F is analytic and when F is of class C^r. The results of this section are based on the study of [3], where the authors use the implicit function theorem to prove them. In Section 3.2 we prove the same result for the case that F is analytic, but now using the Banach fixed point theorem.

In Section 4 we apply the parameterization method to study the existence of a stable invariant curve of an analytic map, $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, with a parabolic fixed point $x_0 \in U$ such that

$$DF(x_0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

namely, a nilpotent parabolic fixed point. The existence of such a curve is already proved in [5], but here it is the first time where it appears as an application of the parameterization method. In this section we also present a method to compute a polynomial approximation of this stable curve. The parameterization method is applied in order to show that once one has computed the approximation of the stable curve up to a high enough order, there exists indeed an invariant curve which is close to the approximated one.

We have also written a code in C language to compute the coefficients of a polynomial approximation of this stable curve based on the recurrences established in Section 4.1. From the results one can observe that the coefficients of the polynomial grow in a factorial way, suggesting that the formal series is of Gevrey type. We also compute an estimation of the Gevrey constant of this series.

2 The parameterization method for invariant manifolds

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a map with $F(0) = 0$, where U is an open set containing the origin. We are interested in invariant manifolds of dimension $m < n$ near the origin and in the dynamics of F restricted to these manifolds.

A natural way to try to find a manifold invariant under F modeled on a vector subspace $E \subset \mathbb{R}^n$ of dimension m is to look for an embedding $K : U_1 \subset E \to \mathbb{R}^n$ and a map $R : U_1 \to U_1$ in such a way that

$$F \circ K = K \circ R. \tag{2.1}$$

This equation ensures that the image of K is invariant under F. Any point $K(x)$ parameterized by $x \in U_1$ has image $F(K(x)) = K(R(x))$ and so it belongs to the image of K, being the right hand side parameterized by $R(x)$. In other words, K gives a conjugation between F restricted to the image of K and R, R is a representation of the dynamics of F restricted to E and $K \circ R$ is the dynamics of F restricted to the invariant manifold.

The fact that the manifold $K(U_1)$ passes through the origin is ensured by requiring $K(0) = 0$. Thus, as $F(0) = 0$, we obtain also from (2.1) that $R(0) = 0$. By differentiating (2.1) at the origin one obtains

$$DF(0)\, DK(0) = DK(0)\, DR(0). \tag{2.2}$$

If K is a one-dimensional invariant manifold, equation (2.2) says that $K'(0)$ is an eigenvector of $DF(0)$ with eigenvalue $R'(0)$.

Conversely, if we want the manifold K to be tangent to E at the origin, that is, if we want that for all $v \in E$, $DK(0)(v) \in E$, then E will be an eigenspace of $DF(0)$.

In this work we will focus on one-dimensional invariant manifolds of two-dimensional maps.

The fact that R is a representation, in some appropriate coordinates, of the dynamics of the map F restricted to the invariant manifold tells us that we need to consider it as a part of the objects to be determined. Observe, indeed, that in the invariance equation (2.1) one has to determine $n + m$ functions but only has n equations.

With this setting, one has the option to look for the simpler expression of K or the simpler expression of R. If we look for the simpler form of K, that will determine the dynamics of R. Conversely, we can look for the normal form of R, which is the simpler form of its expression, and obtain the corresponding form of K for the invariant manifold. This is the approach that we will follow.

An important observation is that if we consider the operator

$$\mathcal{T}(F, K, R) := F \circ K - K \circ R,$$

then equation (2.1) can be written as a functional equation,

$$\mathcal{T}(F, K, R) = 0, \tag{2.3}$$

where given F we look for some K and R that satisfy (2.3). Hence, a natural way to try to solve this problem is to study the properties of the operator $\mathcal{T}$ in suitable Banach spaces, using techniques such as the Banach fixed point theorem or the implicit function theorem. If F is sufficiently differentiable then $\mathcal{T}$ will inherit some differential properties that, in combination with considering K in a suitable space, will lead quickly to some results on existence of the invariant manifold and differentiability with respect to the parameters.

This is the method that we will follow. In order to prove the existence of invariant curves associated to some fixed points of planar maps, we will study some equations of the type of (2.1) with the restrictions given by the hypotheses related to the properties of the fixed points we are interested in.

3 Invariant manifolds associated to an eigenvalue of modulus less than 1

In this section we apply the parameterization method to the study of invariant one-dimensional manifolds associated to a fixed point with an eigenvalue of modulus less than 1 of a planar map.

Let $\Omega \subset \mathbb{R}^2$ be an open set and let $F : \Omega \to \mathbb{R}^2$ be a map of class C^1.

We shall consider two cases, the first one when F is analytic in a neighborhood of the origin, and the second when F is of class C^r in a neighborhood of the origin. In both cases we will prove a stable manifold theorem that establishes the existence of an invariant curve near the origin and its differentiable dependence with respect to the parameters. The results of this section are based on a study of [3].

Before introducing the results that shall be proved, we find it convenient to recall some basic theory on differential calculus in Banach spaces that will be used along the section.

Definition 3.1. Let E, F be two Banach spaces and let $U \subseteq E$ be an open set. An operator $T : U \to F$ is said to be *differentiable* at a point $f_0 \in E$ if there exists a linear operator $DT(f_0) \in \mathcal{L}(E, F)$ such that

$$\lim_{\|f-f_0\|_E \to 0} \frac{\|T(f) - T(f_0) - DT(f_0)(f - f_0)\|_F}{\|f - f_0\|_E} = 0,$$

where $\|\cdot\|_E$ and $\|\cdot\|_F$ denote the norms in E and F, respectively. In the case that such an operator exists, it is unique.

It can be proved that if T is differentiable at a point f_0, then $DT(f_0)$ is a bounded operator (see chapter 2 of [4]).

If T is differentiable in each point of an open set $U \subset E$, then the map

$$\begin{aligned} DT : U &\longrightarrow \mathcal{L}(E, F) \\ f &\longmapsto DT(f) \end{aligned}$$

is called the *derivative* of T. Moreover, if DT is a continuous map in U, we say that T is of class C^1.

The implicit function theorem in Banach spaces and the Banach fixed point theorem are the key results that we will use to apply the parameterization method. The interested reader can see the proofs of these theorems in [4].

Theorem 3.1 (Implicit function theorem)**.** *Let F, G, E be Banach spaces, $U \subseteq F \times G$ an open set and $T : U \to E$ an operator of class C^1. Let $(f_0, g_0) \in U$ such that $T(f_0, g_0) = 0$ and such that $D_2T(f_0, g_0)$ is a homeomorphism from G onto E. Then, there exists a neighborhood U_0 of f_0 in F and a unique map $u : U_0 \to G$, $u \in C^1(U_0)$, such that $u(f_0) = g_0$ and $T(f, u(f)) = 0$ for all $f \in U_0$. Also, the derivative of u is given by*

$$u'(f) = -(D_2T(f, u(f)))^{-1} \circ (D_1T(f, u(f))).$$

Theorem 3.2 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping, that is, such that for all $f, g \in X$, $d(T(f), T(g)) \leq d(f, g)$. Then T has a unique fixed point in X.*

3.1 Analytic stable curves

The following theorem establishes the existence and the regularity properties of a stable curve associated to an eigenvalue λ with $0 < |\lambda| < 1$, under the assumption that the map F is analytic in a neighborhood of the origin. In this case, one expects to find an analytic stable curve tangent to the eigenvector of $DF(0)$ associated to the eigenvalue λ, and where the dynamics on the invariant curve can be represented by the map $t \mapsto \lambda t$.

Theorem 3.3. *Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic map in a neighborhood U of the origin with F(0)=0. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A := DF(0)$, and let $v \in \mathbb{R}^2$, $v \neq 0$, satisfy $Av = \lambda v$. We denote by Spec(A) the set of eigenvalues of A. Assume:*

(a) A is invertible,
(b) $0 < |\lambda| < 1$,
(c) $\lambda^n \notin$ Spec(A) for every integer $n \geq 2$.

Then, there exists an analytic map $K : I \subset \mathbb{R} \to \mathbb{R}^2$, where I is an open neighborhood of 0 in $\mathbb{R}$, satisfying

$$F(K(t)) = K(\lambda t), \quad t \in I,$$

$K(0) = 0$ and $K'(0) = v$. That is, K is the parameterization of an analytic curve invariant under F and tangent to v at the origin and the dynamics on the invariant curve is conjugated to the linear map $t \mapsto \lambda t$ in I, and so $K(t)$ is a stable manifold.

In addition, if $\tilde{K}$ is another analytic solution of $F \circ K = K \circ \lambda$ in a neighborhood of the origin with $\tilde{K}(0) = 0$ and $\tilde{K}'(0) = \beta K'(0)$, then $K(t) = K(\beta t)$ for t small enough, that is, K and $\tilde{K}$ are just two different parameterizations of the same stable curve.

Before start proving the theorem, let us do some remarks on the statement.

Condition (c) on the statement of the theorem is called a *non-resonance condition.* Since by condition (a) one has $0 \notin \text{Spec}(A)$, then all the eigenvalues of A are outside a ball of radius ρ. By condition (b) there is an integer n_0 such that $|\lambda|^{n_0} < \rho$, and so if $n \geq n_0$ then condition (c) holds. That is, even if hypothesis (c) seems to require infinitely many conditions, it requires only n_0 of them.

In the case that $\|A\| < 1$ and λ is a simple eigenvalue and it is the closest one to the unit circle, then under iteration of A, the component along v of an orbit is the one that decays more slowly, as λ is the greatest eigenvalue and all the eigenspaces of A are stable, and hence it is also the one which controls the asymptotic behavior of the dynamics. The invariant manifold associated to this eigenvalue has an analogue behavior, and it is usually called a slow manifold, for clear reasons.

By considering F^{-1} and λ^{-1} in place of F and λ, hypothesis (b) can be changed to $|\lambda| > 1$. With this new setting the theorem establishes the existence of an unstable manifold near the origin under the iteration of F^{-1}.

Proof of Theorem 3.3. We shall consider F as an analytic function in a neighborhood U of 0 in $\mathbb{C}^2$. In order to clarify the notation we will denote z as the variable in $\mathbb{C}$ and w as the variable in $\mathbb{C}^2$. Therefore, we write F as $F(w) = \sum_{n\geq 0} F_n w^n$. Following the main idea of the parameterization method, one has to find a function $K : U \subset \mathbb{C} \to \mathbb{C}^2$ such that

$$F(K(z)) - K(\lambda z) = 0 \tag{3.1}$$

for $z \in \mathbb{D}$ where U is a neighborhood of 0 in $\mathbb{C}$. We write $F(w) = Aw + N(w)$ and so N is the nonlinear part of F. With this notation one has

$$K(\lambda z) = \sum_{n\geq 0} \lambda^n K_n z^n$$

with $K_n \in \mathbb{R}^2$, $\forall\, n$, and

$$\begin{aligned} F(K(z)) = A(K_1 z + K_2 z^2 + \dots) + N((K_1 z + K_2 z^2 + \dots)) = \\ = A(K_1 z + K_2 z^2 + \dots) + F_2(K_1 z + K_2 z^2 + \dots)^2 + \dots \end{aligned}$$

so, equating powers of $K(\lambda z)$ and $F(K(z))$ we have

$$AK_1 = \lambda K_1, \tag{3.2}$$

$$AK_n + R_n(K_1, \dots, K_{n-1}) = \lambda^n K_n, \quad n \geq 2, \tag{3.3}$$

where R_n is a polynomial expression obtained form the previous expansion.

Equation (3.2) does not determine K_1 completely, but only tells us that K_1 is an eigenvector of A with eigenvalue λ. Hence, for any $\delta > 0$, we can fix K_1 such that $|K_1| \leq \delta$.

With this setting, a first a approach to prove the theorem may be the following one. Once K_1 has been chosen, equation (3.3) allows us to determine unequivocally all the other K_n's, as

$$K_n = -(A - \lambda^n)^{-1} R_n(K_1, \dots, K_{n-1}), \quad n \geq 2, \tag{3.4}$$

since $(A - \lambda^n)^{-1}$ exists by the assumption (c). It can be shown, studying the recursion (3.4), that the power series $\sum_{n=1}^{\infty} K_n z^n$ is convergent in some neighborhood of 0. Hence, one obtains an expression for the invariant curve K around the origin, and that proves the existence of an analytic invariant curve satisfying (3.1). However, we will follow another route that will lead us to a complete proof for the theorem, using techniques of functional analysis that can also be adapted to other settings, and thus, that are efficient to prove a wide range of results.

We shall write $K(z) = K_1\, z + K^{>}(z)$, where K_1 has been already chosen. Then the left hand side of equation (3.1) can be written as

$$\begin{aligned} A(K_1\, z + K^{>}(z)) + N(K_1\, z + K^{>}(z)) - K_1 \lambda\, z - K^{>}(\lambda\, z) = \\ = \lambda K_1 z + AK^{>}(z) + N(K_1 z + K^{>}(z)) - K_1 \lambda z - K^{>}(\lambda z) \\ = AK^{>}(z) + N(K_1 z + K^{>}(z)) - K^{>}(\lambda z), \end{aligned}$$

and so equation (3.1) reads

$$AK^{>}(z) + N(K_1 z + K^{>}(z)) - K^{>}(\lambda z) = 0. \tag{3.5}$$

We look for $K^>$ belonging to the following Banach space,

$$H := \Big\{ K^> : \mathbb{D} \to \mathbb{C}^2 \mid K^>(z) = \sum_{n=2}^{\infty} K_n \, z^n, \; \|K^>\| := \sum_{n=2}^{\infty} |K_n| < \infty \Big\},$$

where $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$.

One should note that looking for a solution of equation (3.5) in H does not imply a loss of generality. In fact, we are looking for analytic invariant curves in a small neighborhood of the origin. It can be proved that although the space H is smaller that the space of analytic functions in the unit disk, it is sufficient to consider it for the curves we are looking for. This will be carefully discussed in Section 3.4. The fact that we can look for invariant curves in Banach spaces adapted to the context gives us ease to obtain the desired results.

Now we can reformulate equation (3.5) as an operator equation as

$$\mathcal{T}(K_1, \, K^>) = 0, \tag{3.6}$$

where $\mathcal{T}$ is the nonlinear operator $\mathcal{T} : V \times B \to H$ defined by

$$\mathcal{T}(K_1, \, K^>)(z) = AK^>(z) - K^>(\lambda z) + N(K_1 z + K^>(z)),$$

where V is a ball centered at $0 \in \mathbb{C}^2$ of radius $\delta > 0$ and B is a ball centered at $0 \in H$ with a radius sufficiently small in order that $K_1 z + K^>(z)$ is contained in the domain of N.

We shall define also the linear operator $\mathcal{S} : H \to H$ as

$$(\mathcal{S}K^>)(z) = AK^>(z) - K^>(\lambda z).$$

Thus, one can write the operator $\mathcal{T}$ as

$$\mathcal{T}(K_1, K^>)(z) = (\mathcal{S}K^>)(z) + N(K_1 z + K^>(z)).$$

The fact that $\mathcal{S}$ is linear is clear, as A and the evaluation at λz are so.

With this setting, it will be sufficient to study the existence of a function $K^> \in B$ that satisfies equation (3.6).

This functional equation may be studied in several ways. Here we will use the implicit function theorem in Banach spaces. To study the existence of a solution of (3.6) we shall check that $\mathcal{T}$ verifies the hypotheses of the theorem.

First of all, $\mathcal{T}$ is of class C^1 because it is a composition of several operators of class C^1. Also, it holds that $\mathcal{T}(0,0)(z) = S(0)(z) + N(0) = 0$, and $D_2(0,0) = \mathcal{S}$ (see [8]).

We need to prove then that $\mathcal{S}$, which we already know that is linear and bounded, is also boundedly invertible.

Lemma 3.1. *The operator $\mathcal{S} : H \to H$ defined as*

$$(\mathcal{S}f)(z) = Af(z) - f(\lambda \, z)$$

is boundedly invertible in H.

Proof. Let us see first that $\mathcal{S}: H \to H$ is invertible. Given $\eta \in H$ with $\eta = \sum_{n=2}^{\infty} \eta_n z^n$ we look for $f \in H$ such that $\mathcal{S}(f) = \eta$, that is,

$$\mathcal{S}\Big(\sum_{n=2}^{\infty} f_n z^n\Big) = A\Big(\sum_{n=2}^{\infty} f_n z^n\Big) - \sum_{n=2}^{\infty} f_n (\lambda z)^n = \sum_{n=2}^{\infty} \eta_n z^n.$$

Equating coefficients we are lead to

$$A f_n - f_n \lambda^n = \eta_n, \quad n \geq 2,$$

and hence $f_n = (A - \lambda^n)^{-1} \eta_n$, so the coefficients f_n can be determined if hypotheses (a) and (c) of Theorem 3.3 hold. In this case one has

$$f(z) = \sum_{n=2}^{\infty} (A - \lambda^n)^{-1} \eta_n \, z^n = \mathcal{S}^{-1} \eta(z), \quad z \in \mathbb{D}. \tag{3.7}$$

By hypotheses (a) and (c) we have that $|(A - \lambda^n)^{-1}| \leq C$, for all n, for some constant C, and hence

$$\sum_{n=2}^{\infty} |(A - \lambda^n)^{-1} \eta_n \, z^n| \leq C \sum_{n=2}^{\infty} |\eta_n \, z^n| < \infty, \quad z \in \mathbb{D},$$

which proves that the series expansion (3.7) of f is convergent. Also, by the same property one has that

$$\|\mathcal{S}^{-1}(\eta)\| = \| \sum_{n=2}^{\infty} (A - \lambda^n)^{-1} \eta_n \, z^n \| = \sum_{n=2}^{\infty} |(A - \lambda^n)^{-1} \eta_n| \leq C \sum_{n=2}^{\infty} |\eta_n| = C \, \|\eta\|,$$

that is, $\mathcal{S}^{-1}$ is bounded. □

The theorem follows then applying the implicit function theorem. Indeed, there exists a neighborhood V_0 of 0 in V and a unique function $u : V_0 \to B$, which is of class C^1, such that $\mathcal{T}((K_1), u(K_1)) = 0$ for all $K_1 \in V_0$. That is, there exists a unique solution of equation (3.6), $K^{>} \in B$, which provides an analytic invariant curve for F, and also, $K^{>} = u(K_1)$ is a function of class C^1, what means that the dependence of the invariant curve on the vector K_1 is differentiable.

If $K(z)$ satisfies (3.1) and $\sigma \in \mathbb{C}$, then $\tilde{K}(z) = K(\sigma z)$ also satisfies (3.1). In this case, note that $\tilde{K}_1 = \sigma K_1$. This explains the lack of uniqueness in the modulus of K_1 as a solution of equation (3.2). That is, if we choose $\tilde{K}_1$ differing only by a multiple from K_1, by the uniqueness given by the implicit function theorem we are only choosing another parameterization of the same invariant curve, related to the first one by a linear change of scale. □

3.2 A fixed point equation for the analytic stable curve

Our aim now is to give a new proof of Theorem 3.3 using the Banach fixed point theorem instead of the implicit function theorem. This technique will present us some tools to study functional equations for invariant manifolds. As the statement of the Banach fixed point theorem is less restrictive that the one of the implicit function theorem, it will be less painful to arrive to some results; for example, we will not need the operator $\mathcal{T}$ to be differentiable at any point. Nevertheless, one will need to use more technical machinery to obtain the conditions in the hypotheses of the theorem, and so, to obtain the desired results.

Under the hypotheses of Theorem 3.3 we return to equation (3.6), where we had

$$\mathcal{T}(K_1, K^>) = 0,$$

and we were looking for a function $K^> \in H$ satisfying this equation, that is, satisfying

$$(\mathcal{S}K^>)(z) + N(K_1 z + K^>(z)) = 0, \;\; \forall z \in \mathbb{D}. \tag{3.8}$$

Here we will consider another function space that will be suitable for our purpose. Let us consider

$$\mathcal{A} = \Big\{ f : \overline{\mathbb{D}} \to \mathbb{C}^2 \mid f \in \mathrm{Hol}(\mathbb{D}),\; f \in C^0(\overline{\mathbb{D}}),\; f(0) = f'(0) = 0 \Big\}$$

with the topology of $C^0(\overline{\mathbb{D}})$, that is, $\|f\|_\infty = \sup_{z\in\overline{\mathbb{D}}} |f(z)|$, which is a Banach space.

It is clear that one can consider the operator $\mathcal{S}$ defined as $\mathcal{S}f(z) \mapsto Af(z) - f(\lambda z)$ acting on $\mathcal{A}$, and it is also clear that $\mathcal{S}$ is a linear map from $\mathcal{A}$ to $\mathcal{A}$.

In this case, analogously as in the section before, one has the following result.

Lemma 3.2. *The operator $\mathcal{S} : \mathcal{A} \to \mathcal{A}$ is boundedly invertible in $\mathcal{A}$.*

Proof. Let us consider the equation

$$\mathcal{S}f = \eta, \tag{3.9}$$

where η is any element of $\mathcal{A}$. We need to find a solution, $f \in \mathcal{A}$, for this equation.

We choose L a positive integer such that $|\lambda|^{L+1}\|A^{-1}\| < 1$.

Then we can write

$$\eta(z) = \sum_{n=2}^{L} \eta_n z^n + \tilde{\eta}(z),$$

where $D^i\tilde{\eta}(0) = 0$ for $i = 0, \ldots, L$. We look for a solution of equation (3.9) expressed in a similar way, that is,

$$f(z) = \sum_{n=2}^{L} f_n z^n + \tilde{f}(z)$$

with $D^i\tilde{f}(0) = 0$ for $i = 0, \ldots, L$.

By the linearity of $\mathcal{S}$, one can study separately the equations

$$\mathcal{S}\Big(\sum_{n=2}^{L} f_n z^n\Big) = \sum_{n=2}^{L}(Af_n z^n - f_n \lambda^n z^n) = \sum_{n=2}^{L} \eta_n z^n \tag{3.10}$$

and

$$\mathcal{S}\tilde{f}(z) = A\tilde{f}(z) - \tilde{f}(\lambda z) = \tilde{\eta}(z). \tag{3.11}$$

By the non-resonance assumption (c) of Theorem 3.3, equation (3.10) can be solved equating powers on both sides and so one has

$$f_n = (A - \lambda)^{-1} \, \eta_n, \quad n = 2, \ldots, L.$$

Also, as $|A - \lambda^n|$ is bounded for all n, we have

$$\| \sum_{n=2}^{L} f_n z^n \|_\infty = \| \sum_{n=2}^{L} (A - \lambda^n)^{-1} \eta_n z^n \|_\infty \leq C \, \| \sum_{n=2}^{L} \eta_n z^n \|_\infty \;\leq\; C \, (\, |\eta_2| + \cdots + |\eta_L| \,).$$

Then, applying the Cauchy integral formula on every component of η_n with $\gamma = \partial \mathbb{D}$ one has

$$\begin{aligned} |\eta_n| \; &= \left| \frac{\eta^{n)}(0)}{n!} \right| \; = \left| \frac{1}{2\pi i} \int_\gamma \frac{\eta(z)}{z^{n+1}} \, dz \right| \; = \; \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\eta(e^{i\theta})}{(e^{i\theta})^{n+1}} \, ie^{i\theta} \, d\theta \right| \; \leq \; \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\eta(e^{i\theta})}{(e^{i\theta})^{n+1}} \, ie^{i\theta} \right| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |\eta(re^{i\theta})| \;\leq\; \|\eta\|_\infty, \quad n = 2, \ldots, L, \end{aligned}$$

and so one gets

$$\| \sum_{n=2}^{L} f_n z^n \|_\infty \leq C' \, \|\eta\|_\infty, \tag{3.12}$$

which is a first boundedness condition.

From equation (3.11) one can write recursively

$$\begin{aligned} \tilde{f}(z) &= A^{-1} \, (\tilde{f}(\lambda z) + \tilde{\eta}(z)) = A^{-1} \, (A^{-1}(\tilde{f}(\lambda^2 z) + \tilde{\eta}(\lambda z)) + \tilde{\eta}(z)) = \cdots \\ &= A^{-M} \, \tilde{f}(\lambda^M z) + \sum_{j=0}^{M-1} A^{-j-1} \tilde{\eta}(\lambda^j z), \qquad \forall \, M \in \mathbb{N}, \end{aligned}$$

and so we can consider

$$\tilde{f}(z) = \lim_{M \to \infty} [\, A^{-M} \, \tilde{f}(\lambda^M z) + \sum_{j=0}^{M-1} A^{-j-1} \tilde{\eta}(\lambda^j z) \,].$$

We claim that this limit exists and that

$$\tilde{f}(z) = \sum_{n=0}^{\infty} A^{-n-1} \, \tilde{\eta}(\lambda^n z) \tag{3.13}$$

is the solution to (3.11).

We need to prove that the series (3.13) converges uniformly in $\overline{\mathbb{D}}$.

For every component of η one has, again by the Cauchy integral formula,

$$|\tilde{\eta}(z)| = \frac{1}{(L+1)!} \left| \eta^{L+1)}(\zeta) \right| |z^{L+1}| \;\leq\; \|\eta\|_\infty \, |z|^{L+1}$$

and hence for every element of the series (3.13) one has

$$\|A^{-n-1} \, \tilde{\eta}(\lambda^n z)\| \;\leq\; \|A^{-1}\|^{n+1} |\lambda|^{n(L+1)} |z|^{L+1} \, \|\eta\|_\infty \;=\; \|A^{-1}\| \, (|\lambda|^{L+1} \|A^{-1}\|)^n \, \|\eta\|_\infty,$$

and then, as we assumed that $|\lambda|^{L+1}\|A^{-1}\| < 1$, the series converges uniformly on $\overline{\mathbb{D}}$ by the Weierstrass M-test. Therefore we have that $\tilde{f} \in C^0(\overline{\mathbb{D}})$, and f is analytic in $\mathbb{D}$ as it is the uniform limit of analytic functions on compact sets, and so $f \in \mathcal{A}$.

Also, by a similar argument one has

$$\lim_{M\to\infty} A^{-M}\,\tilde{f}(\lambda^M z) = (0,0),$$

and so we obtain the claimed solution in (3.13).

On the other hand one has

$$\|\tilde{f}\|_\infty = \|\sum_{n=0}^{\infty} A^{-n-1}\,\tilde{\eta}\,(\lambda^n z)\|_\infty \leq \|\eta\|_\infty \sum_{n=0}^{\infty} \|A^{-1}\|\,(|\lambda|^{L+1}\|A^{-1}\|)^n \leq C\|\eta\|_\infty, \tag{3.14}$$

so finally, by the estimates (3.12) and (3.14) we get

$$\|\mathcal{S}^{-1}(\eta)\|_\infty = \|f\|_\infty = \|\sum_{n=2}^{L} f_n x^n + \tilde{f}\,\|_\infty \leq \|\sum_{n=2}^{L} f_n x^n\|_\infty + \|\tilde{f}\,\|_\infty \leq C\|\eta\|_\infty,$$

that is, $\mathcal{S}^{-1}$ is bounded. □

As a result of the previous lemma, one has that equation (3.8) is equivalent to the fixed point equation

$$K^{>}(z) = -\mathcal{S}^{-1}N(K_1\,z + K^{>}(z)). \tag{3.15}$$

We recall that N is the nonlinear part of F, and so N maps a neighborhood of $0 \in \mathbb{C}^2$ to some subset of $\mathbb{C}^2$. As N is analytic and $N(0) = DN(0) = 0$, it is clear that N maps any element $f \in \mathcal{A}$ to $\mathcal{A}$ if we consider it as a composition operator, whenever the range of f is contained in the domain of N. Thus, we can refer to N as a functional operator and in this case we will use the notation $\mathcal{N}$ to refer to it.

Let us consider the closed ball of radius $r \leq 1$ in $\mathcal{A}$, that is,

$$B_r = \{f \in \mathcal{A}\,:\, \|f\|_\infty \leq r\},$$

which is a complete metric space.

With this setting we define $\mathcal{N}$ in the following way,

$$\begin{aligned} \mathcal{N} : B_r \subset \mathcal{A} &\longrightarrow \mathcal{A} \\ f(z) &\longmapsto N(K_1 z + f(z)). \end{aligned}$$

As we consider $\mathcal{N}$ acting on functions defined in the unit disk that represent the nonlinear part of the potential invariant curves for F, we must also fix to the unit disk the domain of definition of the linear part of those invariant curves, that is, we ought to consider

$$\begin{aligned} N \circ (K_1\,\cdot\,+K^{>}) : \overline{\mathbb{D}} &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto N(K_1 z + K^{>}(z)). \end{aligned}$$

Note that the domain of definition of N is in $\mathbb{C}^2$, but the variable z is always in $\mathbb{C}$. As we are studying invariant manifolds of planar maps, the vector K_1 is in the two dimensional space. As we did before, we will denote as w the variable on $\mathbb{C}^2$ in order to clarify the notation.

Let us consider the operator $\Theta: B_r \to \mathcal{A}$ defined as $\Theta := \mathcal{S}^{-1} \circ \mathcal{N}$.

With this setting equation (3.15) can be written as a fixed point equation as

$$K^{>} = -\Theta(K^{>}), \quad K^{>} \in B_r.$$

To study the existence of a solution $K^{>}(z)$ for such an equation we shall prove that Θ is a contraction mapping in a suitable metric space in order to apply the Banach fixed point theorem.

To do so, we will consider the change of variables given by $T(z) = \delta z$, for some $\delta > 0$. Observe that the invariance equation

$$F \circ K - K \circ \lambda = 0$$

is equivalent to

$$(T^{-1} \circ F \circ T) \circ (T^{-1} \circ K \circ T) - (T^{-1} \circ K \circ T) \circ (T^{-1} \circ \lambda \circ T) = 0.$$

Applying such a change of variables it is clear that the operator $\mathcal{S}$ is not affected, for what its norm remains the one found in Lemma 3.2. Also, we have now that

$$(T^{-1} \circ F \circ T) = A + (T^{-1} \circ N \circ T),$$

and hence the operator $\Phi = \mathcal{S}^{-1} \circ \mathcal{N}$ is affected only on what involves $\mathcal{N}$. With the presented scaling of variables, the size of the domain of N changes and we can deal with the new function obtained, $\tilde{N} = T^{-1} \circ N \circ T$. Indeed, if we take δ sufficiently small and we consider

$$\tilde{N}(w) = \frac{1}{\delta} N(\delta w), \tag{3.16}$$

then, as N is analytic in a neighborhood of the origin, $\tilde{N}$ will be analytic in a larger neighborhood, say $B(0, 5/2) \subset \mathbb{C}^2$ and so it will also be analytic in $\overline{B(0, 1+r)} \subset \mathbb{C}^2$.

Lemma 3.3. *Let $N : B(0, a) \subset \mathbb{C}^2 \to \mathbb{C}^2$ be analytic and such that $N(0,0) = (0,0)$, $DN(0,0) = 0 \in \mathbb{C}^4$. Then, for all $\varepsilon > 0$ there exists a transformation as (3.16) such that*

$$\sup_{|w| \le 1+r} \|\nabla \tilde{N}_1(w)\| + \sup_{|w| \le 1+r} \|\nabla \tilde{N}_2(w)\| < \varepsilon,$$

where $\tilde{N}_1$ and $\tilde{N}_2$ are the two components of $\tilde{N}$.

Proof. Let us denote by $w = (w_1, w_2)$ the variable in $\mathbb{C}^2$.

We shall take $\delta < \varepsilon / [(M_1 + M_2)(1 + r)]$ where M_1, M_2 will be determined later, and also $\delta < \frac{2}{5} a$, that is, sufficiently small in order that $\tilde{N}(w) = \frac{1}{\delta} N(\delta w)$ is analytic in $B(0, 5/2) \subset \mathbb{C}^2$.

For all $w \in \overline{B(0,1+r)}$, one has, applying the mean value theorem in several variables,

$$\begin{aligned}
\|\nabla \tilde{N}_1(w)\| &= \left\| \left(\frac{\partial \tilde{N}_1}{\partial w_1}(w_1,w_2), \frac{\partial \tilde{N}_1}{\partial w_2}(w_1,w_2) \right) \right\| = \left\| \left(\frac{\partial N_1}{\partial w_1}(\delta w_1,\delta w_2), \frac{\partial N_1}{\partial w_2}(\delta \omega_1,\delta w_2) \right) \right\| \\
&\le \left| \frac{\partial N_1}{\partial w_1}(\delta w_1,\delta w_2) \right| + \left| \frac{\partial N_1}{\partial w_2}(\delta w_1,\delta w_2) \right| \\
&= \left| \frac{\partial N_1}{\partial w_1}(\delta w_1,\delta w_2) - \frac{\partial N_1}{\partial w_1}(0,0) \right| + \left| \frac{\partial N_1}{\partial w_2}(\delta w_1,\delta w_2) - \frac{\partial N_1}{\partial w_2}(0,0) \right| \\
&\le \left\| \nabla \frac{\partial N_1}{\partial w_1}(\zeta) \right\| \left\| (\delta w_1,\delta w_2) \right\| + \left\| \nabla \frac{\partial N_1}{\partial w_2}(\eta) \right\| \left\| (\delta w_1,\delta w_2) \right\| \\
&\le \delta \|w\| \left[\sup_{|\zeta|\le\delta(1+r)} \left\| \nabla \frac{\partial N_1}{\partial w_1}(\zeta) \right\| + \sup_{|\eta|\le\delta(1+r)} \left\| \nabla \frac{\partial N_1}{\partial w_2}(\eta) \right\| \right] = \delta \|w\| M_1.
\end{aligned}$$

In an analogous way, for all $w \in \overline{B(0,1+r)}$ one can obtain the bound

$$\|\nabla \tilde{N}_2(w)\| \le \delta \|w\| \left[\sup_{|\zeta|\le\delta(1+r)} \left\| \nabla \frac{\partial N_2}{\partial w_1}(\zeta) \right\| + \sup_{|\eta|\le\delta(1+r)} \left\| \nabla \frac{\partial N_2}{\partial w_2}(\eta) \right\| \right] = \delta \|w\| M_2.$$

It is clear that

$$M_i = \sup_{|\zeta|\le\delta(1+r)} \left\| \nabla \frac{\partial N_i}{\partial w_1}(\zeta) \right\| + \sup_{|\eta|\le\delta(1+r)} \left\| \nabla \frac{\partial N_i}{\partial w_2}(\eta) \right\|, \quad i = 1,2,$$

is bounded as N is analytic in a ball of radius $a > \delta\,(5/2) > \delta\,(1+r)$.

Hence, considering the estimations obtained one gets

$$\begin{aligned}
&\sup_{|w|\le 1+r} \|\nabla \tilde{N}_1(w)\| + \sup_{|w|\le 1+r} \|\nabla \tilde{N}_1(w)\| \le \\
&\le \sup_{|w|\le 1+r} \delta \|w\| (M_1 + M_2) = \delta\,(1+r)\,(M_1+M_2) < \varepsilon.
\end{aligned}$$

□

One should note that we can always consider a proper dilatation $\tilde{N}$ of N for our computations, as if the study of $\tilde{N}$ leads us to the existence of some invariant manifolds, they will also exist and have the same regularity properties for the case of N.

With this considerations we can now give the following result.

Theorem 3.4. *The operator $\Theta : B_r \to \mathcal{A}$ is a contraction mapping.*

Proof. We need to see that for all $f_1, f_2 \in B_r$,

$$\|\,\Theta\,(f_1) - \Theta\,(f_2)\,\|_\infty \le Q\,\|\,f_1 - f_2\,\|_\infty,$$

with $Q < 1$.

As $\mathcal{S}^{-1}$ is linear and bounded in $\mathcal{A}$ with $\|\mathcal{S}^{-1}\|_{\mathcal{L}(\mathcal{A},\mathcal{A})} = C$, one has, applying the mean value theorem in several variables,

$$\begin{aligned}
\|\,\Theta\,(f_1) - \Theta\,(f_2)\,\|_\infty &= \|\,\mathcal{S}^{-1}(\mathcal{N}(f_1)) - \mathcal{S}^{-1}(\mathcal{N}(f_1))\,\|_\infty \\
&= \|\,\mathcal{S}^{-1}\,[(\mathcal{N}(f_1)) - (\mathcal{N}(f_1))]\,\|_\infty \\
&\le C\|\,\mathcal{N}(f_1) - \mathcal{N}(f_2)\,\|_\infty \\
&= C \sup_{z\in\overline{\mathbb{D}}} \|\,N(K_1 z + f_1(z)) - N(K_1 z + f_2(z))\,\| \\
&\le C\,[\sup_{z\in\overline{\mathbb{D}}} |N_1(K_1 z + f_1(z)) - N_1(K_1 z + f_2(z))| \\
&\quad + \sup_{z\in\overline{\mathbb{D}}} |N_2(K_1 z + f_1(z)) - N_2(K_1 z + f_2(z))|\,] \\
&\le C\,[\sup_{z\in\overline{D}} \|\nabla N_1(\zeta)\|\,\|f_1(z) - f_2(z)\| \,+\, \sup_{z\in\overline{D}} \|\nabla N_2(\eta)\|\,\|f_1(z) - f_2(z)\|\,],
\end{aligned} \tag{3.17}$$

for some $\zeta,\,\eta \in \{\,t\,(K_1 z + f_1(z)) + (1-t)(K_1 z + f_2(z)) \mid t\in[0,1]\,\}$.

Thus, as N is acting on $K_1 z + f(z)$ with $z\in\mathbb{D}$ and $\|f\|_\infty \le r \le 1$, we can bound the domain of N, ∇N_1 and ∇N_2,

$$|\,K_1 z + f(z)\,| \le |\,K_1\,| + |\,f(z)\,| \le 1 + r$$

as we have previously chosen K_1 sufficiently small.

Hence, returning to (3.17) and using Lemma 3.3 one gets

$$\begin{aligned}
&C\,[\sup_{z\in\overline{D}} \|\nabla N_1(\zeta)\|\,\|f_1(z) - f_2(z)\| \,+\, \sup_{z\in\overline{D}} \|\nabla N_2(\eta)\|\,\|f_1(z) - f_2(z)\|\,] \le \\
&\le C\,[\sup_{z\in\overline{D}}\ \sup_{|\zeta|\le 1+r} \|\nabla N_1(\zeta)\|\,\|f_1(z) - f_2(z)\| + \sup_{z\in\overline{D}}\ \sup_{|\eta|\le 1+r} \|\nabla N_2(\eta)\|\,\|f_1(z) - f_2(z)\|\,] \\
&= C\,\|f_1 - f_2\|_\infty\,[\sup_{|\zeta|\le 1+r} \|\nabla N_1(\zeta)\| + \sup_{|\eta|\le 1+r} \|\nabla N_2(\eta)\|\,] \\
&\le C\,\varepsilon\,\|f_1 - f_2\|_\infty,
\end{aligned}$$

and therefore, as ε is as small as we need, we have

$$\|\,\Theta\,(f_1) - \Theta\,(f_2)\,\|_\infty \;\le\; Q\,\|\,f_1 - f_2\,\|_\infty$$

with $Q = C\,\varepsilon < 1$, and so Θ is a contraction. □

Hence, as $\Theta : B_r \to \mathcal{A}$ is a contraction mapping, we have that in particular, for all $f\in B_r$,

$$\begin{aligned}
\|\,\Theta\,(f)\,\|_\infty &\le \|\,\Theta\,(f) - \Theta\,(0)\,\|_\infty + \|\,\Theta\,(0)\,\|_\infty \\
&\le Q\,\|\,f\,\|_\infty + C\,\|\mathcal{N}(0)\|_\infty \\
&= C\,\varepsilon\,\|\,f\,\|_\infty + \sup_{z\in\mathbb{D}} C\,\|N(K_1 z)\|.
\end{aligned}$$

As we chose $\|K_1\|$ sufficiently small and we have that $N(0,0) = 0$, then the quantity $\sup_{z\in\mathbb{D}} C\,\|N(K_1 z)\|$ is as small as wished. Also, as ε is as small as needed, one has that

$$\|\,\Theta\,(f)\,\|_\infty \le C\,\varepsilon\,\|\,f\,\|_\infty + \sup_{z\in\mathbb{D}} C\,\|N(K_1 z)\| \le r,$$

that is, Θ maps any element of B_r to B_r.

Therefore we are under the assumptions of the Banach fixed point theorem, as taking B_r as our metric space, one has that $\Theta : B_r \to B_r$ is a contraction mapping, and so it has a unique fixed point in B_r. In this way, we get that equation (3.15) has a unique solution $K^>(z)$, which gives the nonlinear part of an analytic invariant curve for F, and so Theorem 3.3 is proved.

3.3 C^r stable curves

The following result establishes the existence and the regularity properties of a stable curve associated to an eigenvalue of modulus less than 1 assuming that the map F is of class C^{r+1}. The path that we will follow to prove it will be similar as the one in Theorem 3.3, but in this case, we will need some technical results when dealing with equations that are set in spaces of finitely differentiable functions.

Theorem 3.5. *Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^{r+1} map in a neighborhood U of the origin with $F(0) = 0$. Let λ be an eigenvalue of $A := DF(0)$ and let $v \in \mathbb{R}^2 \backslash \{0\}$ satisfy $Av = \lambda v$. Assume:*
(a) A is invertible,
(b) $0 < |\lambda| < 1$,

Denote by $L \geq 1$ an integer large enough such that $|\lambda|^{L+1}\|A^{-1}\| < 1$,

(c) $\lambda^n \notin Spec(A)$ for $n = 2, \ldots, L$,
(d) $L + 1 \leq r$.

Then there exists a C^r map, $K : I \subset \mathbb{R} \to \mathbb{R}^2$, where I is a neighborhood of 0, such that

$$F(K(t)) = K(\lambda t), \quad t \in I, \tag{3.18}$$

$K(0) = 0$ and $K'(0) = v$. That is, K is the parameterization of a C^r curve invariant under F and tangent to v at the origin, and the dynamics restricted to the curve is conjugated to the linear map $t \mapsto \lambda t$ in I.

Moreover, if $\tilde{K}$ is another C^r solution of (3.18) in a neighborhood of the origin with $\tilde{K}(0) = 0$ and $\tilde{K}'(0) = \beta K'(0)$ for some $\beta \in \mathbb{R}$, then $\tilde{K}(t) = K(\beta t)$ for t small enough, that is, K and $\tilde{K}$ are two parameterizations of the same invariant curve.

Proof. The setting of this theorem is similar to the one of Theorem 3.3. In that case, the transformations needed to go from (3.1) to (3.6) were purely algebraic manipulations considering $F(x) = Ax + N(x)$ and $K(t) = K_1 t + K^>(t)$. Thus, with the same notation, we can start the current proof from the operator equation

$$\mathcal{T}(K_1, K^>) = 0, \tag{3.19}$$

with

$$\mathcal{T}(K_1, K^>) = (SK^>)(t) + N(K_1 t + K^>(t)),$$

where

$$(Sf)(t) = Af(t) - f(\lambda t).$$

Nevertheless, here we shall consider $\mathcal{T}$ acting on a different space, since we have $F \in C^{r+1}(U)$ and we look for invariant curves of class C^r. Also, note that in this case we do not consider F and K as functions of complex variables, as they are no more analytic. We take the Banach space

$$\Gamma = \{K^> : [-1,\, 1] \to \mathbb{R}^2 \,|\, K^> \in C^r([-1,\, 1])\,,\; K^>(0) = (K^>)'(0) = 0\}$$

endowed with the C^r topology, that is,

$$\|K^>\|_{C^r} \;=\; \max_{i \le r} \; \sup_{|t| \le 1} |D^i K^>(t)|.$$

Thus, we consider $\mathcal{S} : \Gamma \to \Gamma$ and $\mathcal{T} : V \times B \to \Gamma$, where $V \subset \mathbb{R}^2$ is a ball centered at 0 of radius $\delta > 0$ and $B \subset \Gamma$ is a ball centered at 0 with radius sufficiently small in order that $K_1 t + K^>(t)$ is contained in the domain of N.

As in Theorem 3.3, one has that $\mathcal{T}$ is of class C^1 in a neighborhood V of $(0,0)$. Also, $\mathcal{T}(0,0) = 0$ and $D_2\mathcal{T}(0,0) = \mathcal{S}$, and so $\mathcal{S}$ is a bounded linear operator. From this, to apply the implicit function theorem to equation (3.19), it only remains to establish the invertibility of $\mathcal{S}$. The following lemma gives the desired result, and the basic tool for proving it is the fact that we can invert $\mathcal{S}$ in spaces of functions that vanish at the origin to high enough order, and that the lower order terms can be obtained equating powers due to the non-resonance assumptions, (c).

Lemma 3.4. *The operator $\mathcal{S} : \Gamma \to \Gamma$ defined as*

$$(\mathcal{S}f)(t) = Af(t) - f(\lambda t)$$

is boundedly invertible in Γ.

Proof. The proof is similar to the one of Lemma 3.2.

Let us consider the equation

$$\mathcal{S}f = \eta \tag{3.20}$$

where η is any element of Γ. We need to find a solution, $f \in \Gamma$, for this equation. We can write

$$\eta(t) = \sum_{n=2}^{L} \eta_n t^n + \tilde{\eta}(t),$$

where $D^i\tilde{\eta}\,(0) = 0$ for $i = 0, \ldots, L$, as we took $L + 1 \le r$. We look for a solution of equation (3.20) expressed in a similar way, that is,

$$f(t) = \sum_{n=2}^{L} f_n t^n + \tilde{f}(t)$$

with $D^i\tilde{f}\,(0) = 0$ for $i = 0, \ldots, L$. By the linearity of $\mathcal{S}$, we can study separately the equations

$$\mathcal{S}\Big(\sum_{n=2}^{L} f_n t^n\Big) = \sum_{n=2}^{L}(Af_n t^n - f_n\lambda^n t^n) = \sum_{n=2}^{L} \eta_n t^n \tag{3.21}$$

and

$$\mathcal{S}\tilde{f}(t) = A\tilde{f}(t) - \tilde{f}(\lambda t) = \tilde{\eta}(t). \tag{3.22}$$

By the non-resonance assumption (c), equation (3.21) can be solved equating powers on both sides and so one has

$$f_n = (A - \lambda)^{-1}\,\eta_n, \quad n = 2, \ldots, L.$$

Also, as $|A - \lambda^n|$ is bounded for all $n = 2, \ldots L$, we have that

$$\|\sum_{n=2}^{L} f_n t^n\|_{C^r} = \|\sum_{n=2}^{L}(A-\lambda^n)^{-1}\eta_n t^n\|_{C^r} \le C\sum_{n=2}^{L}\|\eta_n t^n\|_{C^r} \le C'\,\|\eta\|_{C^r} \tag{3.23}$$

which is a first boundedness condition. The last inequality is clear since, by Taylor's theorem,

$$\begin{aligned}\|\sum_{n=2}^{L}\eta_n t^n\|_{C^r} \;\le\; \sum_{i=0}^{r}\|D^i\sum_{n=2}^{L}\eta_n t^n\|_\infty \;\le\; (r+1)\,C\,\|\sum_{n=2}^{L}\eta_n t^n\|_\infty \\ \le\; C'\sum_{n=2}^{L}|\eta_n||t^n| = C'\frac{|t^n|}{n!}\,|\eta^{n)}(0)| \;\le\; C'\frac{|t^n|}{n!}\|\eta\|_{C^r} \;\le\; C''\|\eta\|_{C^r}.\end{aligned}$$

Now we go back to equation (3.22) and we claim that a solution for this equation is given by

$$\tilde{f}(t) = \sum_{n=0}^{\infty} A^{-n-1}\,\tilde{\eta}(\lambda^n t). \tag{3.24}$$

First, we need to prove that the series (3.24) converges uniformly in $[-1,\,1]$.

By Taylor's theorem one has, for every component of $\tilde{\eta}(t)$,

$$|\tilde{\eta}(t)| \;\le\; \sup_{|\zeta|\le 1}\frac{1}{(L+1)!}\left|\eta^{L+1)}(\zeta)\,t^{L+1}\right| \;\le\; \frac{1}{(L+1)!}\|\eta\|_{C^r}\,|t|^{L+1} = C\,\|\eta\|_{C^r}\,|t|^{L+1},$$

and hence for every element of the series (3.24) we have

$$|A^{-n-1}\,\tilde{\eta}(\lambda^n t)| \;\le\; C\|A^{-1}\|^{n+1}|\lambda|^{n(L+1)}|t|^{L+1}\,\|\eta\|_{C^r} \;\le\; C\,(|\lambda|^{L+1}\|A^{-1}\|)^n\,\|\eta\|_{C^r},$$

and then, as we supposed that $|\lambda|^{L+1}\|A^{-1}\| < 1$, the series (3.24) converges uniformly on $[-1,\,1]$ by the Weierstrass M-test. Therefore we have that $\tilde{f} \in C^0([-1,\,1])$.

Also, due to the uniform convergence, it can be checked that (3.24) is indeed a solution of equation (3.22), replacing it into the equation and rearranging the terms, as seen in Lemma 3.2.

Next, we need to see that the series obtained taking derivatives up to order r in (3.24) is also uniformly convergent in $[-1,\,1]$, and so we will obtain that $\tilde{f}$ is of class C^r and it is indeed an element of Γ. Also, we will use this fact to show that $\|\tilde{f}\|_{C^r} \le C\|\eta\|_{C^r}$.

Again by Taylor's theorem,

$$|D^i\tilde{\eta}(t)| \;\le\; C\|\eta\|_{C^r}|t|^{(L+1-i)_+}, \quad 0 \le i \le r,$$

where $(L+1-i)_+ = \max(L+1-i,0)$, as we can have $i \geq L+1$ if $L+1<r$. Thus, using the previous inequality one has

$$\begin{aligned}\|D^i A^{-n-1}\tilde{\eta}(\lambda^n t)\|_\infty &= \|A^{-n-1}D^i\tilde{\eta}(\lambda^n t)\lambda^{in}\|_\infty \\ &\leq C\|A^{-1}\|^n \, \|\eta\|_{C^r}\, |\lambda|^{in}\, |\lambda|^{n(L+1-i)_+} \\ &\leq C\,(|\lambda|^{L+1}\|A^{-1}\|)^n\, \|\eta\|_{C^r}, \quad 0 \leq i \leq r.\end{aligned}$$

Hence, as $|\lambda|^{L+1}\|A^{-1}\| < 1$, we have that the series of the derivatives converges uniformly by the Weierstrass M-test. Then one has $\tilde{f} \in \Gamma$ and also

$$\|\tilde{f}\|_{C^r} = \sup_{i\leq r} \|\sum_{n=0}^{\infty} D^i A^{-n-1}\tilde{\eta}(\lambda^n t)\|_\infty \leq C\,\|\eta\|_{C^r}\sum_{n=0}^{\infty}(|\lambda|^{L+1}\|A^{-1}\|)^n \leq C'\|\eta\|_{C^r}. \tag{3.25}$$

Finally, by the estimates (3.23) and (3.25) we get

$$\|\mathcal{S}^{-1}(\eta)\|_{C^r} = \|f\|_{C^r} = \|\sum_{n=2}^{L} f_n x^n + \tilde{f}\,\|_{C^r} \leq \|\sum_{n=2}^{L} f_n x^n\|_{C^r} + \|\tilde{f}\,\|_{C^r} \leq C\|\eta\|_{C^r},$$

that is, $\mathcal{S}^{-1}$ is bounded. □

The theorem follows from the lemma, as we can apply the implicit function theorem to the operator $\mathcal{T}$ to show that for a small enough K_1, we can find $K^> \in B$ in such a way that (3.19) is satisfied. Hence, once K_1 is fixed, there exists a C^r invariant curve $K^>$ for the map F near the origin, and also $K^> = u(K_1)$ is a function of class C^1, which means that the dependence of the invariant curve on the vector K_1 is differentiable.

The last statement of the theorem follows analogously as in Theorem 3.3. □

3.4 Adapting Banach spaces

As it has been seen up to now, the parameterization method consists on studying the existence of the solution of a functional equation that establishes the condition for a manifold to be invariant under the dynamics of a certain map. Such an equation is set in a certain function space, and so one looks for the solution in that space.

Along this work we deal with invariant curves for planar discrete dynamical systems. We are interested in invariant curves in a neighborhood of a fixed point and in their regularity properties, but *a priori* this curves are not considered in a fixed function space. That gives us the freedom of adapting to the context the spaces on which the functional equations are set.

In Section 3 we have considered an equation which is set in the Banach space

$$H := \Big\{K^> : \mathbb{D} \to \mathbb{C}^2 \;\Big|\; K^>(z) = \sum_{n=2}^{\infty} K_n\, z^n,\; \|K^>\| := \sum_{n=2}^{\infty} |K_n| < \infty\Big\}.$$

We need to see that if the given map F has an analytic invariant curve $K = K_1 \cdot + K^>$ defined in a small neighborhood of 0, it will be sufficient to consider $K^>$ belonging to H.

Let us consider the following change of variables. If $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, take

$$\begin{aligned} \tilde{F} :\ \tilde{U} \subseteq \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{1}{\delta} F(\delta x) \end{aligned}$$

with $U \subset \tilde{U}$.

If such a dilatation is applied to F, the new mapping $\tilde{F} = \frac{1}{\delta} F(\delta x)$ will have now an invariant curve $\tilde{K} = K_1 x + \tilde{K^>}$ with $\tilde{K^>} = \frac{1}{\delta} K^>(\delta x)$, that is,

$$\tilde{K^>} = \frac{1}{\delta} \sum_{n=2}^{\infty} K_n\, \delta^n\, x^n.$$

Taking the complex extension of F and K, let R be the radius of convergence of $K^>$. Then the radius of convergence of $\tilde{K^>}$ is

$$\tilde{R} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{|K_n \delta^n|}} = \limsup_{n\to\infty} \frac{1}{\delta \sqrt[n]{|K_n|}} = \frac{R}{\delta}, \tag{3.26}$$

that is, the function $\tilde{K^>}$ has an augmented radius of convergence with respect to $K^>$ in a factor $\frac{1}{\delta}$. Thus, if we take $\delta > R$, the function $\tilde{K^>}(z)$ will be holomorphic in a disk that contains the unit disc. In such a situation we can see that $\tilde{K^>}$ is indeed contained in the space H. In other words, it is sufficient to do a proper dilatation of the map F to ensure that we can look for the solution of equation (3.5) in H without loss of generality.

Les us see that if $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is a holomorphic function in $D(0,r)$ with $r \geq 1+\varepsilon$, for any $\varepsilon > 0$, and such that $f(0) = f'(0) = 0$, then f belongs to H.

If $f(z) = \sum_{n=2}^{\infty} a_n\, z^n$, for $|z| < 1+\varepsilon$, we have to see that $\sum_{n=2}^{\infty} |a_n| < \infty$.

By the Cauchy integral formula, one has

$$f^n(0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz,$$

where $\gamma = \partial D(0,r)$, and then

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^{n+1}}\, ire^{i\theta}\, d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(re^{i\theta})}{(re^{i\theta})^{n+1}}\, ire^{i\theta} \right| d\theta \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} |f(re^{i\theta})| \leq \frac{1}{r^n} \sup_{z\in\gamma} |f(z)| \leq \frac{M}{r^n}, \end{aligned}$$

for some constant $M \in \mathbb{R}$. Hence, as $r \geq 1+\varepsilon$,

$$\sum_{n=2}^{\infty} |a_n| \leq M \sum_{n=2}^{\infty} \frac{1}{r^n} < \infty,$$

and so f belongs to H.

In the case of analytic invariant curves we have used the particular function space H because its norm was suitable to prove the invertibility of the linear operator $\mathcal{S}$. Nevertheless, one can take any other Banach space whenever it can be justified that, under a suitable change of variables in F, we can set in that space the functional equation we are interested in and no generality is lost for the class of invariant manifolds we are looking for.

In the case of C^r invariant curves, where the object to study is an equation which is set in the space

$$\Gamma = \{K^> : [-1,\, 1] \to \mathbb{R}^2 \,|\, K^> \in C^r([-1,\, 1])\,,\, K^>(0) = (K^>)'(0) = 0\}$$

a dilatation $x \mapsto \frac{1}{\delta}F(\delta x)$ for some δ is sufficient to ensure that an invariant manifold of F belongs to Γ.

4 Invariant manifolds associated to a nilpotent parabolic point

This section is devoted to the study of stable one-dimensional invariant manifolds associated to a class of parabolic fixed points of planar maps.

We recall that if $F : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is a map of class C^1 and x_0 is a fixed point of F, then x_0 is said to be parabolic if both the eigenvalues of $DF(x_0)$ have modulus equal to 1.

Here we will consider the case when the origin is a nilpotent parabolic fixed point of F. More concretely, we will take the origin as a fixed point of F with

$$DF(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The study of invariant curves near such a fixed point has been done in [5] using a normal form of F. The aim of this section is to study the same problem but now using the parameterization method.

Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^∞ map in a neighborhood of the origin, and where the origin is a parabolic nilpotent fixed point. One can write

$$F(x, y) = \begin{pmatrix} x + y \\ y \end{pmatrix} + \begin{pmatrix} a_{20}\, x^2 + a_{11}\, xy + a_{02}\, y^2 + O(\|(x,\, y)\|^3) \\ b_{20}\, x^2 + b_{11}\, xy + b_{02}\, y^2 + O(\|(x,\, y)\|^3) \end{pmatrix}. \tag{4.1}$$

We look for a formal invariant curve, $K : I \subseteq \mathbb{R} \to \mathbb{R}^2$ of the form $K(t) = (O(t^2), O(t^3))$ because [5] shows that if one looks for an invariant curve being a graph on the variable x, one obtains formally that $y(x) = cx^{3/2} + O(x^2)$, and so if we want $K(t)$ to be expressed as a power series, we must take $t^2 = x$ and then we obtain $K(t) = (O(t^2), O(t^3))$. Also, since there is some freedom we take $K''(0) = (2, 0)$.

Then, one can write

$$K(t) = \begin{pmatrix} t^2 + K_3^x\, t^3 + K_4^x t^4 + O(t^5) \\ K_3^y\, t^3 + K_4^y\, t^4 + O(t^5) \end{pmatrix}. \tag{4.2}$$

Even if we look for a formal curve K, we can think of its coefficients as Taylor coefficients. Then, in (4.2), K_i^x is the i-th Taylor coefficient of the first component of K and K_i^y is the i-th Taylor coefficient of the second component of K.

Following the main idea of the parameterization method, we should look for a curve K satisfying

$$F(K(t)) = K(R(t)), \quad t \in I, \tag{4.3}$$

where $R : I \to I$ is the restriction of F on I. We look for $R(t)$ being of the form

$$R(t) = t + R_2\, t^2 + R_3\, t^3 + O(t^4)$$

as we have $R(0) = 0$ and $R'(0) = 1$ as 1 is an eigenvalue of $DF(0)$. We ought to lo look for some K and R satisfying equation (4.3). As we said previously, the approach that we will follow is to take the simplest expression of $R(t)$.

As we supposed F of class C^∞ and we look for some formal expressions of K and R, we can expand both terms in equation (4.3) in order to find the coefficients of K and R.

4.1 Approximation of the invariant curve

Our scope now is to expand $F \circ K$ and $K \circ R$ as a power series in order to find the first n Taylor coefficients of the invariant curve, $K(t)$, and the restricted dynamics $R(t)$.

To get started with this process we will write the Taylor expansion of $G(t)$ up to order 5, supposing that the coefficient b_{20} of the Taylor expansion of $F(x,y)$ is such that $b_{20} \neq 0$. Thus, we have

$$F(K(t)) = \begin{pmatrix} t^2 + t^3\,(K_3^x + K_3^y) + t^4\,(K_4^x + K_4^y + a_{20}) + t^5\,(K_5^x + K_5^y + 2\,a_{20}\,K_3^x + a_{11}\,K_3^y) \\ K_3^y\,t^3 + t^4\,(K_y^4 + b_{20}) + t^5\,(K_5^y + 2\,b_{20}\,K_3^x + b_{11}\,K_3^y) \end{pmatrix} + \begin{pmatrix} O(t^6) \\ O(t^6) \end{pmatrix}, \tag{4.4}$$

$$K(R(t)) = \begin{pmatrix} t^2 + t^3(2R_2 + K_3^x) + t^4(R_2^2 + 2R_3 + 3K_3^x R_2 + K_4^x) + \\ t^3\,K_3^y + t^4\,(3\,K_3^y\,R_2 + K_4^y) \end{pmatrix} + \begin{pmatrix} t^5(2R_4 + 2R_2R_3 + 3K_x^3 R_3 + 3K_3^x R_2^2 + 4K_4^x R_2 + K_5^x) \\ t^5\,(3\,K_3^y\,R_3 + 3\,K_3^y\,R_2^2 + 4\,K_4^y\,R_2 + K_5^y) \end{pmatrix} + \begin{pmatrix} O(t^6) \\ O(t^6) \end{pmatrix}. \tag{4.5}$$

Now, equating every component of (4.4) to the corresponding one in (4.5) we get the following equations, called *cohomological equations* for the map F.

Equating coefficients of degree 3 we are lead to

$$K_3^y = 2\,R_2; \tag{4.6}$$

equating the coefficients of degree 4 we have

$$K_4^y + a_{20} = R_2^2 + 2\,R_3 + 3\,K_3^x\,R_2, \tag{4.7}$$
$$b_{20} = 3\,K_3^y\,R_2; \tag{4.8}$$

and for the coefficients of degree 5 we get

$$K_5^y + 2\,a_{20}\,K_3^x + a_{11}\,K_3^y = 2\,R_4 + 2\,R_2\,R_3 + 3\,K_3^x\,R_3 + 3\,K_3^x\,R_2^2 + 4\,K_4^x\,R_2, \tag{4.9}$$
$$2\,K_3^x\,b_{20} + b_{11}\,K_3^y = 3\,R_3\,K_3^y + 3\,R_2^2\,K_3^y + 4\,R_2\,K_4^y. \tag{4.10}$$

From equations (4.6) and (4.8) one obtains

$$\begin{cases} R_2 = \pm\sqrt{\frac{b_{20}}{6}}, \\ K_3^y = \pm 2\,\sqrt{\frac{b_{20}}{6}}. \end{cases} \tag{4.11}$$

Thus, the coefficient b_{20} of F has to be positive in order to proceed, that is, F cannot have an invariant curve with the given conditions if $b_{20} < 0$.

Note that if we take $R_2 = -\sqrt{\frac{b_{20}}{6}}$ we are fixing the restricted dynamics as $R(t) = t - \sqrt{\frac{b_{20}}{6}}\, t^2 + O(t^3)$, and so the iterates of t under R tend to the origin. Thus, R(t) will be the restricted dynamics of a stable curve, whose parameterization starts with $K_3^y = 2\sqrt{\frac{b_{20}}{6}}$. If instead we fix $R_2 = +\sqrt{\frac{b_{20}}{6}}$, we are imposing the dynamics of an unstable curve.

From equations (4.7) and (4.10) we get

$$\begin{cases} K_4^y - 3\,R_2\,K_3^x = R_2^2 - a_{20} + 2\,R_3, \\ -4\,R_2\,K_4^y + 2\,b_{20}\,K_3^x = K_3^y\,(3\,R_3 + 3\,R_2^2 - b_{11}). \end{cases} \tag{4.12}$$

This linear system for K_3^x and K_4^y cannot have a unique solution as the matrix of coefficients has null determinant,

$$\begin{vmatrix} 1 & -3\,R_2 K_3^x \\ -4\,R_2\,K_4^y & 2\,b_{20}\,K_3^x \end{vmatrix} = 2\,b_{20} - 12\frac{b_{20}}{6} = 0.$$

In order that the system is compatible, the expanded matrix of coefficients and independent terms must have rank equal to 1, that is, we must have

$$K_3^y\,(3\,R_3 + 3\,R_2^2 - b_{11}) = -4\,R_2\,(R_2^2 - a_{20} + 2\,R_3),$$

that from the values obtained in (4.11), leads us to

$$R_3 = \frac{1}{7}\Big(2\,a_{20} + b_{11} - \frac{5}{6}\,b_{20}\Big),$$

which is the only value of R_3 for which system (4.12) has a solution.

Observe, though, that system (4.12) has a free parameter as the two equations are linearly dependent, but we cannot be sure that we can give an arbitrary value to K_3^x at the moment, because when writing the cohomological equations given by higher order terms, we might obtain new constraints for the coefficients K_3^x and K_4^y.

From now on, let us denote $G = F \circ K - K \circ R$.

Observe that equations (4.12) for K_3^x and K_4^y come from equating to 0 the coefficient of t^4 of the first component of G and the coefficient of t^5 of the second component of G. On what follows, we will determine an algorithm to compute the coefficients K_n^x and K_{n+1}^y of $K(t)$ supposing that we have obtained previously the coefficients up to a lower order, that is, up to K_{n-1}^x and K_n^y.

Let us write

$$G(t) = \begin{pmatrix} G_2^x\,t^2 + \ldots + G_n^x\,t^n + t^{n+1}\,(\hat{G}_{n+1}^x + \tilde{G}_{n+1}^x) + O(t^{n+2}) \\ G_3^y\,t^3 + \cdots + G_{n+1}^y\,t^{n+1} + t^{n+2}\,(\hat{G}_{n+2}^y + \tilde{G}_{n+2}^y) + O(t^{n+3}) \end{pmatrix}, \tag{4.13}$$

where, in the first component, $\hat{G}_{n+1}^x$ are the coefficients of t^{n+1} that only contain known parameters (that is, that contain coefficients of $K(t)$ until K_{n-1}^x and K_n^y and that therefore they have been determined previously) and $\tilde{G}^x_{n+1}$ are the coefficients of t^{n+1} that contain the parameters K^x_n and K^y_{n+1}; and similarly for the second component.

We suppose that the coefficients of K and R have been determined in such a way that

$$\begin{aligned} G^x_2\, t^2 + \ldots + G^x_n\, t^n &= 0, \\ G^y_3\, t^3 + \cdots + G^y_{n+1}\, t^{n+1} &= 0. \end{aligned} \tag{4.14}$$

If now we look for a better approximation of $K(x)$ we have to impose

$$\begin{aligned} G^x_2\, t^2 + \ldots + G^x_n\, t^n + t^{n+1}\,(\hat{G}^x_{n+1} + \tilde{G}^x_{n+1}) &= 0, \\ G^y_3\, t^3 + \cdots + G^y_{n+1}\, t^{n+1} + t^{n+2}\,(\hat{G}^y_{n+2} + \tilde{G}^y_{n+2}) &= 0, \end{aligned}$$

and from hypothesis (4.14) we get

$$\begin{aligned} t^{n+1}\,(\hat{G}^x_{n+1} + \tilde{G}^x_{n+1}) &= 0, \\ t^{n+2}\,(\hat{G}^y_{n+2} + \tilde{G}^y_{n+2}) &= 0, \end{aligned} \tag{4.15}$$

where $\hat{G}^x_{n+1}$, $\hat{G}^y_{n+2}$ are known.

Proceeding similarly as in (4.4) and (4.5) one obtains

$$\tilde{G}^x_{n+1} = K^y_{n+1} - n\, R_2\, K^x_n, \tag{4.16}$$

$$\tilde{G}^y_{n+2} = 2\, b_{20}\, K^x_n - (n+1)\, R_2\, K^y_{n+1}, \tag{4.17}$$

and so we are led to the following linear system for K^x_n and K^y_{n+1},

$$\begin{pmatrix} -n\, R_2 & 1 \\ 2\, b_{20} & -(n+1)\, R_2 \end{pmatrix} \begin{pmatrix} K^x_n \\ K^y_{n+1} \end{pmatrix} = \begin{pmatrix} -\hat{G}^x_{n+1} \\ -\hat{G}^y_{n+2} \end{pmatrix}. \tag{4.18}$$

To discuss the compatibility of such a system we consider the determinant of the matrix of coefficients, M,

$$\det(M) = (n+1)\, n\, R_2^2 - 2\, b_{20} = b_{20}\left(\frac{n(n+1)}{6} - 2\right) = 0 \quad \Leftrightarrow \quad n = 3.$$

Therefore, we can obtain recurrently the coefficients K^x_n and K^y_{n+1} if we have determined previously the coefficients of $K(t)$ up to K^x_{n-1} and K^y_n, except for the case $n = 3$, where the solution of (4.18) is not unique; but that special case is precisely the one we obtained in (4.15), and so one has that system (4.15) is indeed an undetermined linear system. That means that for every chosen value of K^x_3 we will obtain a value for K^y_4, and those values will determine unequivocally the next coefficients of $K(t)$ proceeding with equations (4.18).

Observe also that the coefficients R_n, $n = 4, 5, \ldots$, of the restricted dynamics $R(t)$ do not appear on the left side of equations (4.18), that is, they appear only in the independent terms. As system (4.18) is compatible and determined whatever are the values of $\hat{G}^x_{n+1}$ and $\hat{G}^y_{n+2}$, except for $n = 3$, we can give any value to R_n, $n \geq 4$. As we are looking for the simplest expression of $R(t)$ we choose $R_n = 0$, $\forall n \geq 4$.

Hence, we can obtain approximated solutions to equation (4.3) with

$$R(t) = t - 2\sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2\, a_{20} + b_{11} - \frac{5}{6}\, b_{20}\Big)\, t^3$$

and the corresponding parameterization of $K(t)$, which is a stable curve, or

$$R(t) = t + 2\sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2\, a_{20} + b_{11} - \frac{5}{6}\, b_{20}\Big)\, t^3$$

and the corresponding parameterization of $K(t)$, which is an unstable curve.

If we had chosen other values for the coefficients $R_n,\ n \geq 4$, we would have obtained different values for the coefficients of $K(t)$, but in all cases, the restricted dynamics $R(t)$ and the corresponding parameterization of $K(t)$ correspond to the same invariant manifold (see [5]).

In order to study the existence of a stable curve asymptotic to a nilpotent parabolic fixed point, we will consider that the coefficients of $K(t)$ have been computed up to high enough order and we will study the existence of the remainder of $K(t)$ using functional analysis techniques as in Section 3.

Let us denote $K(t)$ as in (4.2), and

$$\tilde{K}(t) = \begin{pmatrix} t^2 + K_3^x\, t^3 + \cdots + K_n\, t^n \\ K_3^y\, t^3 + \cdots + K_{n+1}^y\, t^{n+1} \end{pmatrix}, \tag{4.19}$$

where all the coefficients have been determined previously in order that

$$\begin{aligned} &G_2^x\, t^2 + \ldots + G_n^x\, t^n + t^{n+1}\,(\hat{G}_{n+1}^x + \tilde{G}_{n+1}^x) = 0, \\ &G_3^y\, t^3 + \cdots + G_{n+1}^y\, t^{n+1} + t^{n+2}\,(\hat{G}_{n+2}^y + \tilde{G}_{n+2}^y) = 0. \end{aligned}$$

With this setting let us define

$$E(t) := F(\tilde{K}(t)) - \tilde{K}(R(t)) = \begin{pmatrix} O(t^{n+2}) \\ O(t^{n+3}) \end{pmatrix}. \tag{4.20}$$

We want to find a curve $K(t) = \tilde{K}(t) + \Delta(t)$ such that

$$F \circ K - K \circ R = 0 \tag{4.21}$$

with $R(t) = t - \sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2a_{20} + b_{11} - \frac{5}{6} b_{20}\Big)\, t^3$, where

$$\Delta(t) = \begin{pmatrix} O(t^{n+1}) \\ O(t^{n+2}) \end{pmatrix}.$$

Notice that in order to look for the existence of an unstable curve near the fixed point one should replace $R_2 = -\sqrt{\frac{b_{20}}{6}}$ by $R_2 = \sqrt{\frac{b_{20}}{6}}$ to proceed with the study.

We shall deal with the functional equation

$$F \circ (\tilde{K} + \Delta) - (\tilde{K} + \Delta) \circ R = 0,$$

where Δ is the unknown. Observe that the left hand side of this equality can be written as

$$\begin{aligned} F \circ K - K \circ R &= F \circ \tilde{K} + (DF \circ \tilde{K})\, \Delta + N(\tilde{K}, \Delta) - \tilde{K} \circ R - \Delta \circ R \\ &= E + (DF \circ \tilde{K})\, \Delta - \Delta \circ R + N(\tilde{K},\, \Delta), \end{aligned} \tag{4.22}$$

where $N(\tilde{K},\, \Delta)$ denotes the nonlinear terms of $F \circ (\tilde{K} + \Delta)$, that is, $N(\tilde{K}, \Delta) = \frac{1}{2}\,(D^2 F \circ \tilde{K})\,(\Delta, \Delta) + O(\|\Delta\|^3)$.

4.2 Analytic stable curves

Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a map of class C^∞ and assume that it is analytic in a neighborhood of the origin. The aim of this section is to prove the existence and the analyticity of an invariant stable curve of F associated to the origin when it is a parabolic nilpotent fixed point, that is, if $F(0) = 0$ and

$$DF(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The main result of this section is Theorem 4.2 and the whole section is devoted to introduce the setting and the preliminary results that we will need.

We will deal with equation (4.22) in the case that F is analytic in a neighborhood of the origin. In this case we will look for an invariant stable curve K being also analytic, taking the restricted dynamics of F inside the curve as

$$R(t) = t - \sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2a_{20} + b_{11} - \frac{5}{6} b_{20}\Big)\, t^3. \tag{4.23}$$

Let β, ρ be positive real numbers and let us consider the following domain,

$$S = S(\beta, \rho) = \Big\{ z \in \mathbb{C} \mid |\arg(z)| < \frac{\beta}{2},\ 0 < |z| < \rho \Big\}.$$

With this setting we consider the following family of Banach spaces,

$$\mathcal{X}_p = \{ f : S \to \mathbb{C} \mid f \in \mathrm{Hol}(S),\ \|f\|_p = \sup_{z \in S} \frac{|f(z)|}{|z|^p} < \infty \}, \quad p \in \mathbb{N}.$$

Note that $\mathcal{X}_{p+1} \subset \mathcal{X}_p$, for all $p \in \mathbb{N}$.

We shall define also the spaces $\mathcal{X}_p \times \mathcal{X}_{p+1}$ of vector-valued complex functions, containing functions whose first component belongs to $\mathcal{X}_p$ and whose second component belongs to $\mathcal{X}_{p+1}$, endowed with the norm

$$\|f\|_{p,\, p+1} = \|f_1\|_p + \|f_2\|_{p+1},$$

where f_1 and f_2 denote the two components of f, respectively.

Now we state a particular case of Lemma 7.1 of [1] that we will need for our purposes.

Lemma 4.1. *Let $R : S(\beta, \rho) \to \mathbb{C}$ be a map of the form $R(z) = z - az^2 + O(|z|^3)$, with $a > 0$. Assume that $\beta < \pi$. Then, for any $\nu \in (0,\, a\,\cos(\beta/2))$, there exists $\rho > 0$ small enough such that*

$$|R^n(z)| \leq \frac{|z|}{1 + n\,\nu\,|z|}, \quad \forall\, n \in \mathbb{N}.$$

In addition, R maps $S(\beta, \rho)$ into itself.

We will consider the function R as in (4.23) defined on the set $S(\beta, \rho)$, for some fixed values of β and ρ for which Lemma 4.1 holds. With the notation of Lemma 4.1 we have $a = \sqrt{\frac{b_{20}}{6}}$ and ν any fixed value between 0 and $a\cos(\beta/2)$.

We are interested on the existence of a function Δ defined on $S(\beta,\, \rho)$ for which we have $F \circ K - K \circ R = 0$, in order that then $K = \tilde{K} + \Delta$ is a stable invariant curve associated to the parabolic fixed point of F which we are dealing with. This would provide an analytic stable curve on the interval $(0,\, \rho)$. It is known that in general we can not expect to obtain analyticity at the origin.

We shall consider Δ set in the space $\mathcal{X}_p \times \mathcal{X}_{p+1}$ for a certain p that will be determined later. Observe that as we obtained a recursive algorithm to compute an approximation $\tilde{K}$ of K up to an arbitrary order, we can consider Δ in $\mathcal{X}_p \times \mathcal{X}_{p+1}$ for any p.

Given $p \in \mathbb{N}$ we define the linear operator $\mathcal{S}$ as

$$\begin{aligned} \mathcal{S} :\, \mathcal{X}_p &\longrightarrow \mathcal{X}_p \\ f &\longmapsto f - f \circ R. \end{aligned}$$

It ts clear that $\mathcal{S}$ indeed maps any element of $\mathcal{X}_p$ to $\mathcal{X}_p$, as if $f(z) = O(|z|^p)$, then it also holds that $f(R(z)) = O(|z|^p)$, and $f(R(z))$ is well defined by Lemma 4.1.

One can also consider the operator $\mathcal{S}$ acting on vector-valued functions in the following way,

$$\begin{aligned} \mathcal{S} :\, \mathcal{X}_p \times \mathcal{X}_{p+1} &\longrightarrow \mathcal{X}_p \times \mathcal{X}_{p+1} \\ (f_1,\ f_2) \quad &\longmapsto (f_1, f_2) - (f_1 \circ R,\ f_2 \circ R). \end{aligned} \tag{4.24}$$

We use the same symbol $\mathcal{S}$ hoping that its domain will be clear from the context.

Now observe that from equation (4.22) we can write

$$\begin{aligned} F \circ K - K \circ R &= E + (DF \circ \tilde{K} - I)\,\Delta + \Delta - \Delta \circ R + N(\tilde{K},\, \Delta) = \\ &= E + (DF \circ \tilde{K} - I)\,\Delta + \mathcal{S}\Delta + N(\tilde{K},\, \Delta). \end{aligned} \tag{4.25}$$

The following lemma establishes the pseudo-invertibility of $\mathcal{S}$, in the sense that it has a right inverse $\mathcal{S}^{-1} :\, \mathcal{X}_p \to \mathcal{X}_{p-1}$ such that $\mathcal{S} \circ \mathcal{S}^{-1} = I$ in $\mathcal{X}_p$, and gives a bound for the norm of $\mathcal{S}^{-1}$.

Lemma 4.2. *Given $p \in \mathbb{N}$, the operator $S :\, \mathcal{X}_p \to \mathcal{X}_p$ has a right inverse, $S^{-1} :\, \mathcal{X}_p \to \mathcal{X}_{p-1}$, which is a bounded operator, that is, $\|\mathcal{S}^{-1}\eta\|_{p-1} \leq C\|\eta\|_p$, for all $\eta \in \mathcal{X}_p$, with $C \leq \left(\rho + \frac{1}{\nu\,(p-1)}\right)$.*

Proof. Let us consider the equation

$$\mathcal{S}f = \eta, \tag{4.26}$$

where η is an element of $\mathcal{X}_{p+q}$. We will look for a solution $f \in \mathcal{X}_p$ for the equation and then we will determine the value of q. From equation (4.26), as one has $f - f \circ R = \eta$, one can write recursively

$$\begin{aligned} f = \eta + f \circ R = \eta + (\eta + f \circ R) \circ R = f \circ R \circ R + \eta + \eta \circ R = \cdots \\ = f \circ R^M + \sum_{j=0}^{M-1} \eta \circ R^j, \quad \forall\, M \in \mathbb{N}, \end{aligned}$$

and so, formally, one can take

$$f = \lim_{M \to \infty} [\, f \circ R^M + \sum_{j=0}^{M-1} \eta \circ R^j \,].$$

We claim that this limit exists and that the solution to equation (4.26) is

$$f(z) = \sum_{j=0}^{\infty} \eta(R^j(z)). \tag{4.27}$$

First, observe that

$$\lim_{M \to \infty} f \circ R^M = 0,$$

as by Lemma 4.1 one has

$$|R^M(z)| \leq \frac{|z|}{1 + M\nu|z|} \sim \frac{1}{M},$$

and so R^M tends to 0 as M tends to infinity, and also by the definition of the space $\mathcal{X}_p$, one has that $f(0) = 0$.

Let us see now that the series (4.27) converges uniformly on $S(\beta, \rho)$.

As, by definition, one has $\|\eta\|_{p+q} = \sup_{z \in S} \dfrac{|\eta(z)|}{|z|^{p+q}}$, then

$$|\eta(R^j(z))| \leq \|\eta\|_{p+q}\, |R^j(z)|^{p+q}, \quad \forall\, q \in \mathbb{N},$$

and so one obtains, again by Lemma 4.1,

$$|\eta(R^j(z))| \,\leq\, \|\eta\|_{p+q}\, |R^j(z)|^{p+q} \leq\, \|\eta\|_{p+q} \left(\frac{|z|}{1 + j\nu|z|} \right)^{p+q} \,\leq\, C\, \|\eta\|_{p+q}\, \frac{1}{j^{p+q}},$$

and hence, as $p + q > 1$, the series (4.27) converges uniformly by the Weierstrass M-test, and so it defines a function which is holomorphic in S and which gives a solution to (4.26).

In order to prove that $\|\mathcal{S}^{-1}(\eta)\|_p < C\, \|\eta\|_{p+q}$, let us start computing $\|\mathcal{S}^{-1}(\eta)\|_p$.

We have

$$\begin{aligned} \|\mathcal{S}^{-1}(\eta)\|_p = \sup_{z \in S} \frac{|\mathcal{S}^{-1}(\eta(z))|}{|z|^p} \,= \sup_{z \in S} \frac{1}{|z|^p} \Big| \sum_{j=0}^{\infty} \eta(R^j(z)) \Big| \\ \leq\, \|\eta\|_{p+q} \sup_{z \in S} \frac{1}{|z|^p} \sum_{j=0}^{\infty} |R^j(z)|^{p+q}, \end{aligned}$$

where the last inequality follows since one has

$$\Big|\sum_{j=0}^{\infty}\eta(R^j(z))\Big| \,\le\, \sum_{j=0}^{\infty}|\eta(R^j(z))| \,\le\, \sum_{j=0}^{\infty}\|\eta\|_{p+q}\,|R^j(z)|^{p+q}, \quad \forall\, q\in\mathbb{N}.$$

Now, by Lemma 4.1 one has

$$\|\eta\|_{p+q}\,\sup_{z\in S}\frac{1}{|z|^p}\sum_{j=0}^{\infty}|R^j(z)|^{p+q} \,\le\, \|\eta\|_{p+q}\,\sup_{z\in S}\frac{1}{|z|^p}\sum_{j=0}^{\infty}\frac{|z|^{p+q}}{(1+j\,\nu\,|z|)^{p+q}},$$

and using the integral criterion for series one gets

$$\begin{aligned}
\frac{1}{|z|^p}\sum_{j=0}^{\infty}\frac{|z|^{p+q}}{(1+j\,\nu\,|z|)^{p+q}} &= |z|^q\sum_{j=0}^{\infty}\frac{1}{(1+j\,\nu\,|z|)^{p+q}}\\
&\le |z|^q\left(1+\int_0^{\infty}\frac{1}{(1+x\,\nu\,|z|)^{p+q}}\,dx\right)\\
&= |z|^q\left(1+\int_0^{\infty}\frac{1}{(1+y)^{p+q}}\,\frac{1}{\nu\,|z|}\,dy\right)\\
&\le\, |z|^{q-1}\left(\rho+\frac{1}{\nu\,(p+q-1)}\right).
\end{aligned}$$

Therefore, we have obtained

$$\|\mathcal{S}^{-1}(\eta)\|_p \le \|\eta\|_{p+q}\,\sup_{z\in S}\,|z|^{q-1}\left(\rho+\frac{1}{\nu\,(p+q-1)}\right), \quad \eta\,\in\,\mathcal{X}_{p+q}. \tag{4.28}$$

We want to establish the boundedness of $\mathcal{S}^{-1}$ between spaces as similar as possible. It is clear that, from the previous estimates, we cannot have $\|\mathcal{S}^{-1}(\eta)\|_p \le C\,\|\eta\|_p$, as if we take $q=0$ in (4.28), then the right hand side of the expression is not bounded. The minimum value we can take for q is $q=1$, and hence we get

$$\|\mathcal{S}^{-1}(\eta)\|_p \le \|\eta\|_{p+1}\,\big(\rho+\frac{1}{\nu\,p}\big),$$

which shows that for all $\eta\in\mathcal{X}_{p+1}$, equation (4.26) has a solution $f\in\mathcal{X}_p$. In other words, the operator $\mathcal{S}:\mathcal{X}_p\to\mathcal{X}_p$ has a right inverse, $\mathcal{S}^{-1}:\mathcal{X}_p\to\mathcal{X}_{p-1}$, which is bounded, with norm $\|\mathcal{S}^{-1}\|_{\mathcal{L}(\mathcal{X}_p,\,\mathcal{X}_{p-1})} \le \big(\rho+\dfrac{1}{\nu(p-1)}\big)$. □

One can apply Lemma 4.2 to vector valued functions of $\mathcal{X}_p\times\mathcal{X}_{p+1}$, because $\mathcal{S}$ acts componentwise, as defined in (4.24).

With this result we can return to equation (4.25) and we can rewrite $F\circ K - K\circ R = 0$ as a fixed point equation,

$$\Delta = -\mathcal{S}^{-1}\,[E+N(\tilde{K},\Delta)+(DF\circ\tilde{K}-I)\,\Delta] \tag{4.29}$$

Supposing that we have computed the coefficients of $\tilde{K}(z)$ in (4.18) up to order L and $L+1$, respectively for every component of $\tilde{K}$, we can consider Δ as an element of $\mathcal{X}_{L+1} \times \mathcal{X}_{L+2}$.

If $\Delta \in \mathcal{X}_{L+1} \times \mathcal{X}_{L+2}$, then it is clear that $E \in \mathcal{X}_{L+2} \times \mathcal{X}_{L+3}$, as shown in (4.20), and also one has that

$$N(\tilde{K}, \Delta) = \frac{1}{2}\,(D^2 F \circ \tilde{K})(\Delta, \Delta) + O(\|\Delta\|^3) \, \in \, \mathcal{X}_{2L+2} \times \mathcal{X}_{2L+2},$$

as we have

$$(D^2 F \circ \tilde{K})(\Delta, \Delta)(z) + O(\|\Delta\|^3)(z) = (O(|z|^{2L+2}),\, O(|z|^{2L+2})),$$

and

$$(DF \circ \tilde{K} - I)\,\Delta \, \in \, \mathcal{X}_{2L+2} \times \mathcal{X}_{2L+2}$$

as we have

$$(DF \circ \tilde{K} - I)(z) = (O(|z|^{2L+2}),\, O(|z|^{2L+2})).$$

Now, let us fix $r > 0$ and let us consider the closed ball

$$B^r_{L+1,\,L+2} = \{ f \in \mathcal{X}_{L+1} \times \mathcal{X}_{L+2} \mid \|f\|_{L+1,L+2} \leq r \},$$

which is a complete metric space.

Recall that we have $\mathcal{X}_p \subset \mathcal{X}_{p-1}$, for all p, and so we can define the operators

$$\begin{aligned} \mathcal{J}_1 \,:\, B^r_{L+1,\,L+2} &\longrightarrow \mathcal{X}_{L+2} \times \mathcal{X}_{L+3} \\ \Delta \quad &\longmapsto N(\tilde{K}, \Delta), \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_2 \,:\, B^r_{L+1,\,L+2} &\longrightarrow \mathcal{X}_{L+2} \times \mathcal{X}_{L+3} \\ \Delta \quad &\longmapsto (DF \circ \tilde{K} - I)\,\Delta. \end{aligned}$$

The value of r must be chosen to be smaller than the radius of analyticity of F, in order that the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ can be defined properly.

In what follows, we shall fix the domain of $\tilde{K}$ to S, which is the same that for Δ. This is natural since $\tilde{K}$ and Δ are two terms of the same function, K, which we assume to be defined in S.

With this setting we can see that the fixed point equation (4.29) is set in a suitable fashion, in the following sense. Let us define the operator $\mathcal{J}$ as

$$\mathcal{J}(\Delta) := E + \mathcal{J}_1(\Delta) + \mathcal{J}_2(\Delta)$$

for any $\Delta \in B^r_{L+1,L+2}$, and observe that then $\mathcal{J}(\Delta) \in \mathcal{X}_{L+2} \times \mathcal{X}_{L+3}$ and so $(\mathcal{S}^{-1} \circ \mathcal{J})(\Delta) \in \mathcal{X}_{L+1} \times \mathcal{X}_{L+2}$. Hence, we can define the operator

$$\Phi :\; B^r_{L+1,\,L+2} \longrightarrow \mathcal{X}_{L+1,\,L+1}$$

as

$$\Phi := \mathcal{S}^{-1} \circ \mathcal{J}. \tag{4.30}$$

With this definitions, equation (4.29) turns into the fixed point equation

$$\Delta = -\Phi\,(\Delta), \quad \Delta \in B^r_{L+1,L+2}.$$

Our aim is to prove that Φ is a contraction mapping, in order to be able to apply the Banach fixed point theorem and obtain the existence of a solution Δ for equation (4.29).

Lemma 4.3. *Fix $L \geq 1$. The operator $\mathcal{J}_1 : B^r_{L+1,\,L+2} \to \mathcal{X}_{L+2} \times \mathcal{X}_{L+3}$ satisfies*

$$\|\mathcal{J}_1\,(\Delta_1) - \mathcal{J}_1\,(\Delta_2)\|_{L+2,\,L+3} < \mathcal{C}\,\|\Delta_1 - \Delta_2\|_{L+1,\,L+2}, \quad \forall\; \Delta_1,\, \Delta_2\, \in B^r_{L+1,\,L+2},$$

for some constant $\mathcal{C}$ that depends neither on L nor on the radius ρ of $S(\beta,\,\rho)$ for decreasing values of ρ.

Proof. Let us denote by (Δ^x, Δ^y) the components of Δ and by $(\mathcal{J}^x_1, \mathcal{J}^y_1)$ the components of the image of $\mathcal{J}_1$.

We need to see that for all $\Delta_1,\, \Delta_2\, \in B^r_{L+1,\,L+2}$ one has

$$\|\mathcal{J}^x_1(\Delta_1) - \mathcal{J}^x_1(\Delta_2)\|_{L+2} + \|\mathcal{J}^y_1(\Delta_1) - \mathcal{J}^y_1(\Delta_2)\|_{L+3} < \mathcal{C}\,(\|\Delta^x_1 - \Delta^x_2\|_{L+1} + \|\Delta^y_1 - \Delta^y_2\|_{L+1}).$$

Let us bound $\mathcal{J}_1(\Delta)$ as the remainder of a Taylor expansion,

$$|\mathcal{J}_1(\Delta)| = |F \circ (\tilde{K} + \Delta) - DF \circ \tilde{K} - (DF \circ \tilde{K})\,\Delta| \leq |\frac{1}{2}\,(D^2F \circ g)\,(\Delta,\,\Delta)|,$$

with $g \in \{(1-t)\tilde{K} + t(\tilde{K} + \Delta) \mid t \in [0,\,1]\}$.

From the definition of $\mathcal{J}_1$ one has

$$\frac{1}{2}\,D^2F^x(g(z)) = \begin{pmatrix} a_{20} + O(|z|^2) & \frac{1}{2}a_{11} + O(|z|^2) \\ \frac{1}{2}a_{11} + O(|z|^2) & a_{02} + O(|z|^2) \end{pmatrix}$$

and

$$\frac{1}{2}\,D^2F^y(g(z)) = \begin{pmatrix} b_{20} + O(|z|^2) & \frac{1}{2}b_{11} + O(|z|^2) \\ \frac{1}{2}b_{11} + O(|z|^2) & b_{02} + O(|z|^2) \end{pmatrix},$$

where F^x and F^y denote the first and the second component of F.

Since $O(|z|^2) \leq M\,|z|^2$, for some positive constant M, one obtains

$$\begin{aligned} &\|\mathcal{J}^x_1(\Delta_1) - \mathcal{J}^x_1(\Delta_2)\|_{L+2} + \|\mathcal{J}^y_1(\Delta_1) - \mathcal{J}^y_1(\Delta_2)\|_{L+3} \leq \\ &\leq \sup_{z\in S} \frac{1}{|z|^{L+2}} \Big[(|a_{20}| + M|z|^2)\,|(\Delta^x_1)^2(z) - (\Delta^x_2)^2(z)| \\ &\quad + (|a_{11}| + M|z|^2)\,|\Delta^x_1(z)\Delta^y_1(z) - \Delta^x_2(z)\Delta^y_2(z)| + (|a_{02}| + M|z|^2)\,|(\Delta^y_1)^2(z) - (\Delta^y_2)^2(z)|\Big] \\ &\quad + \sup_{z\in S} \frac{1}{|z|^{L+3}} \Big[(|b_{20}| + M|z|^2)\,|(\Delta^x_1)^2(z) - (\Delta^x_2)^2(z)| \\ &\quad + (|b_{11}| + M|z|^2)\,|\Delta^x_1(z)\Delta^y_1(z) - \Delta^x_2(z)\Delta^y_2(z)| + (|b_{02}| + M|z|^2)\,|(\Delta^y_1)^2(z) - (\Delta^y_2)^2(z)|\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq (|a_{20}| + M\rho^2)\, C_1\, \|\Delta_1^x - \Delta_2^x\|_{L+1} \\
&\quad + (|a_{11}| + M\rho^2)\, (C_2\, \|\Delta_1^x - \Delta_2^x\|_{L+1} + C_3\, \|\Delta_1^y - \Delta_2^y\|_{L+2}) + (|a_{02}| + M\rho^2)\, C_4\, \|\Delta_1^y - \Delta_2^y\|_{L+2} \\
&\quad + (|b_{20}| + M\rho^2)\, C_5\, \|\Delta_1^x - \Delta_2^x\|_{L+1} \\
&\quad + (|b_{11}| + M\rho^2)\, (C_6\, \|\Delta_1^x - \Delta_2^x\|_{L+1} + C_7\, \|\Delta_1^y - \Delta_2^y\|_{L+2}) + (|b_{02}| + M\rho^2)\, C_8\, \|\Delta_1^y - \Delta_2^y\|_{L+2} \\
&= [C_1\, (|a_{20}| + M\rho^2) + C_2\, (|a_{11}| + M\rho^2) + C_5\, (|b_{20}| + M\rho^2) \\
&\quad + C_6\, (|b_{11}| + M\rho^2)]\, \|\Delta_1^x - \Delta_2^x\|_{L+1} \\
&\quad + [C_3\, (|a_{11}| + M\rho^2) + C_4\, (|a_{02}| + M\rho^2) + C_7\, (|b_{11}| + M\rho^2) \\
&\quad + C_8\, (|b_{02}| + M\rho^2)]\, \|\Delta_1^y - \Delta_2^y\|_{L+2}.
\end{aligned}
$$

Note that in order that the constants C_i, $i = 1, \ldots, 8$ exist, we need to consider $L \geq 1$. Indeed, for the case of C_5 one has

$$
C_5 = \sup_{z \in S} \frac{1}{|z|^2} |\Delta_1^x(z) + \Delta_2^x(z)|,
$$

so one needs Δ_1^x, Δ_2^x to belong to $\mathcal{X}_{L+1}$ with $L + 1 \geq 2$. Also, one has

$$
\begin{aligned}
C_5 = \sup_{z \in S} \frac{1}{|z|^2} |\Delta_1^x(z) + \Delta_2^x(z)| &\leq \sup_{z \in S} \frac{1}{|z|^2} |\Delta_1^x(z)| + \sup_{z \in S} \frac{1}{|z|^2} |\Delta_2^x(z)| \\
&= \|\Delta_1^x\|_{L+1} \sup_{z \in S} |z|^{L-1} + \|\Delta_2^x\|_{L+1} \sup_{z \in S} |z|^{L-1} \leq 2\, r\, \rho^{L-1} \leq 2\, r\, \rho,
\end{aligned}
$$

and similarly for the other constants, so all of them are bounded by above independently of L and they decrease as ρ decreases.

Now, from the obtained estimates we have

$$
\begin{aligned}
&\|\mathcal{J}_1^x(\Delta_1) - \mathcal{J}_1^x(\Delta_2)\|_{L+2} + \|\mathcal{J}_1^y(\Delta_1) - \mathcal{J}_1^y(\Delta_2)\|_{L+3} \\
&\leq [C_1\, (|a_{20}| + M\rho^2) + C_2\, (|a_{11}| + M\rho^2) + C_4\, (|b_{20}| + M\rho^2)]\, \|\Delta_1^x - \Delta_2^x\|_{L+1} \\
&\quad + [C_4\, (|a_{02}| + M\rho^2) + C_5\, (|b_{11}| + M\rho^2) + C_8\, (|b_{02}| + M\rho^2)]\, \|\Delta_1^y - \Delta_2^y\|_{L+2} \\
&\leq \mathcal{C}_1\, \|\Delta_1^x - \Delta_2^x\|_{L+1} + \mathcal{C}_2\, \|\Delta_1^y - \Delta_2^y\|_{L+2} \\
&\leq \mathcal{C}\, (\|\Delta_1^x - \Delta_2^x\|_{L+1} + \|\Delta_1^y - \Delta_2^y\|_{L+2}),
\end{aligned}
$$

as we wanted to see, and from the previous computations it is clear that the constant $\mathcal{C}$ does not depend on L and that it does not increase when ρ decreases. $\square$

Lemma 4.4. *Fix $L \geq 1$. The linear operator $\mathcal{J}_2 : B^r_{L+1,\, L+2} \to \mathcal{X}_{L+2} \times \mathcal{X}_{L+3}$ satisfies*

$$
\|\mathcal{J}_2\, (\Delta)\|_{L+2,\, L+3} < \mathcal{D}\, \|\Delta\|_{L+1,\, L+2}, \quad \forall\, \Delta \in B^r_{L+1,\, L+2},
$$

for some constant $\mathcal{D}$ that depends neither on L nor on the radius ρ of $S(\beta,\, \rho)$ for decreasing values of ρ. That is, $\mathcal{J}_2$ is a bounded operator for all $L \geq 1$, and the family of operators named after $\mathcal{J}_2$ and indexed by $L \geq 1$ is uniformly bounded.

Proof. As in Lemma 4.3, let us denote by (Δ^x, Δ^y) the components of Δ and by $(\mathcal{J}_2^x, \mathcal{J}_2^y)$ the components of the image of $\mathcal{J}_2$.

We need to prove that for all $\Delta \in \mathcal{X}_{L+1,\,L+2}$ one has

$$\|\mathcal{J}_2^x(\Delta)\|_{L+2} + \|\mathcal{J}_2^y(\Delta)\|_{L+3} < \mathcal{D}\,(\|\Delta^x\|_{L+1} + \|\Delta^y\|_{L+2}).$$

From the definition of $\mathcal{J}_2$ one has

$$\mathcal{J}_2(\Delta) = \begin{pmatrix} [2a_{20}z^2 + O(|z|^3)]\,\Delta^x + [1 + a_{11}z^2 + O(|z|^3)]\,\Delta^y \\ [2b_{20}z^2 + O(|z|^3)]\,\Delta^x + [b_{11}z^2 + O(|z|^3)]\,\Delta^y \end{pmatrix}.$$

Now, since $O(|z|^3) \leq M\,|z|^3$, for some positive constant M, we have

$$\begin{aligned}
\|\mathcal{J}_2\,\Delta\|_{L+2,\,L+3} &= \|\mathcal{J}_2^x(\Delta)\|_{L+2} + \|\mathcal{J}_2^y(\Delta)\|_{L+3} \\
&\leq \sup_{z\in S} \frac{1}{|z|^{L+2}} \Big[(2|a_{20}z^2| + M|z|^3)\,|\Delta^x(z)| + (1 + |a_{11}z^2| + M|z|^3)\,|\Delta^y(z)|\Big] \\
&\quad + \sup_{z\in S} \frac{1}{|z|^{L+3}} \Big[(2|b_{20}z^2| + M|z|^3)\,|\Delta^x(z)| + (|b_{11}z^2| + M|z|^3)\,|\Delta^y(z)|\Big] \\
&\leq \sup_{z\in S} [\,(2|a_{20}z| + M|z|^2)\,\|\Delta^x\|_{L+1} + (1 + |a_{11}z^2| + M|z|^3)\,\|\Delta^y\|_{L+2}\,] \\
&\quad + \sup_{z\in S} [\,(2|b_{20}| + M|z|)\,\|\Delta^x\|_{L+1} + (|b_{11}z| + M|z|^2)\,\|\Delta^y\|_{L+2}\,) \\
&\leq\ (2|a_{20}|\rho + M\rho^2 + 2|b_{20}| + M\rho)\,\|\Delta^x\|_{L+1} \\
&\quad + (1 + |a_{11}|\rho^2 + M\rho^3 + |b_{11}|\rho + M\rho^2)\,\|\Delta^y\|_{L+2}
\end{aligned}$$

Therefore it is clear that we have

$$\begin{aligned}
\|\mathcal{J}_2\,\Delta\|_{L+2,\,L+3} &\leq (2|a_{20}|\rho + M\rho^2 + 2|b_{20}| + M\rho)\,\|\Delta^x\|_{L+1} \\
&\quad + (1 + |a_{11}|\rho^2 + M\rho^3 + |b_{11}|\rho + M\rho^2)\,\|\Delta^y\|_{L+2} \\
&< \mathcal{D}_1\,\|\Delta^x\|_{L+1} + \mathcal{D}_2\,\|\Delta^y\|_{L+2} \\
&< \mathcal{D}\,(\|\Delta^x\|_{L+1} + \|\Delta^y\|_{L+2}),
\end{aligned}$$

as we wanted to see. Also, from the previous computations it is clear that the constant $\mathcal{D}$ does not depend on L and that it does not increase when ρ decreases. □

With the previous two lemmas we can show now that Φ defines a contraction on $B^r_{L+1,\,L+2}$ for some range of values of L.

Theorem 4.1. *The operator $\Phi\,:\,B^r_{L+1,\,L+2} \to \mathcal{X}_{L+1} \times \mathcal{X}_{L+2}$ is a contraction mapping provided that L is sufficiently large and the radius ρ of $S(\beta,\,\rho)$ is sufficiently small.*

Proof. From Lemmas 4.3 and 4.4 one has, for all $\Delta_1,\,\Delta_2 \in B^r_{L+1,\,L+2}$,

$$\begin{aligned}
\|\mathcal{J}(\Delta_1) - \mathcal{J}(\Delta_2)\|_{L+2,\,L+3} &= \|\mathcal{J}_1(\Delta_1) - \mathcal{J}_1(\Delta_2) + \mathcal{J}_2(\Delta_1) - \mathcal{J}_2(\Delta_2)\|_{L+2,\,L+3} \\
&\leq \|\mathcal{J}_1(\Delta_1) - \mathcal{J}_1(\Delta_2)\|_{L+2,\,L+3} + \|\mathcal{J}_2(\Delta_1) - \mathcal{J}_2(\Delta_2)\|_{L+2,\,L+3} \\
&\leq (\mathcal{C} + \mathcal{D})\,\|\Delta_1 - \Delta_2\|_{L+2,\,L+3}.
\end{aligned}$$

Now, by Lemma 4.2 one has

$$\begin{aligned}
\|\Phi(\Delta_1) - \Phi(\Delta_2)\|_{L+1,\,L+2} &= \|\mathcal{S}^{-1}(\mathcal{J}(\Delta_1)) - \mathcal{S}^{-1}(\mathcal{J}(\Delta_2))\|_{L+1,\,L+2} \\
&= \|\mathcal{S}^{-1}(\mathcal{J}^x(\Delta_1)) - \mathcal{S}^{-1}(\mathcal{J}^x(\Delta_2))\|_{L+1} \\
&\quad + \|\mathcal{S}^{-1}(\mathcal{J}^y(\Delta_1)) - \mathcal{S}^{-1}(\mathcal{J}^y(\Delta_2))\|_{L+2} \\
&\le \Big(\rho + \frac{1}{\nu(L+1)}\Big) \|\mathcal{J}^x(\Delta_1) - \mathcal{J}^x(\Delta_2)\|_{L+2} \\
&\quad + \Big(\rho + \frac{1}{\nu(L+2)}\Big) \|\mathcal{J}^y(\Delta_1) - \mathcal{J}^y(\Delta_2)\|_{L+3} \\
&\le \Big(\rho + \frac{1}{\nu(L+1)}\Big) \|\mathcal{J}(\Delta_1) - \mathcal{J}(\Delta_2)\|_{L+2,\,L+3} \\
&\le \Big(\rho + \frac{1}{\nu(L+1)}\Big) (\mathcal{C} + \mathcal{D}) \|\Delta_1 - \Delta_2\|_{L+1,\,L+2}.
\end{aligned}$$

Then, as the constants $\mathcal{C}$ and $\mathcal{D}$ do not depend on L and do not increase when ρ decreases, one can fix the radius ρ of $S(\beta,\,\rho)$ small enough in order to get $\rho\,(\mathcal{C} + \mathcal{D}) < 1/2$, and also one can take L sufficiently large so that we have $\dfrac{(\mathcal{C} + \mathcal{D})}{\nu(L+1)} < 1/2$, and hence, it follows that Φ is a contraction. $\square$

Recall that L defines the degree of the polynomial approximation $\tilde{K}$, and so one has to compute this approximation to high enough order in order to obtain that Φ is a contraction. Also, it is necessary to fix previously the radius of the domain of definition of the function spaces $\mathcal{X}_p$. Note that these values of ρ and L depend only of the coefficients of the map F and that they can be computed easily.

From Theorem 4.1 we can show that with a proper choice of ρ and L, in fact Φ maps the ball $B^r_{L+1,\,L+2}$ into itself, and so Φ is an operator from $B^r_{L+1,\,L+2}$ to $B^r_{L+1,\,L+2}$. Indeed, if $\Delta \in B^r_{L+1,\,L+2}$, then one has, for some $Q < 1$,

$$\begin{aligned}
\|\Phi(\Delta)\|_{L+1,\,L+2} &\le \|\Phi(\Delta) - \Phi(0)\|_{L+1,\,L+2} + \|\Phi(0)\|_{L+1,\,L+2} \\
&\le Q\,\|\Delta\|_{L+1,\,L+2} + \|\Phi(0)\|_{L+1,\,L+2} \\
&= Q\,\|\Delta\|_{L+1,\,L+2} + \|\mathcal{S}^{-1}(E)\|_{L+1,\,L+2} \\
&\le Q\,\|\Delta\|_{L+1,\,L+2} + \Big(\rho + \frac{1}{\nu(L+1)}\Big)\|E\|_{L+2,\,L+3},
\end{aligned}$$

and the second term is as small as needed by the same argument as in Theorem 4.1, so we get

$$\|\Phi(\Delta)\|_{L+1,\,L+2} \le Q\,\|\Delta\|_{L+1,\,L+2} + \Big(\rho + \frac{1}{\nu(L+1)}\Big)\|E\|_{L+2,\,L+3} \le r,$$

that is,

$$\Phi(\Delta) \in B^r_{L+1,\,L+2}.$$

Thus, we can apply the Banach fixed point theorem to $\Phi : B^r_{L+1,\,L+2} \to B^r_{L+1,\,L+2}$ and so the existence of a solution to (4.29) is obtained. With this setting we can give now the main result of this section.

Theorem 4.2. *Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic map in a neighborhood U of the origin such that $F(0) = 0$. Suppose that the Taylor expansion of F around the origin is given by*

$$F(x,y) = \begin{pmatrix} x\, +\, y \\ y \end{pmatrix} + \begin{pmatrix} a_{20}\, x^2 + a_{11}\, xy + a_{02}\, y^2 + O(\|(x,\, y)\|^3) \\ b_{20}\, x^2 + b_{11}\, xy + b_{02}\, y^2 + O(\|(x,\, y)\|^3) \end{pmatrix},$$

with $b_{20} > 0$, and consider the polynomial

$$R(t) = t - \sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2a_{20} + b_{11} - \frac{5}{6} b_{20}\Big)\, t^3.$$

Suppose that there exists a formal polynomial $\tilde{K}(t) = (\tilde{K}^x(t),\, \tilde{K}^y(t))$ of degree $(M, M+1)$, with $\tilde{K}(0) = \tilde{K}'(0) = (0, 0)$ and $\tilde{K}''(0) = (2,\, 0)$, such that

$$F(\tilde{K}(t)) - \tilde{K}(R(t)) = \begin{pmatrix} O(t^{M+2}) \\ O(t^{M+3}) \end{pmatrix}.$$

Then, if M is sufficiently large, there exists a unique analytic map $K : I \subset \mathbb{R} \to \mathbb{R}^2$, where I is an interval of the form $(0,\, \rho) \subset \mathbb{R}$, satisfying

$$F(K(t)) = K(R(t)), \quad t \in I,$$

such that

$$K(t) - \tilde{K}(t) = \begin{pmatrix} O(t^{M+1}) \\ O(t^{M+2}) \end{pmatrix}.$$

Proof. Given F and R as in the statement, we need to see that there exists a solution $K : I \subset \mathbb{R} \to \mathbb{R}^2$, $0 \in I$, for the equation

$$F \circ K - K \circ R = 0.$$

Supposing that there exists the polynomial $\tilde{K}(t)$ given in the statement, one can write this equation as

$$\begin{aligned} &F \circ (\tilde{K} + \Delta) - (\tilde{K} + \Delta) \circ R = \\ &= E + (DF \circ \tilde{K})\, \Delta - \Delta \circ R + N(\tilde{K},\, \Delta), \end{aligned} \tag{4.31}$$

where $E = F \circ \tilde{K} - \tilde{K} \circ R$ and $N(\tilde{K},\, \Delta) = F \circ (\tilde{K} + \Delta) - F \circ \tilde{K} - (DF \circ \tilde{K})\, \Delta$. The unknown is now $\Delta = K - \tilde{K} = (O(t^{M+1}),\, O(t^{M+2}))$.

If we consider Δ as an element of the space $B^r_{M+1,\, M+2}$, then (4.31) can be seen as a functional equation,

$$\Delta = \mathcal{S}^{-1}\, [E + N(\tilde{K},\, \Delta) + (DF \circ \tilde{K})\, \Delta], \quad \Delta \in B^r_{M+1,\, M+2},$$

with $\mathcal{S}$ defined in (4.24), or equivalently,

$$\Delta = -\Phi\, (\Delta), \quad \Delta \in B^r_{M+1,\, M+2}, \tag{4.32}$$

with $\Phi : B^r_{M+1,\,M+2} \to B^r_{M+1,\,M+2}$ defined in (4.30).

Theorem 4.1 shows that Φ is a contraction mapping, and as $B^r_{M+1,\,M+2}$ is a complete metric space, the Banach fixed point theorem shows that (4.32) has a unique solution Δ^* in $B^r_{M+1,\,M+2}$. Then, the function given by $K(t) = \tilde{K}(t) + \Delta^*(t)$ is the one that we are looking for, and defines an analytic invariant curve under F, satisfying $K(0) = K'(0) = (0,0)$, $K''(0) = (2,0)$, and so $K(t)$ gives a conjugation between F restricted to the image of K and the map $R(t) = t - \sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2a_{20} + b_{11} - \frac{5}{6}b_{20}\Big)\, t^3$. □

This result has been presented as an *a posteriori* theorem, which means that one provides the existence of a mathematical object that has been previously estimated by some means, for instance numerically. This class of statements are usually the basis of computer assisted proofs, which consist on using a computer program to perform lengthy computations, and providing sufficiently approximated solutions so that one can use the *a posteriori* result to obtain the existence of an exact solution.

Hence, Theorem 4.2 ensures that a numerical computation of an invariant manifold using the algorithm described in Section 4.1 provides indeed a polynomial approximation of an invariant stable curve under F associated to the origin, and it also establishes that, once this approximation is fixed, the correction to obtain a parameterization of the invariant curve is unique.

It is clear, though, that one can give a result of existence of a stable manifold as a direct corollary of Theorem 4.2. Such a result is the one that we state now.

Theorem 4.3. *Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic map in a neighborhood U of the origin such that $F(0) = 0$. Suppose that the Taylor expansion of F around the origin is given by*

$$F(x,y) = \begin{pmatrix} x\, +\, y \\ y \end{pmatrix} + \begin{pmatrix} a_{20}\, x^2 + a_{11}\, xy + a_{02}\, y^2 + O(\|(x,\, y)\|^3) \\ b_{20}\, x^2 + b_{11}\, xy + b_{02}\, y^2 + O(\|(x,\, y)\|^3) \end{pmatrix},$$

with $b_{20} > 0$.

Then, there exists an analytic map $K : I \subset \mathbb{R} \to \mathbb{R}^2$, where I is an interval of the form $(0,\, \rho) \subset \mathbb{R}$, satisfying

$$F(K(t)) = K(R(t)), \quad t \in I,$$

with

$$R(t) = t - \sqrt{\frac{b_{20}}{6}}\, t^2 + \frac{1}{7}\Big(2a_{20} + b_{11} - \frac{5}{6}b_{20}\Big)\, t^3,$$

and $K(0) = K'(0) = (0,0)$, $K''(0) = (2,0)$.

That is, K is the parameterization of an analytic curve invariant under F and tangent to the $x-$axis, and the dynamics on K is conjugate to the map $R(t)$ in I, and so $K(t)$ is a stable manifold associated to the origin.

We have proved the existence of an invariant stable curve associated to a parabolic nilpotent point of an analytic map F, being the curve defined on the interval $(0,\, \rho)$, where ρ has been fixed to be small enough. Nevertheless, such an invariant curve is defined in a bigger domain.

As F is analytic in a neighbourhood of the fixed point, taking the successive preimages of the curve defined in $(0, \rho)$ we obtain a global invariant curve that will be analytic in the whole domain of analyticity of F.

4.3 Numerical estimates for the analytic stable curve

The aim of this section is to perform a numerical simulation in order to compute an approximation, $\tilde{K}$, of a stable curve, K, associated to a nilpotent parabolic fixed point using the algorithm described in Section 4.1. Our scope is to estimate numerically if the formal approximation of such a stable curve is a series of Gevrey type.

Definition 4.1. Given $\gamma > 0$, we say that a formal series of the form $\sum_{n=0}^{\infty} a_n t^n$ is $\gamma-$*Gevrey* if there exist positive constants C, D such that

$$|a_n| \leq C\, D^n\, (n!)^\gamma, \quad \forall\, n \in \mathbb{N}.$$

This class of series was first introduced and studied in [6], and in [1] and [2] the authors study the Gevrey properties of invariant parabolic curves of analytic maps.

In order to perform the numerical simulation, we have written a code in C language that, given a polynomial $F : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$F(x,y) = \begin{pmatrix} x + y + x^2 + xy + y^2 + O(\|(x,\, y)\|^3) \\ y + x^2 + xy + y^2 + O(\|(x,\, y)\|^3) \end{pmatrix}, \tag{4.33}$$

computes the coefficients of a polynomial approximation, $\tilde{K}$, of the stable invariant curve associated to the origin, which we name K, up to degree 300.

For the simulation we have chosen the following two maps,

$$F_1(x,y) = \begin{pmatrix} x + y + x^2 + xy + y^2 \\ y + x^2 + xy + y^2 \end{pmatrix},$$

and

$$F_2(x,y) = \begin{pmatrix} x + y + \sum_{n=0}^{2} x^{2-n} y^n + \cdots + \sum_{n=0}^{8} x^{8-n} y^n \\ y + 3\,x^2 \end{pmatrix}.$$

From Definition 4.1, one has that if a series $\sum_{n=0}^{\infty} a_n\, t^n$ is $\gamma-$Gevrey, then

$$\log|a_n| \leq \log C + n\, \log D + \gamma\, \log(n!),$$

and so

$$\frac{\log|a_n| - \log C}{\log(n!)} \leq \frac{n\, \log D}{\log(n!)} + \gamma,$$

which shows that γ can be bounded as

$$\gamma \geq \lim_{n\to\infty} \frac{\log|a_n|}{\log(n!)},$$

in the case that such a limit exists. Hence, the quantity $\gamma^* = \lim_{n\to\infty} \frac{\log|a_n|}{\log(n!)}$ is a lower bound of the Gevrey constant of the formal series. If $\gamma^* > 0$, then the coefficients $\{a_n\}_n$ grow in a factorial way, and in this case we say that the series is *strictly* of Gevrey type.

Our program computes the quantities

$$\alpha_n = \frac{\log|K_n^x|}{\log(n!)}, \quad \beta_n = \frac{\log|K_n^y|}{\log(n!)}, \quad n \leq 300, \tag{4.34}$$

where $\{K_n^x,\ K_n^y\}_n$ are the coefficients of each component of $\tilde{K}$.

In what follows we present an overview of the C code that we have written.

The program admits as input any function F of the form (4.33) with rational coefficients up to degree 10. From the coefficients of $F(x, y)$, it computes the coefficients of $R(t)$, namely

$$R_2 = 2\sqrt{\frac{b_{20}}{6}}, \quad R_3 = \frac{1}{7}\Big(2\,a_{20} + b_{11} - \frac{5}{6}\,b_{20}\Big),$$

and the coefficients K_n^x and K_{n+1}^y up to $n = 3$ with the formula given in (4.12). Recall that the value of K_3^x is free, so one has to chose it before running the program. We have performed the simulation with $K_3^x = 0$, $K_3^x = 1$ and $K_3^x = -1$ for both F_1 and F_2.

The program computes the next coefficients of $\tilde{K}$ solving the linear system given in (4.18), and recurrently, it computes K_n^x and K_{n+1}^y using the values K_{i-1}^x, K_i^x, for $i \leq n$. The main difficulty of the program is to compute the independent terms of such a system, $\hat{G}_{n+1}^x$ and $\hat{G}_{n+2}^y$, for every n. In order to compute these quantities, several functions are implemented.

Recall from Section 4.1 that $\hat{G}_{n+1}^x$ and $\hat{G}_{n+2}^y$ are the coefficients of the series expansion of $F \circ K - K \circ R$ of degree $n+1$ and $n+2$, respectively, which only contain coefficients of K up to K_{n-1}^x and K_n^y.

In order to obtain $\hat{G}_{n+1}^x$ and $\hat{G}_{n+2}^y$ the program computes separately an approximation of $F \circ K$ and $K \circ R$. To obtain the approximation of $F \circ K$, it computes all the powers of the form $(K^x(t))^m \cdot (K^y(t))^{(10-m)}$ for $m \leq 10$ and composes them with the expression of F. To obtain the approximation of $K \circ R$, all the powers of R up to order 300 are computed and composed with the expression of $\tilde{K}$ already known, that is, up to order $(n-1,\, n)$.

The program obtains the approximation $\tilde{K}$ up to degree 300 for any given value of K_3^x and computes the sequence of values $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ defined in (4.34). In order to estimate the Gevrey behavior of the invariant curve K, we are interested in the values of α_n and β_n for n large.

In Figures 3 and 4 we have represented the values of α_n and β_n, respectively, versus n, being F_1 the input function, and in Figures 3 and 4 we have represented the values of α_n and β_n, respectively, versus n, being F_2 the input function.

From the results plotted in the figures it appears that the values of α_n and β_n may tend respectively to some constants α and β as n tends to infinity. Hence, we suggest that the invariant curves associated to the origin for the given maps may be functions of Gevrey type. For the case of F_1 we have $\alpha, \beta \in (0.4, 0.5)$ and for the case of F_2 we have $\alpha, \beta \in (0.5, 0.6)$.

Observe that in all cases, the values of α_n and β_n, for n big enough, do not seem to depend on the initial value chosen for K_3^x. That is, different parameterizations of the same stable curve have the same Gevrey constant.

Also, it holds that for both F_1 and F_2 the limits of $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ appear to be the same quantity, that is, the Gevrey constants are the same for both components of K.

The fact that the obtained polynomial approximations for the invariant curves associated to F_1 and F_2 give functions of strictly Gevrey type shows that the series associated to these curves can not converge in any neighborhood of the origin, due to the factorial growth of the coefficients. This implies that the invariant curve associated to a nilpotent parabolic point given in Theorem 4.3 can not be a holomorphic function in any neighborhood of the origin, if considered as a function of a complex variable. This is indeed the reason for which in Section 4.2 one has to consider spaces of functions defined on a sector $S(\beta, \rho)$. Otherwise, it would not be possible to obtain an analytic function Δ satisfying the functional equation established in (4.29). If such an equation had as solution an holomorphic function in a neighbourhood of the origin, then the invariant curve $K = \tilde{K} + \Delta$ would not be of Gevrey type, in contradiction with the numerical results that we have obtained.

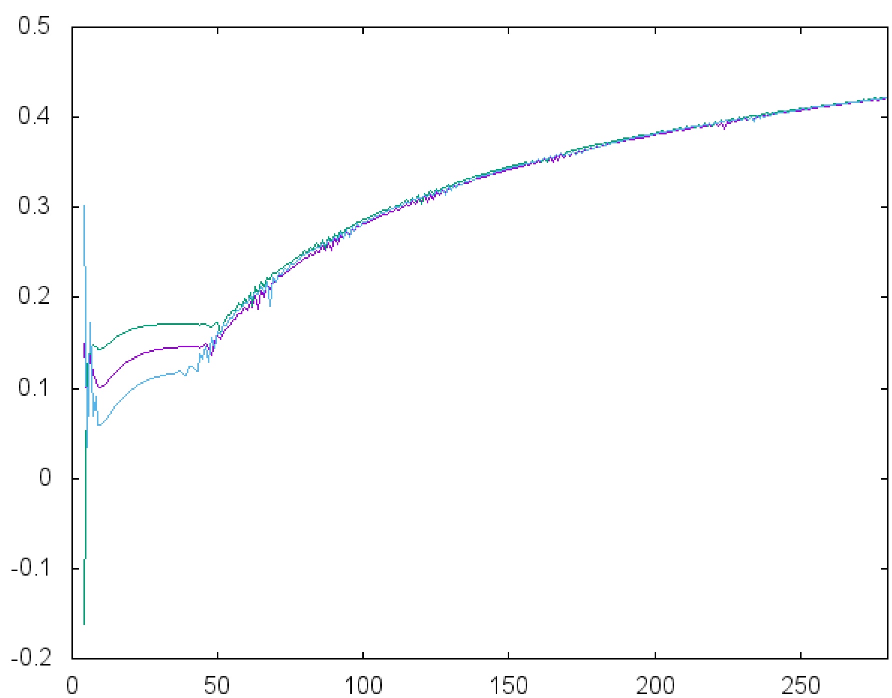

Figure 1: Representation of the constants α_n versus n for the map F_1. The three different plots correspond to the simulation starting with $K_3^x = 1$, $K_3^x = 0$ and $K_3^x = -1$.

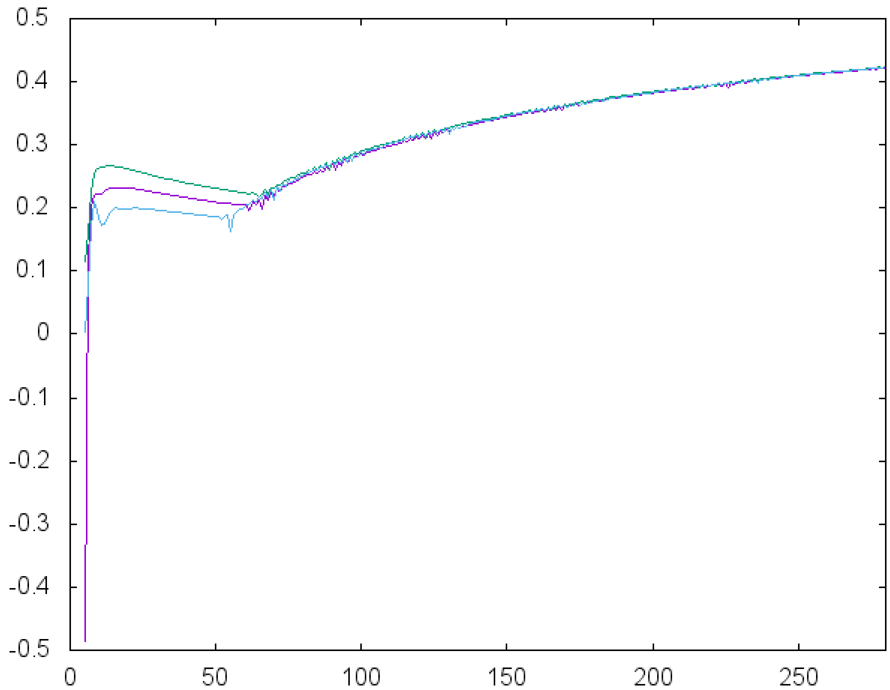

Figure 2: Representation of the constants β_n versus n for the map F_1. The three different plots correspond to the simulation starting with $K_3^x = 1$, $K_3^x = 0$ and $K_3^x = -1$.

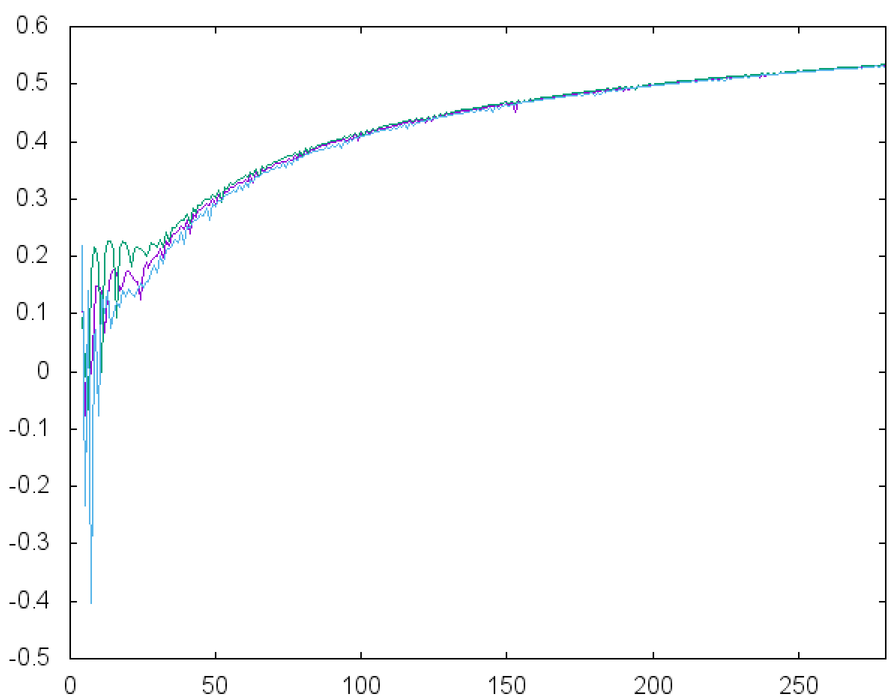

Figure 3: Representation of the constants α_n versus n for the map F_2. The three different plots correspond to the simulation starting with $K_3^x = 1$, $K_3^x = 0$ and $K_3^x = -1$.

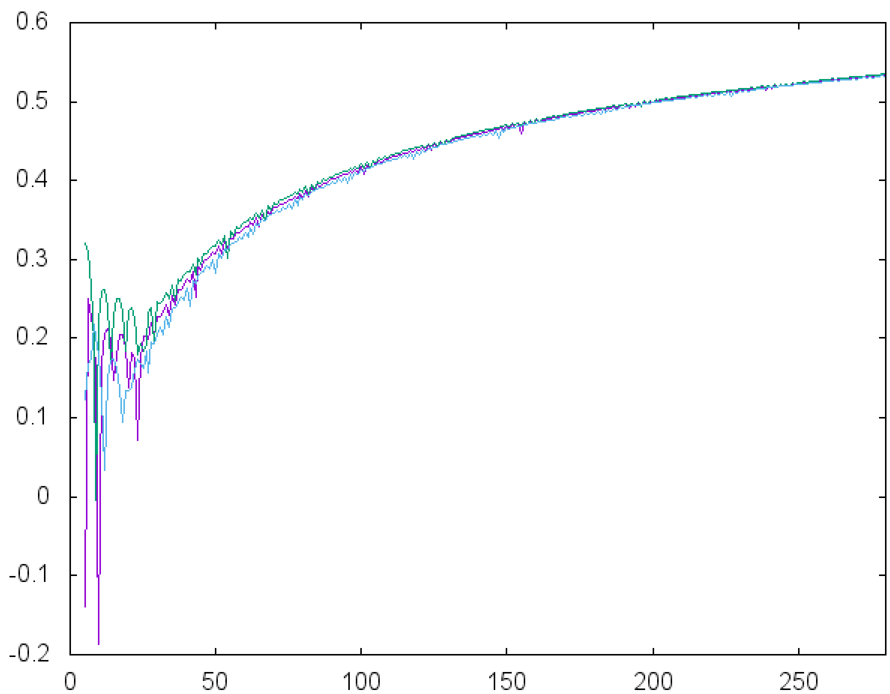

Figure 4: Representation of the constants β_n versus n for the map F_2. The three different plots correspond to the simulation starting with $K_3^x = 1$, $K_3^x = 0$ and $K_3^x = -1$.

References

[1] Baldomá, I., Fontich, E., Martín, P. Gevrey estimates for one dimensional parabolic invariant manifolds of non-hyperbolic fixed points. Discrete and Continuous Dynamical Systems, vol. 38 n. 8 (2017) pp. 4159-4190.

[2] Baldomá, I., Haro, À. One dimensional invariant manifolds of Gevrey type in real-analytic maps. Discrete and Continuous Dynamical Systems, Ser. B 10, n. 2-3 (2008), pp. 295–322.

[3] Cabré, X., Fontich, E., de la Llave, R. The parameterization method for invariant manifolds III: overview and applications. Journal of Differential Equations 218 (2005), pp. 444-515.

[4] Dieudonné, J. Élements d'analyse. Fondements de l'analyse moderne. (French) Cahiers scientifiques, Gauthier-Villars éditeur, 1972.

[5] Fontich, E. Stable curves asymptotic to a degenerate fixed point. Nonlinear Analysis 35 (1999), pp. 711-733.

[6] Gevrey, M. Sur la nature analytique des solutions des équations aux dérivées partielles. Premier mémoire. (French) Annales Scientifiques de l'École Normale Supérieure (3) 35 (1918), pp. 129–190.

[7] Luque, A. El mètode de la parametrització en sistemes dinàmics. (Catalan) Butlletí de la Societat Catalana de Matemàtiques, vol 27, n. 1 (2012), pp 5-38.

[8] Meyer, K. R. The implicit function theorem and analytic differential equations. Dynamical systems - Warwick 1974; pp. 191–208. Lecture Notes in Mathematics, Vol. 468, Springer, Berlin, 1975.

[9] Palis, J., de Melo, W. Geometric theory of dynamical systems. An introduction. Springer-Verlag, New York-Berlin, 1982.