

# Cyclicity of a fake saddle inside the quadratic vector fields

P. De Maesschalck<sup>a</sup>, S. Rebollo-Perdomo<sup>b</sup>, J. Torregrosa<sup>c,\*</sup>

<sup>a</sup>*Hasselt University, Martelarenlaan 42, B-3500 Hasselt, Belgium*

<sup>b</sup>*Centre de Recerca Matemàtica, 08193 Bellaterra, Barcelona, Spain*

<sup>c</sup>*Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Spain*

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## Abstract

This paper concerns the study of small-amplitude limit cycles that appear in the phase portrait near an unfolded fake saddle singularity. This degenerate singularity is also known as a impassable grain. The normal form of the unperturbed vector field is like a degenerate flow box. Near the singularity, the phase portrait consists of parallel fibers, all of which but one have no singular points, and at the singular fiber, there is one node. We study different techniques in order to show that the cyclicity is bigger or equal than two when the normal form is quadratic.

*Keywords:* Cyclicity, Fake saddle, Impassable grain, Degenerate critical point, Limit cycle, Bifurcation, Singularity unfolding.

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## 1. Introduction

This paper concerns the study of small-amplitude limit cycles that appear in the phase portrait near an unfolded degenerate singularity. More specifically, we assume that the unperturbed vector field can be put in a normal form that is like a degenerate flow box: near the singularity, the phase portrait consists of parallel fibers, all of which but one have no singular points, and the singular fiber has a semi-stable equilibrium point. This singularity is known as a fake saddle or a impassable grain, see [23]. In fact, it is a singularity with exactly two saddle sectors.

Though the paper deals with more general vector fields in local normal form, to present the ideas, think of the following typical model:

$$X_0: \{\dot{x} = 0, \quad \dot{y} = x^2 + y^2\}.$$

whose local phase portrait is shown in Figure 1.

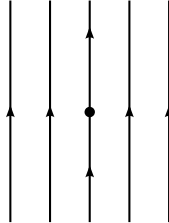


Figure 1: Phase portrait of  $X_0$ .

In any unfolding, the fibers  $\{x = \text{const}\}$  away from the origin will smoothly perturb to fibers without singular points. Close to the origin, more complicated phenomena may occur: we show the presence of Hopf bifurcations, Bogdanov–Takens bifurcations, slow-fast (canard) behavior, homoclinic and heteroclinic orbits. All the above phenomena are well-known mechanisms near which limit cycles can be born, and in fact the study of periodic orbits near the degenerate point is the principle goal of this paper. We use the aforementioned mechanisms to show the presence of up to two small amplitude limit cycles, and give evidence that by using these mechanisms this is the best one can get.

Determining an upper bound for the number of limit cycles turned out to be too difficult, as it revealed that a multi-parameter global study of phase portraits was needed, going far beyond the traditional perturbative methods to create limit cycles.

In a study of unfoldings of a vector field like  $X_0$ , it is best to make a homogeneous (family) blow-up of the perturbed family of vector fields, thereby focusing on the behavior at the blow-up locus. The behavior at the blow-up locus is shown to be mostly determined by perturbation terms of degree two and lower. *We will therefore focus our attention to perturbations of at most degree two.* Though this restriction immediately shows a relation between the Hilbert 16th problem in degree 2, the study of the singularity at  $X_0$  has in fact no contribution in the degree-2 programme outlined by Dumortier, Roussarie and Rousseau [13]: in that programme, homogeneous vector fields could be avoided using rescalings. Setting this remark aside, the study of the cyclicity of  $X_0$  at the origin has a relevance on its own.

In Section 2 we consider forms for the unperturbed system under some additional generic and geometric constraints, and show that the normal form depends on two parameters  $(A, B)$ . Next, we consider normal forms for unfoldings. It is shown that, up to the quadratic part, any such unfolding can be written as a 6-parameter family of vector fields: two parameters altering  $(A, B)$ , and 4 additional parameters  $(\mu_1, \mu_2, \mu_3, \mu_4)$ . We infer that the study of any smooth normal form for the singularity and its unfolding will most dominantly be affected by its quadratic part. More precisely, we prove that the normal form for

unfoldings is

$$\begin{cases} \dot{x} &= ax^2 + bxy + \mu_1 + \mu_2x + \mu_3y + O(\|(x, y)\|^3), \\ \dot{y} &= x^2 + y^2 - \mu_4 + O(\|(x, y)\|^3), \end{cases} \quad (1)$$

where  $a = A + o(1)$  and  $b = B + o(1)$ .

Next, we restrict to quadratic vector fields and present several results on the existence of limit cycles using perturbative arguments. In fact the maximum number of limit cycles obtained in this way is two, with configurations  $(2 : 0)$  and  $(1 : 1)$ . In Section 3 we study the cyclicity and simultaneity properties near isolated singularities. First perturbing weak foci by computing Lyapunov coefficients, secondly studying the presence of cusp points and their unfolding in a Bogdanov–Takens bifurcation diagram (in fact we prove the simultaneous existence of two BT-diagrams). A characterization of the centers of (1) restricted to quadratic case can be found in Section 4. The quadratic perturbations of some centers included in these families is studied by many authors. The Hamiltonian case is studied in [19] but the reversible non-Hamiltonian case only have been considered in few particular cases, see [3, 5, 16, 20], and in all cases the cyclicity is two. A short review of them is also given in this section. In Section 5, we study slow-fast families of vector fields appearing in the model.

In a final step, inspired by [5], we consider in Section 6 a class of symmetric unfoldings of the singular point, allowing us to reduce the dimension of the parameter space. As the unperturbed vector field is invariant under the transformation  $(x, y, t) \mapsto (-x, -y, -t)$ , we take the perturbations which are invariant under this transformation as well. Thus  $\mu_2 = \mu_3 = 0$ , and the restricted quadratic family (1) can be written as

$$\begin{cases} \dot{x} &= ax^2 + bxy + \mu, \\ \dot{y} &= x^2 + y^2 - 1. \end{cases} \quad (2)$$

We prove that this family has at most two limit cycles in configuration  $(1 : 1)$ . We show the limitations of the perturbative techniques as they fail to provide a global bifurcation diagram for the number of limit cycles even for this simple family of vector fields.

## 2. Local normal forms

### 2.1. Normal form of the unperturbed fake saddle

We consider a smooth vector field  $X_0$  having a smooth invariant curve  $x = \phi(y)$  (with  $\phi(0) = 0$ ), and for which the reduction of  $X_0$  to this curve is given by the equation

$$\dot{y} = cy^2 + O(y^3), \quad c > 0.$$

The conditions in the following lemma determine precisely the kind of singularity we examine in this paper.

**Lemma 1.** *Assume, under the above condition, that the origin is a degenerate singular point having exactly two separatrices, both of which are the boundary of two hyperbolic sectors. Then there exists a smooth local change of coordinates bringing the vector field in the form*

$$\begin{cases} \dot{x} &= Ax^2 + Bxy + O(\|(x, y)\|^3), \\ \dot{y} &= x^2 + y^2 + O(\|(x, y)\|^3), \end{cases} \quad (3)$$

where  $A \geq 0$ ,  $B < 1$  and  $A^2 < 4(1 - B)$ . (When the invariant fiber is a straight line, the change of coordinates is linear.)

*Proof.* We assume that  $X_0 = (P, Q)$  has a degenerate singular point at the origin, implying that  $P(0, 0) = Q(0, 0) = 0$ . We also assume that the degenerate point has no other separatrices beside the ones on the fiber  $y = 0$ , and that  $\{y = 0\}$  separates two hyperbolic sectors. Since the origin is a degenerate singularity, a blow-up analysis reveals the nature of singular points. We write

$$(x, y) = (r \cos \theta, r \sin \theta).$$

Imposing that  $\{y = 0\}$  is the border of a hyperbolic sector implies that  $P_y(0, 0) < 1$ . (This is done by requiring the Jacobian matrix of the blow-up vector field at  $(r, \theta) = (0, \pm\pi/2)$  to have a saddle structure.) Under this condition, we can apply a linear transformation  $(x, y) \mapsto (x, \rho x + y)$  to make  $Q_y(0, 0) = 0$ . Furthermore, we can then exclude the case  $Q_x(0, 0) = 0$  from the study (since there are always extra separatrices then), after which we can scale the coefficient to 1. In short, we can assume

$$P(x, y) = Ax^2 + Bxy, \quad Q(x, y) = x^2 + y^2.$$

Additional search for separatrices leads to the property that  $A^2 < 4(1 - B)$ . Symmetry allows us to assume  $A \geq 0$ .  $\square$

As a special case, we will sometimes consider those vector fields (3) that are invariant under the symmetry  $(x, y, t) \mapsto (-x, -y, -t)$ . The quadratic part is of course always invariant under the symmetry, but for some vector fields the higher order terms can break symmetry.

The vector field (3) has the so-called *degenerate flow-box property*: there exists two boxes  $B_i \subset B_e$  (interior and exterior) with the following properties. Each of the boxes are diffeomorphic to a square, and its four boundaries consist of two orbits, an inset and an outset. Along the inset, the vector field is transverse and points inwards the box, and along the outside, the vector field is transverse and points outwards. The orbit-edges of  $B_i$  reach the outset of  $B_e$  in positive time and the inset of  $B_e$  in negative time. Furthermore,  $B_e \setminus B_i$  has no singular points, and  $B_i$  can be chosen arbitrarily small around the origin.

**Lemma 2.** *Any smooth perturbation of vector field (3) retains the degenerate flow-box property.*

*Proof.* The boxes consist of regular orbits, which perturb to regular orbits, and insets and outsets. The transversality along the inset and outset also persists for small perturbations.  $\square$

Of course, the most interesting thing to study is what happens inside the interior box  $B_i$  for perturbations of (3). In the next section, we study unfoldings.

## 2.2. Normal forms for unfoldings of the fake saddle

We consider now a perturbation

$$\begin{cases} \dot{x} &= Ax^2 + Bxy + \varepsilon P(x, y), \\ \dot{y} &= x^2 + y^2 + \varepsilon Q(x, y). \end{cases}$$

We let

$$x = X + pY + q, \quad y = Y + rX + s,$$

where  $p, q, r, s$  are to be determined implicitly below and keeping in mind all 4 ought to be  $O(\varepsilon)$ . For the new equations in  $(X, Y)$ , we will require that

$$\frac{\partial \dot{Y}}{\partial X}(0, 0) = \frac{\partial \dot{Y}}{\partial Y}(0, 0) = \frac{\partial^2 \dot{Y}}{\partial X \partial Y}(0, 0) = \frac{\partial^2 \dot{X}}{\partial Y^2}(0, 0) = 0.$$

Considering the mapping  $\Psi: (p, q, r, s, \varepsilon) \mapsto (\frac{\partial \dot{Y}}{\partial X}(0, 0), \dots, \frac{\partial^2 \dot{X}}{\partial Y^2}(0, 0))$ , then it is clear that  $\Psi(0, 0, 0, 0, 0) = (0, 0, 0, 0)$ . On the other hand, it is a tedious but easy exercise to show that

$$\frac{\partial \Psi}{\partial (p, q, r, s)}(0) = \begin{vmatrix} 2(B-1) & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 2-B & 0 \end{vmatrix} = 8(B-1)(B-2),$$

so we can apply the Implicit Function Theorem to prepare the perturbation in the required form. This implies that we may consider

$$\begin{cases} \dot{x} &= ax^2 + bxy + \mu_1 + \mu_2 x + \mu_3 y + O(\|(x, y)\|^3), \\ \dot{y} &= x^2 + y^2 - \mu_4 + O(\|(x, y)\|^3), \end{cases}$$

where  $a = A + o(1)$ ,  $b = B + o(1)$ , and  $a^2 < 4(1 - b)$ .

We now write

$$(\mu_1, \mu_2, \mu_3, \mu_4) = (v^2 M_1, v M_2, v M_3, v^2 M_4), \quad v \geq 0.$$

In a perturbation scheme where  $\mu \approx 0$ , we can keep  $\|(M_1, M_2, M_3, M_4)\| = 1$  and let  $v \approx 0$ .

**Lemma 3.** *The  $((v, M_1, M_2, M_3, M_4)$ -families of) boxes  $B_i \subset B_o$  in  $(x, y)$ -space can be chosen so that  $B_i$  lies in a  $O(v)$ -neighbourhood of the origin. (In the language of blow-up: in a family blow-up procedure, the interior box can be chosen as a compact  $\mathcal{K}$  in the family chart.)*

*Proof.* Use phase directional blow-up and the fact that the equator only contains singularities at the poles.  $\square$

We now blow up the origin using a homogeneous family blow-up and consider the phase directional chart:

$$(x, y, v) = (vX, vY).$$

We find

$$\begin{cases} \dot{X} &= aX^2 + bXY + M_1 + M_2X + M_3Y + O(v), \\ \dot{Y} &= X^2 + Y^2 - M_4 + O(v). \end{cases} \quad (4)$$

**Remark 4.** *It is clear that the behaviour for  $v = 0$  is the important piece of information needed to determine the behaviour of (4). Furthermore, the vector field for  $v = 0$  is a normal form in the space of quadratic perturbations. In addition, the system (4) with  $v = 0$  has no singular points for  $M_4 < 0$ , and for  $M_4 = 0$ , it has a unique singular point at the origin, whose index is zero. Therefore in both cases the restricted system has no limit cycles.*

### 2.3. Parameter charts of the quadratic normal form

The normal form obtained from (4), after reducing to  $v = 0$ , that can have limit cycles is

$$\begin{cases} \dot{X} &= aX^2 + bXY + M_1 + M_2X + M_3Y, \\ \dot{Y} &= X^2 + Y^2 - M_4, \end{cases} \quad (5)$$

with  $M_4 \geq 0$ .

Recall also that  $(M_1, M_2, M_3, M_4)$  lies on a sphere and cannot be zero simultaneously. It is important to realize that the local problem that was initially posed in this paper, i.e. study the degenerate singular point, now has become a *global problem, both in phase space and parameter space*. The control on the number of limit cycles is typically not easy in this situation.

Removing the capitals in the notation, we have reduced to the global study of the following family of vector fields:

$$\begin{cases} \dot{x} &= ax^2 + bxy + \mu_1 + \mu_2x + \mu_3y, \\ \dot{y} &= x^2 + y^2 - \mu_4, \end{cases} \quad (6)$$

where  $(x, y)$  is to be considered in a large compact set in the plane, and where the parameters  $(\mu_1, \mu_2, \mu_3, \mu_4)$  lie on a sphere and cannot be zero simultaneously, and  $(a, b) \in \Omega := \{(a, b) \in \mathbb{R}^2 \mid a \geq 0, a^2 + 4b - 4 < 0\}$ , see Figure 2.

In some cases instead of working on a sphere, we prefer to work in one of the charts of the sphere:  $\mu_4 = 1$  and  $(\mu_1, \mu_2, \mu_3) \subset K \subset \mathbb{R}^3$ , that is, we can consider the system

$$\begin{cases} \dot{x} &= ax^2 + bxy + \mu_1 + \mu_2x + \mu_3y, \\ \dot{y} &= x^2 + y^2 - 1, \end{cases} \quad (7)$$

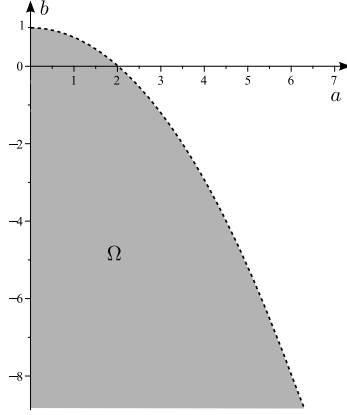


Figure 2: Region of the parameters  $a$  and  $b$ .

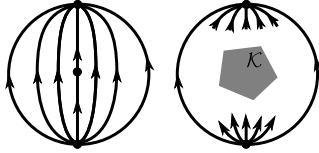


Figure 3: Phase portrait of system (6) for the unperturbed one (left) and near the infinity (right).

where  $(x, y)$  lies in a large compact set in the plane,  $(a, b) \in \Omega$ , and  $(\mu_1, \mu_2, \mu_3) \subset K \subset \mathbb{R}^3$  with  $K$  a large compact set as well. By choosing to work in this chart, we avoid the situation where  $\mu_4 \approx 0$  and  $(\mu_1, \mu_2, \mu_3)$  lies on a unit sphere. This choice is made for the sake of convenience and because it is to be expected that no additional phenomena are present there. With techniques similar to the ones proposed here, and by using extra desingularizations (blow-ups), a comprehensive study of the parameter chart  $\mu_4 \approx 0$  is equally possible.

Following the techniques described in [11] the phase portraits near the infinity of the unperturbed and perturbed system (6) is drawn in Figure 3. In particular the limit cycles will appear in a compact region  $\mathcal{K}$  in the phase space, in correspondence with the findings in Lemma 3. Moreover the total index of finite singularities is zero because the index of the fake saddle singularity of the unperturbed system (6) is also zero.

### 3. Limit cycles bifurcating from singularities

This section is devoted to the research of limit cycles that bifurcate from singularities. From previous section we know that the total index of the finite

singularities is 0. This research is divided in two subsections. In the former we will study the maximum order of a weak focus (index +1). The latter deals with the study of Bogdanov–Takens bifurcation (index 0). In both cases we first study each point separately and afterwards the simultaneous bifurcation.

### 3.1. Bifurcation from weak foci

Let  $X$  be the vector field associated to system (6). Then its linearization matrix at a point  $(x, y)$  is

$$DX(x, y) = \begin{pmatrix} 2ax + by + \mu_2 & bx + \mu_3 \\ 2x & 2y \end{pmatrix}.$$

Thus, the trace and the determinant of  $DX(x, y)$  are denoted by

$$\text{tr } DX(x, y) = 2(ax + by) + (2 - b)y + \mu_2, \quad (8)$$

$$\det DX(x, y) = 4y(ax + by) - 2b(x^2 + y^2) - 2\mu_3x + 2\mu_2y. \quad (9)$$

**Lemma 5.** *For a weak focus  $(x_0, y_0)$  of system (6) we have:*

- (i) *The value  $x_0$  is nonzero.*
- (ii) *The first Lyapunov quantities are given by*

$$V_3 = -\frac{v_{30} - v_{31}a - v_{32}a^2}{8x_0\alpha_0^5}, \quad (10)$$

$$V_5 = -\frac{2x_0y_0v_{50} + v_{51}a + x_0y_0v_{52}a^2 - v_{53}a^3}{16x_0^3\alpha_0^7}, \quad (11)$$

where  $\alpha_0$  is the positive square root of  $\det DX(x_0, y_0)$ , and

$$\begin{aligned} v_{30} &= 2x_0y_0(b+2)(4(1-b)x_0^2 + \alpha_0^2 + 4y_0^2), \\ v_{31} &= (\alpha_0^2 + 4y_0^2)^2 - 2b(12y_0^2 + \alpha_0^2)x_0^2, \\ v_{32} &= 4x_0y_0(\alpha_0^2 + 4y_0^2), \\ v_{50} &= (2x_0^2 + 4y_0^2 + \alpha_0^2)(4x_0^2 + 4y_0^2 + \alpha_0^2)(b+2), \\ v_{51} &= 8(b-2)(b-3)x_0^6 + (112(2-b)y_0^2 + 2\alpha_0^2(b^2 + 10 - 9b))x_0^4 \\ &\quad - 2(4y_0^2 + \alpha_0^2)(\alpha_0^2(1+b) - (13b+14)y_0^2)x_0^2 - (4y_0^2 + \alpha_0^2)^3, \\ v_{52} &= 48bx_0^4 - ((32 - 152b)y_0^2 + 2\alpha_0^2(4-b))x_0^2 - 13(4y_0^2 + \alpha_0^2)^2, \\ v_{53} &= 12x_0^2(x_0^2 + 3y_0^2)(4y_0^2 + \alpha_0^2). \end{aligned}$$

*Proof.* When  $x_0 = 0$  we have

$$\begin{aligned} \text{tr } DX(0, y_0) &= \mu_2 + by_0 + 2y_0 = 0, \\ \det DX(0, y_0) &= 2y_0(by_0 + \mu_2) = -4y_0^2 \leq 0. \end{aligned}$$



Consequently  $(0, y_0)$  is not a weak focus (see [24]). This proves (i).

For proving statement (ii), we will first use a suitable linear transformation to bringing system (6) in its Poincaré normal form.

By hypothesis  $\text{tr } DX(x_0, y_0) = 0$  and  $\det DX(x_0, y_0) = \alpha_0^2$  for a value  $\alpha_0 > 0$ . Hence, from (8) and (9) it follows that

$$\mu_2 = -2(ax_0 + by_0) - y_0(2 - b) \quad \text{and} \quad \mu_3 = -\frac{2bx_0^2 + \alpha_0^2 + 4y_0^2}{2x_0}.$$

On the other hand, since  $(x_0, y_0)$  is a singular point, from (6) we can easily see that

$$\mu_1 = \frac{2x_0^2(ax_0 + 2y_0) + 2bx_0^2 + \alpha_0^2 + 4y_0^2}{2x_0} \quad \text{and} \quad \mu_4 = (x_0^2 + y_0^2).$$

By using the linear transformation

$$u = \frac{x_0(x - x_0) + y_0(y - y_0)}{x_0}, \quad v = \frac{\alpha_0(y - y_0)}{2x_0}$$

and the time reparametrization  $d\tau = \alpha_0 dt$ , the system (6) becomes

$$\begin{cases} u' &= -v + a_{20}u^2 + a_{11}uv + a_{02}v^2, \\ v' &= u + b_{20}u^2 + b_{11}uv + b_{02}v^2, \end{cases} \quad (12)$$

where the prime denotes the derivative with respect to  $\tau$ , and

$$\begin{aligned} a_{20} &= \frac{ax_0 + y_0}{x_0 \alpha_0}, & a_{11} &= \frac{2bx_0^2 - 4y_0(y_0 + ax_0)}{x_0 \alpha_0^2}, \\ b_{20} &= \frac{1}{2x_0}, & a_{02} &= \frac{4y_0(y_0(y_0 + ax_0) + x_0^2(1 - b))}{x_0 \alpha_0^3}, \\ b_{11} &= \frac{2y_0}{x_0 \alpha_0}, & b_{02} &= \frac{2(x_0^2 + y_0^2)}{x_0 \alpha_0^2}. \end{aligned}$$

The origin of this system corresponds to the point  $(x_0, y_0)$  of (6).

Finally, the expressions of the Lyapunov quantities  $V_3$  and  $V_5$  given in (10) and (11) follows after rewriting system (12) as

$$z' = iz + Az^2 + Bz\bar{z} + C\bar{z}^2,$$

(with  $z := u + iv$ ), applying [25, Proposition 1.2] and simplifying  $V_5|_{V_3 \equiv 0}$ .  $\square$

**Theorem 6.** *Let  $(x_0, y_0)$  be a weak focus of the perturbed system (6).*

(i) *if  $a \neq 0$ , then  $(x_0, y_0)$  is of order at most 2.*

(ii) *if  $a = 0$ , then  $(x_0, y_0)$  is of order at most 1 if  $y_0 \neq 0$  and  $b \neq -2$ ; otherwise  $(x_0, y_0)$  is a center.*

*Proof.* (i) It sufficient to show that if  $V_3 = 0$ , then  $V_5 \neq 0$ . This follows if the resultant of the numerators of  $V_3$  and  $V_5$  respect to  $y_0$  is a non-vanishing function. This resultant factors as

$$\text{Res}(V_3, V_5, y_0) = a^{10} \alpha_0^{16} x_0^{20} (b-2)^2 R_1 R_2^2 R_3,$$

where

$$R_1 = 4(a^2 + b^2)x_0^2 + b^2\alpha_0^2, \quad R_2 = 2a^2b + (b-1)(b+2)^2$$

and

$$R_3 = 5(3-b)(3(b+1)a^2 + 4(2b-1)^2)x_0^2 + \alpha_0^2(10b-5+3a^2)^2.$$

By the assumption  $a > 0$ , and the fact  $x_0 \neq 0$  (Lemma 5), it is clear that  $R_1 > 0$ . To complete the proof we will prove that  $R_2$  and  $R_3$  do not vanish in  $\Omega$ .

If  $(a, b) \in \Omega$ , then  $a^2 < -4(b-1)$ . Thus,  $R_2 < (b-1)(b-2)^2 < 0$  in  $\Omega$  because  $b-1 < 0$  and  $b-2 < -1$  in  $\Omega$ .

Showing that  $3(b+1)a^2 + 4(2b-1)^2$  and  $3-b$  are positive in  $\Omega$ , is sufficient to show that  $R_3 > 0$  in  $\Omega$  because  $x_0 \neq 0$ . If  $(a, b) \in \Omega$ , then it is clear that  $3-b > 0$ . Moreover, if  $b \geq -1$ , then  $b+1 \geq 0$ , which implies that  $3(b+1)a^2 + 4(2b-1)^2 > 0$ ; and if  $b < -1$ , then  $-4ba^2 < -16b(b-1)$ , whereby

$$3(b+1)a^2 + 4(2b-1)^2 = 3(b+1)a^2 - 16b(1-b) + 4 > 3a^2 - ba^2 > 0.$$

(ii) We consider the equivalent system (12). For  $a = 0$ , the Lyapunov quantity  $V_3$  simplifies to

$$V_3 = \frac{y_0(2+b)(4(1-b)x_0^2 + \alpha_0^2 + 4y_0^2)}{4\alpha_0^5}.$$

Since in  $\Omega$  we have  $1-b > 0$ , the polynomial  $4(1-b)x_0^2 + \alpha_0^2 + 4y_0^2$  is positive because  $\alpha_0 > 0$ . Hence, as we are assuming that  $y_0 \neq 0$  and  $b \neq -2$  we can conclude that  $V_3 \neq 0$ . Therefore, the origin of (12) or equivalently the weak focus  $(x_0, y_0)$  of (6) is of order at most 1.

To complete the proof we will prove that if  $y_0 = 0$  or  $b = -2$ , then  $(x_0, y_0)$  is a center. We will consider two cases.

*Case 1.* Suppose  $y_0 = 0$ . From the expression of (6) we get that  $\mu_1 + \mu_2 x_0 = 0$  because  $a = 0$ . On the other hand, from (8) we obtain  $\text{tr } DX(x_0, 0) = \mu_2$ , which is zero because  $(x_0, 0)$  is a weak focus. Thus, we obtain  $\mu_1 = \mu_2 = 0$ . Hence the system (6) becomes

$$\begin{cases} \dot{x} &= y(bx + \mu_3), \\ \dot{y} &= x^2 + y^2 - \mu_4. \end{cases} \quad (13)$$

Thus, from (9) it follows that  $\det DX(x_0, 0) = -2(bx_0^2 + \mu_3 x_0)$  which must be positive because  $(x_0, 0)$  is a weak focus. Hence  $b^2 + \mu_3^2 \neq 0$ . We will prove that this system has a first integral.

When  $b = 0$  we can assume that  $\mu_3 \neq 0$  and  $e^{-2x/\mu_3}$  is an integrating factor that provides

$$\frac{\mu_3 (2\mu_4 - 2y^2 - \mu_3^2 - 2x\mu_3 - 2x^2)}{4e^{2x/\mu_3}}$$

as a first integral, which has either a minimum or a maximum at  $(x_0, 0)$ . Therefore  $(x_0, 0)$  is a center of the system.

If  $b \neq 0$ , then  $(bx + \mu_3)^{-(b+2)/b}$  is an integrating factor and, consequently,

$$\frac{(b-2)(x^2 + (y^2 - \mu_4)(1-b)) - \mu_3^2 - 2x\mu_3}{2(b-1)(b-2)(bx + \mu_3)^{\frac{2}{b}}}$$

is a first integral. Hence, if  $(x_0, 0)$  is a weak focus for the system, then the first integral has either a minimum or a maximum at  $(x_0, 0)$  which implies that  $(x_0, 0)$  is a center.

Case 2. Suppose  $b = -2$ . From (8) we get  $\text{tr } DX(x_0, y_0) = \mu_2$ , which must be zero by the assumption on  $(x_0, y_0)$ . Thus (6) becomes

$$\begin{cases} \dot{x} &= -2xy + \mu_1 + \mu_3y, \\ \dot{y} &= x^2 + y^2 - \mu_4, \end{cases} \quad (14)$$

which is a Hamiltonian system with Hamiltonian function

$$H(x, y) = \frac{x^3}{3} + xy^2 - \frac{\mu_3}{2}y^2 - \mu_4x - \mu_1y.$$

Therefore, if  $(x_0, y_0)$  is a weak focus of the system, it is a center.  $\square$

**Corollary 7.** *If  $a \neq 0$  then the perturbed system (6) has no centers bifurcating from the origin of the unperturbed system.*

**Theorem 8.** *If system (6) has two simultaneous weak foci, then each one of them has order at most one.*

*Proof.* If  $a = 0$ , then the assumption follows from statement (ii) of Theorem 6. Hence, for the rest of the proof we assume that  $a \neq 0$ .

Let  $(x_0, y_0)$  and  $(x_1, y_1)$  be two different weak foci of system (6). As they are singular points, and from (8), (9) we have

$$\begin{aligned} ax_i^2 + bx_iy_i + \mu_2x_i + \mu_3y_i &= 0, \\ x_i^2 + y_i^2 - \mu_4 &= 0, \\ \mu_2 - (2ax_i + (b+2)y_i) &= 0, \\ \alpha_i^2 - 2x_i\mu_3 + 2bx_i^2 + 4y_i^2 &= 0, \end{aligned}$$

where  $\alpha_i = \det DX(x_i, y_i) > 0$  for  $i = 0, 1$ . From these expressions we can write  $\mu_1, \mu_2, \mu_3, \mu_4, x_1, y_1, \alpha_0$ , and  $\alpha_1$  as functions of  $(x_0, y_0, a, b)$ . In particular we get

$$\begin{aligned} x_1 &= \frac{(4a^2 - (b+2)^2)x_0 + 4a(b+2)y_0}{4a^2 + (b+2)^2}, \\ y_1 &= \frac{4a(b+2)x_0 - (4a^2 - (b+2)^2)y_0}{4a^2 + (b+2)^2}. \end{aligned}$$

We denote by  $V_{j0}$  and  $V_{j1}$  the Lyapunov quantities of the weak focus  $(x_0, y_0)$  and  $(x_1, y_1)$ , respectively. We note that the expression of  $V_{j1}$  for  $j = 3, 5$  comes from (10) and (11) in Lemma 5 if we replace  $x_0, y_0$ , and  $\alpha_0$  by  $x_1, y_1$ , and  $\alpha_1$ , respectively. In particular,

$$V_{30} = \frac{\tilde{V}x_0((b+2)x_0 - 2ay_0)^2}{\alpha_0^5(4a^2 + (b+2)^2)^2}$$

and

$$V_{31} = \frac{\tilde{V}((4a^2 - (b+2)^2)x_0 + 4a(b+2)y_0)((b+2)x_0 - 2ay_0)^2}{\alpha_1^5(4a^2 + (b+2)^2)^3}$$

where  $\tilde{V} = -a(b-2)(2ba^2 + 3b^2 + b^3 - 4)$ .

We note that if  $(b+2)x_0 - 2ay_0 = 0$ , then that implies that  $y_1 = y_0$  and  $x_1 = x_0$ , which contradicts our assumption  $(x_0, y_0) \neq (x_1, y_1)$ . Hence we have  $(b+2)x_0 - 2ay_0 \neq 0$ .

On the other hand, it is easy to see that the zero locus of  $\tilde{V}$  is outside of  $\Omega \setminus \{a = 0\}$ , and we know that  $x_0 \neq 0$  and  $x_1 \neq 0$ , that is,  $(4a^2 - (b+2)^2)x_0 + 4a(b+2)y_0 \neq 0$ . Moreover, this implies that  $V_{30} \neq 0$  and  $V_{31} \neq 0$ . Therefore,  $(x_0, y_0)$  and  $(x_1, y_1)$  are weak foci of order at most one.  $\square$

### 3.2. Bifurcation from nilpotent cusps

We consider cusp singularities in (7), i.e. singularities  $(x_0, y_0)$  where the determinant and the trace of the linearization is zero, but the linearization itself is not. It is an elementary computation to show that nilpotent singularities are located at  $(x_0, y_0) = (\cos \theta, \sin \theta)$  when

$$\mu_1 = a \cos^2 \theta + (2 + b \cos^2 \theta) \tan \theta, \quad (15)$$

$$\mu_2 = -(b+2) \sin \theta - 2a \cos \theta, \quad (16)$$

$$\mu_3 = (2-b) \cos \theta - 2 \cos^{-1} \theta. \quad (17)$$

**Proposition 9.** *Around a nilpotent singularity and for any  $N \geq 2$ , there exists a local set of coordinates bringing (7) locally in the normal form*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= \sum_{k=2}^N (r_k x^k + s_k x^{k-1} y) + O(\|(x, y)\|^{N+1}), \end{cases}$$

where in particular

$$r_2 = 4 \tan \theta (1 + a \sin \theta \cos \theta - b \cos^2 \theta), \quad s_2 = 2b \cos \theta - 4a \sin \theta + 4 \cos \theta.$$

The two coefficients  $r_2$  and  $s_2$  vanish only simultaneously when both  $b = -2$  and  $\theta = 0 \pmod{\pi}$ .

*Proof.* This proposition is even valid up to  $N = \infty$  (this is a reduction to Liénard form, see for example [22]), and is well-known. Here we restrict to giving the procedure to normalize up to cubic terms, the general case being a direct generalization. We first write  $(x, y) = (\tilde{y} + x_0, \tilde{x} + y_0)$  to put the singularity at the origin. Notice the exchange of the roles of  $x$  and  $y$  to go towards normal form. After this, a linear change of coordinates  $(\tilde{x}, \tilde{y}) = (2 \cos \theta \bar{x}, -2 \sin \theta \bar{x} + \bar{y})$  changes the linear part to  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Finally, we write

$$\bar{x} = X + a_2 X^2, \quad \bar{y} = Y + b_0 X^2 + b_1 XY + b_2 Y^2$$

and for suitable choices of  $(a_2, b_0, b_1, b_2)$  (easily found with the help of a symbolic math program) one can eliminate the quadratic terms in  $\bar{X}$  and the term with  $y^2$  in the  $\bar{Y}$ -equation. Let us finally mention that the system of equations  $\{r_2 = 0, s_2 = 0\}$  is solved when either  $\sin \theta = 0$  (in which case  $b = -2$  follows) or  $b = 2 + 2 \cos^{-2} \theta > 2$ , which is out of the parameter regime that we consider.  $\square$

When both  $r_2$  and  $s_2$  are nonzero, the nilpotent singularity is of codimension 2, and upon varying the parameters it unfolds in a complete Bogdanov–Takens diagram. This is easy to see: the trace at the singular point is given by  $2ax + (b + 2)y + \mu_2$ , and a similar expression is found for the determinant at the nilpotent point. Computing the Jacobian determinant of the mapping  $(x_0, y_0, \mu_1, \mu_2) \mapsto (\dot{x}, \dot{y}, \text{tr}, \det)|_{x=x_0, y=y_0}$ , evaluated at a point  $(x_0, y_0) = (\cos \theta, \sin \theta)$ , and given the conditions (15), (16), and (17), gives  $-4 \tan \theta ((b - 2) \cos^2 \theta - 2) \neq 0$  in the parameter domain under study and when  $\theta \neq 0$ . In other words when  $\theta \neq 0$ , the two parameters  $(\mu_1, \mu_2)$  can be used as versal parameters completely unfolding the nilpotent point. We conclude that (7) contains all elements appearing in Bogdanov–Takens diagrams.

**Remark 10.** Similarly, the mapping  $(x_0, y_0, \mu_1, \mu_2) \mapsto (\dot{x}, \dot{y}, \text{tr}, \det)|_{x=x_0, y=y_0}$  is easily verified to be regular at nilpotent singularities, especially when  $\mu_2 = \mu_3 = 0$ , i.e. the case that appears in the reversible system (2). This shows that not only cusp singularities appearing in (2) unfold in full Bogdanov–Takens diagrams, but they even do so inside the family (2).

Moreover, we can show that simultaneous BT-bifurcations occur:

**Proposition 11.** A nilpotent singularity  $(x_0, y_0) = (\cos \theta, \sin \theta)$  occurs simultaneously with another nilpotent singularity at  $(x_1, y_1) = (\cos \phi, \sin \phi)$  only when  $\theta = \phi + \pi \pmod{2\pi}$  and

$$a = \frac{(2 \sin^2 \theta - 1) \sin \theta}{\cos^3 \theta}, \quad b = -2 \tan^2 \theta, \quad \mu_1 = \tan \theta, \quad \mu_2 = \mu_3 = 0.$$

The quadratic normal forms of Proposition 9 at both points are then the same, and  $r_2 = 4 \sin \theta \cos^{-3} \theta$  and  $s_2 = 4(1 - 2 \sin^2 \theta) \cos^{-3} \theta$ . In the parameter domain  $\{a \neq 0\}$ , simultaneously occurring nilpotent singularities are always of codimension 2 and they unfold completely and independently upon varying the four parameters  $(a, b, \mu_2, \mu_3)$ . In particular this implies the presence of two small-amplitude limit cycles or saddle-homoclinics near the two cusp singular points.

*Proof.* We express  $(\dot{x}, \dot{y}, \text{tr}, \det)$  at  $(x_0, y_0)$  and at  $(x_1, y_1)$  and get a system of 8 polynomial equations, which we consider in variables  $(y_0, x_1, y_1, a, b, \mu_1, \mu_2, \mu_3)$ , and treating for example  $x_0 = \cos \theta$  as a parameter. A cumbersome computation shows that  $(x_0, y_0)$  necessarily equals  $(\cos \theta, \sin \theta)$  and  $(x_1, y_1)$  equals  $(\cos \phi, \sin \phi)$  with  $\phi = \theta$  or  $\phi = \theta + \pi$ . From the same computation, we derive expressions for  $a$  and  $b$ , and using those expressions, we can simplify the known expressions for the quadratic coefficients  $r_2$  and  $s_2$ . It is clear that  $r_2 \cdot s_2 = 0$  only when  $a = 0$ . Finally, assume  $\phi = \theta + \pi$  and both points are nilpotent. Computing  $(\dot{x}, \dot{y}, \text{tr}, \det)$  at both points  $(x, y) = (\cos \theta, \sin \theta)$  and  $(x, y) = (-\cos \theta, -\sin \theta)$  leads to a map from  $(x_0, y_0, x_1, y_1, a, b, \mu_2, \mu_3)$  into  $\mathbb{R}^8$ . The Jacobian determinant of this map can be easily verified to be nonzero which implies that  $(a, b, \mu_2, \mu_3)$  independently control the two bifurcation parameters of the Bogdanov–Takens bifurcation plane for both points.  $\square$

Two simultaneously appearing nilpotent singular points of codimension  $> 2$  hence could occur only along  $a = 0$ . Let us finally state a result concerning the maximal codimension of any appearing nilpotent singularity in our system through the next two propositions:

**Proposition 12.** *A nilpotent singularity for which  $r_2 = 0$  and  $s_2 \neq 0$  is of at most codimension three and around this point, (7) can be locally brought into the normal form*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= s_2 xy + r_3 x^3 + \tilde{s}_3 x^2 y + O(\|(x, y)\|^5), \end{cases}$$

with  $s_2$  as before and nonzero  $r_3 = 4(b-2)$  and with  $\tilde{s}_3 = s_3 - 3s_2 r_4 / (5r_3)$ . The singularity is of nilpotent saddle, focus or elliptic type (see [15]), depending on parameters  $(s_2, \tilde{s}_3, r_3)$ .

*Proof.* Starting from the Liénard form up to some degree  $N \geq 5$ , one can consider yet another transformation

$$x = X + c_2 X^2 + c_3 X^3.$$

A time change allows to keep  $\dot{X} = y$ . Tracking down the effect on the  $\dot{y}$  equation, we can choose  $c_2$  and  $c_3$  to annihilate the terms of order 4. With this change, a term of order 3 however changes from  $s_3 x^2 y$  to  $\tilde{s}_3 x^2 y$ . The explicit expression for  $r_3$  is obtained exactly like  $r_2$  and  $s_2$  are obtained in the proof of Proposition 9. (In fact, one finds that  $r_3 = 8a \sin \theta \cos \theta + 4b(2 \sin^2 \theta - 1)$  which simplifies to  $4(b-2)$  along  $r_2 = 0$ .)  $\square$

**Proposition 13.** *At a nilpotent singularity for which  $s_2 = 0$  and  $r_2 \neq 0$  (7) can be locally brought into the normal form*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= r_2 x^2 + \tilde{s}_4 x^3 y + O(\|(x, y)\|^5), \end{cases}$$

with  $\tilde{s}_4 = s_4 - s_3 r_3 / r_2$ . Except when  $(a, b) = (0, -2)$ ,  $\tilde{s}_4$  is never zero together with  $s_2$  in the relevant parameter domain.

*Proof.* Just like in the previous proposition, we start with the Liénard form up to some degree  $N \geq 5$ , and we consider  $x = X + c_2 X^2 + c_3 X^3$ , this time combined with  $y = Y + d_2 Y^2 + d_3 Y^3$ . A time rescaling again allows to keep  $\dot{X} = Y$ , and the coefficients  $c_2, c_3, d_2, d_3$  are determined in order get  $\dot{Y}$  in the required form. The exact expression for  $\tilde{s}_4$  is quite long and we have chosen not to include it, as there are precise instructions on how to compute it. Algebraically, there are two disjoint curves where  $\tilde{s}_4 = s_2 = 0$ , both of them lie outside the domain  $\{b < 1, a^2 + 4(b-1) < 0\}$ .  $\square$

We leave it to the interested reader to check whether or not a versal unfolding of the two above treated codimension 3 situations are found in (7). Versal unfoldings of the situations described in Propositions 12 and 13 are found in [15] and [14], respectively. For  $(a, b) = (0, -2)$ , and for

$$(\mu_1, \mu_2, \mu_3) = \left( \frac{2 \sin^3 \theta}{\cos \theta}, 0, \frac{2 \cos 2\theta}{\cos \theta} \right),$$

the normal form in the above proposition around the singular point  $(x_0, y_0) = (\cos \theta, \sin \theta)$  degenerates further as  $\tilde{s}_4 = 0$ , and the study remains inconclusive. It is easy to conclude that topologically, the singularities are cusps, but the unfolding is less clear. From the previous section, we know however that since also  $\mu_2 = 0$ , the system is Hamiltonian in that case.

#### 4. Configuration of centers and their perturbations

This section deals with the centers of the family (7) and a review of known results about their quadratic perturbations.

**Proposition 14.** *System (7) can have a center only if it has the form*

$$\begin{cases} \dot{x} = y(bx + \mu_3), \\ \dot{y} = x^2 + y^2 - 1, \end{cases} \quad \text{with } b < 1 \text{ and } \mu_3 \in \mathbb{R}; \quad (18)$$

or

$$\begin{cases} \dot{x} = -2xy + \mu_1 + \mu_3 y, \\ \dot{y} = x^2 + y^2 - 1, \end{cases} \quad \text{with } \mu_1, \mu_3 \in \mathbb{R}. \quad (19)$$

The bifurcation diagrams of systems (18) and (19), as well as their different topological phase portraits on the Poincaré disc, are shown in Figures 4 and 5, respectively. In particular,

(i) system (18) has a center if and only if

$$(\mu_3, b) \in \{b < 1\} \cap (\{\mu_3 - b > 0\} \cup \{\mu_3 + b < 0\}) \subset \mathbb{R}^2, \text{ and}$$

(ii) system (19) has a center if and only if

$$(\mu_1, \mu_3) \in \{\mu_1^2 < 1\} \cup (\{\mathcal{D} < 0\} \cap \{\mu_1 \geq 1\}) \subset \mathbb{R}^2,$$

$$\text{where } \mathcal{D} = 64\mu_1^4 + \mu_1^2(\mu_3^4 - 80\mu_3^2 - 128) - (\mu_3 - 2)^3(\mu_3 + 2)^3.$$

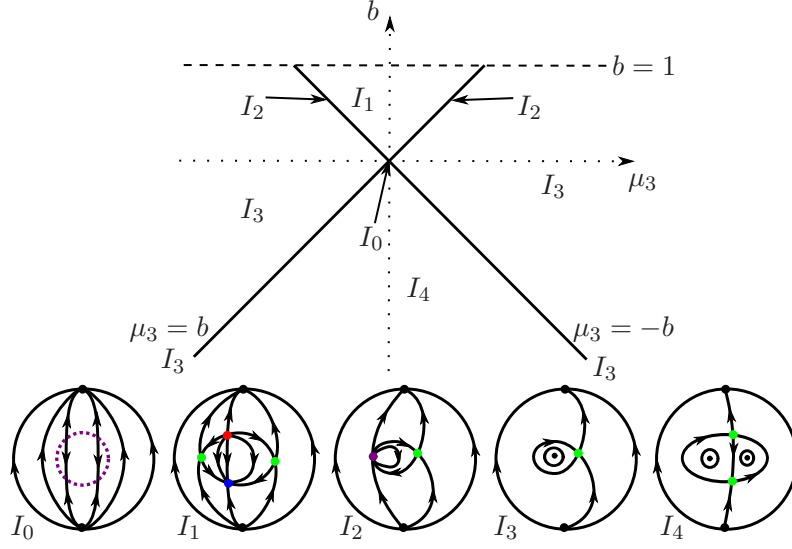


Figure 4: Bifurcation diagram of system (18) and its different topological phase portraits.

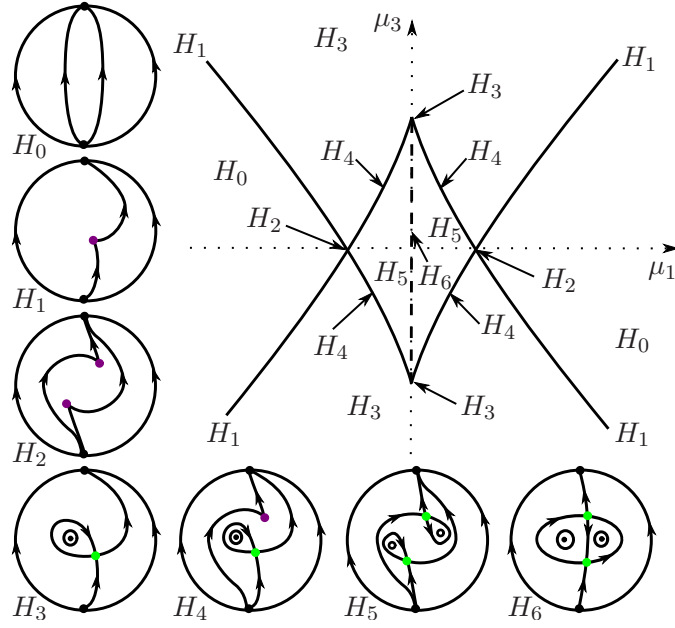


Figure 5: Bifurcation diagram of system (19) and its different topological phase portraits. The curve  $\mathcal{D} = 0$  is drawn as a continuous line.



From this result we can classify the centers in three families: The non-reversible Hamiltonian ((19) with  $\mu_1 \neq 0$ ), the reversible Hamiltonian ((18) with  $b = -2$  or (19) with  $\mu_1 = 0$ ), and the reversible non-Hamiltonian ((18) with  $b \neq -2$ ). As we show in the proof below, if the Hamiltonian associated with (19) has four or two different real singular points, then it has four singular values (in the complex plane) for  $\mu_1 \neq 0$ , then from [17] we have that at most two limit cycles can be born from  $H_3$  and  $H_5$  under quadratic perturbations; in particular the cyclicity of the period annuli is at most 2. Moreover, in [21] it is proved that the cyclicity of the period annulus of  $H_4$  is at most 2. The reversible Hamiltonian is studied in [4] proving that again the cyclicity of the period annuli of  $H_6$  is two, when both are considered separately or simultaneously. However, the global cyclicity of  $H_4$  and  $H_6$  cannot be determined since simultaneously limit cycles could bifurcate from the cusp or from the heteroclinic connections, respectively. This problem remains open. The cyclicity of the period annuli in the reversible non-Hamiltonian case only have been considered in few particular cases and in all the cases again the cyclicity is two. See [3, 5, 16, 20].

*Proof of Proposition 14.* From the proof of Theorem 6 we obtain that system (6) can have a center only if it writes as (13) or (14) which, after the rescaling introduced in Section 2.3, become (18) and (19), respectively. Moreover, by applying the map  $(x, y, t) \rightarrow (-x, -y, -t)$  we can assume, if necessary,  $\mu_3 \geq 0$  in systems (18) and (19). Finally, as the singularities at infinity are hyperbolic we restrict our analysis to the finite singularities that globally have total index zero, see Section 3.

We first study the bifurcation diagram and the phase portraits for system (18) considering the division of the parameter space according to the number and type of its singularities.

If  $b = \mu_3 = 0$ , then the circle  $x^2 + y^2 = 1$  is a curve of singularities of the system, and every vertical straight line is invariant. Thus, the phase portrait is topologically conjugated to  $I_0$  in Figure 4.

For the remaining cases the points  $(-1, 0)$  and  $(1, 0)$ , the intersection points of  $y = 0$  with the circle  $x^2 + y^2 = 1$ , are finite singularities of the system. Moreover, the system can have two additional singularities  $(-\mu_3/b, y_i)$  for  $i = 1, 2$  with  $y_1 > 0$  and  $y_2 < 0$ , which are the intersection points of the invariant straight line  $x = -\mu_3/b$  with the circle  $x^2 + y^2 = 1$ .

The next table shows the classification of the singularities of the system.

	$(-1, 0)$	$(1, 0)$	$(-\frac{\mu_3}{b}, y_1)$	$(-\frac{\mu_3}{b}, y_2)$
$I_1$	saddle	saddle	unstable node	stable node
$I_2$	$h-e$	saddle	—	—
$I_3$	center	saddle	—	—
$I_4$	center	center	saddle	saddle

where  $h-e$  means one hyperbolic sector and one elliptic sector,

$$I_1 = \{(\mu_3, b) \mid b < 1, b > \mu_3\}, \quad I_3 = \{(\mu_3, b) \mid b < 1, |b| < \mu_3, -b = \mu_3\},$$

$$I_2 = \{(\mu_3, b) \mid b < 1, b = \mu_3\}, \quad I_4 = \{(\mu_3, b) \mid b < 1, \mu_3 < -b\}.$$

See Figure 4.

The proof of the above classification follows from the Hartman–Grobmann Theorem for hyperbolic singularities and from Theorem 6 for centers. This finishes the local study of the phase portraits in regions  $I_1$ ,  $I_3$  for  $\mu_3 \neq -b$ , and  $I_4$ . The same can be applied for  $(1, 0)$  in  $I_2$  and  $(-1, 0)$  in  $I_3$  when  $\mu_3 = -b$ . The remaining cases, i.e.  $(1, 0)$  in  $I_3$  with  $\mu_3 = -b$  and  $(-1, 0)$  in  $I_2$ , are nilpotent singularities. For determining the local behavior we must apply the results about the characterization of nilpotent singularities. See Section 3.4 and Theorem 3.5 in [11]. Alternatively, see [1, Ch. IX, Theorem 66].

The local phase portraits of the singularities determine the global ones for regions  $I_2$  and  $I_3$ . The global phase portraits for regions  $I_1$  and  $I_4$  are determined by using the local phase portraits of the singularities and the existence of the invariant straight line  $x = -\mu_3/b$ . See Figure 4.

Secondly, we will study the bifurcation diagram and the phase portraits for system (19), which is a Hamiltonian system with Hamiltonian function

$$H(x, y) = \frac{x^3}{3} + xy^2 - \frac{\mu_3}{2}y^2 - x - \mu_1y. \quad (20)$$

As in previous case, we first divide the parameter space into regions according to the number of singularities. This can be done studying the intersection points of the two components of the vector field. That is a hyperbola or the product of two straight lines with the unit circle. Consequently the number of finite singularities is 0, 1, 2, 3, 4. Alternatively, such a division is given by the zero-locus of the discriminant of the resultant between the components of the vector field:

$$\begin{aligned} \Delta(\text{Res}(-2xy + \mu_1 + \mu_3y, x^2 + y^2 - 1, x), y) = \\ \Delta(4y^4 + (\mu_3^2 - 4)y^2 + 2\mu_1\mu_3y + \mu_1^2, y) = \\ 256\mu_1^2(64\mu_1^4 + \mu_1^2(\mu_3^4 - 80\mu_3^2 - 128) - (\mu_3^2 - 4)^3) = 256\mu_1^2\mathcal{D}(\mu_1, \mu_3). \end{aligned} \quad (21)$$

See Figure 5.

We will describe the phase portraits and the bifurcation diagram in terms of the number of singularities. For simplicity each region and the corresponding phase portrait are denoted by the same symbol.

In  $H_0 = \{\mathcal{D} > 0, |\mu_1| > 1\}$  the conics do not intersect. Consequently, there are no finite singularities and the phase portrait is topologically equivalent to  $H_0$  in Figure 5.

The conics are tangent in only one point in  $H_1 = \{\mathcal{D} = 0, |\mu_1| > 1\}$ . The unique singularity is nilpotent and it is a cusp point, because the vector field is Hamiltonian. The global phase portrait is topologically equivalent to  $H_1$  in Figure 5.

When there are at least two singularities we prove that the value of the Hamiltonian on two of them coincide only when  $\mu_1 = 0$ . Hence, only in this case, two singularities can be connect by an invariant curve. Writing the singularities

as

$$(x_i, y_i) = \left( \frac{1 - t_i^2}{1 + t_i^2}, \frac{2t_i}{1 + t_i^2} \right), \quad i = 1, 2,$$

with  $t_1 \neq t_2$ , the values of  $\mu_1$  and  $\mu_3$  are uniquely determined in terms of  $t_1, t_2$ . Thus, the difference of the Hamiltonian at these points is

$$\frac{4(t_1 + t_2)((3t_1^2 + 1)t_2 - 2t_1)^2 + 3(t_1^2 + 1)^2(t_1 - t_2)^3}{3(t_1^2 + 1)^3(t_2^2 + 1)^3(3t_1^2 + 1)},$$

which vanishes only when  $t_1 = -t_2$ , that is  $x_1 = x_2$  and  $y_1 = -y_2$ , or equivalently  $\mu_1 = 0$ .

There are two different situations with two singular points. The first, in  $H_2 = \{\mu_1 = \pm 1, \mu_3 = 0\}$ , when the conics are tangent in two different points and the second, in  $H_3 = \{\mathcal{D} < 0\} \cup \{\mu_3 = \pm 2, \mu_1 = 0\}$ , when the conics intersect transversally. In  $H_2$  the singularities are of cusp type, using the same argument as in region  $H_1$ , and the global phase portrait is topologically equivalent to  $H_2$  in Figure 5 because the cusps are not connected. In  $H_3$  as the system is Hamiltonian both singularities are simple, because the transversality, and of index +1 and -1, respectively. Consequently, they are of saddle and center type, the global phase portrait is totally determined and it is topologically equivalent to  $H_3$  in Figure 5.

In  $H_4 = \{\mathcal{D} = 0, 0 < |\mu_1| < 1\}$  the conics intersect in two transversal points and one tangent. The local behavior of these points is the same as the equivalent points in the previously studied cases. Then there are a cusp, a saddle and a center point. As the Hamiltonian at the cusp and the saddle does not coincide, the global phase portrait is topologically equivalent to  $H_4$  in Figure 5.

The remaining cases are the ones with four singularities. As the conics intersect transversally, the system is Hamiltonian and the total index is 0, we have two points with index +1 (centers) and two with index -1 (saddles). The global phase portrait depends on the value of the Hamiltonian at the saddle points. These value do not coincide in region  $H_5 = \{\mathcal{D} > 0, 0 < |\mu_1| < 1\}$  and hence the saddles are disconnected. They coincide in region  $H_6 = \{\mu_1 = 0, |\mu_3| < 2\}$ . Additionally, in the last region, the system has a vertical invariant straight line that connect the saddles. The global phase are topologically equivalent to  $H_5$  and  $H_6$  in Figure 5, respectively.  $\square$

## 5. Singular perturbations

We consider

$$\begin{cases} \dot{x} &= \varepsilon(ax^2 + bxy + \mu_1 + \mu_2x + \mu_3y), \\ \dot{y} &= x^2 + y^2 - 1, \end{cases} \quad (22)$$

where  $\varepsilon$  is a small perturbation parameter. This system is obtained from (7) after rescaling the parameters by  $\varepsilon$  (and using the same symbols for the rescaled variants). Note that  $(\varepsilon a, \varepsilon b) \in \Omega$  when  $\varepsilon \geq 0$  is small enough.

This kind of systems will be studied using techniques from geometric singular perturbation theory. This consists of studying two limiting systems

$$\begin{cases} \dot{x} &= 0, \\ \dot{y} &= x^2 + y^2 - 1, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} &= ax^2 + bxy + \mu_1 + \mu_2 x + \mu_3 y, \\ 0 &= x^2 + y^2 - 1, \end{cases}$$

These two systems are called the fast reduced system and the slow reduced system. We define  $C = \{(x, y) : x^2 + y^2 = 1\}$  as the critical curve, a circle. Away from the singular points of the fast system, i.e. away from  $C$ , the dynamics of (22) is a perturbation of the dynamics of the fast system, whereas close to  $C$ , the dynamics of the slow system plays a role. Of interest in  $C$  are two so-called contact points, located at  $(x, y) = (\pm 1, 0)$ . There,  $C$  is tangent to the vertical fibers of the fast subsystem, and the fast vector field has a nilpotent singular point (at other points of  $C$ , we have a partially hyperbolic singular point). The presence of contact points allows the possibility of existence of slow-fast cycles of canard type. Before checking the conditions on the presence of canard cycles in detail, we prepare the computations by presenting expressions in polar coordinates. In polar coordinates  $(x, y) = (r \cos \theta, r \sin \theta)$ , we have

$$\begin{cases} \dot{r} &= (r^2 - 1) \sin \theta + \varepsilon R(r, \theta) \cos \theta, \\ \dot{\theta} &= \frac{r^2 - 1}{r} \cos \theta - \frac{\varepsilon}{r} R(r, \theta) \sin \theta, \end{cases}$$

with  $R(r, \theta) = ar^2 \cos^2 \theta + br^2 \cos \theta \sin \theta + \mu_1 + \mu_2 r \cos \theta + \mu_3 r \sin \theta$ . Around partially hyperbolic points of  $C$ , the critical curve  $C$  perturbs to an  $\varepsilon$ -dependent invariant manifold. Using formal methods, it is easy to see that such invariant manifolds have the expression

$$r = 1 - \varepsilon \frac{\cos \theta}{2 \sin \theta} (a \cos^2 \theta + b \cos \theta \sin \theta + \mu_1 + \mu_2 \cos \theta + \mu_3 \sin \theta) + O(\varepsilon^2).$$

Note that the invariant manifolds potentially break down when  $\sin \theta = 0$ , e.g. at the contact points. For canard solutions to exist around the point  $(x, y) = (+1, 0)$ , it is clear that  $a + \mu_1 + \mu_2 = o(1)$  as  $\varepsilon \rightarrow 0$ . We will prove in the next lemma that this is indeed a necessary condition. The fast subsystem is simply the reduction to  $\varepsilon = 0$ , while the slow subsystem can now be expressed in terms of  $\theta$ :

$$\dot{\theta} = - \frac{a \cos^2 \theta + b \cos \theta \sin \theta + \mu_1 + \mu_2 \cos \theta + \mu_3 \sin \theta}{\sin \theta}.$$

(We have plugged in the expression for the invariant manifold into the vector field to obtain this formula.) Checking near  $\theta = 0$ , this yields

$$\dot{\theta} = - \frac{a + \mu_1 + \mu_2}{\theta} - (b + \mu_3) + O(\theta). \quad (23)$$

We have already obtained heuristically that the first term should be zero, in that case it is clear that the second term should be positive: for a periodic orbit of slow-fast type to appear, the orbit has to go up along the circle slowly, and then go along a fast vertical fiber downwards to close the loop. These heuristical remarks can be made exact:

**Lemma 15.** *There exists a smooth function  $\lambda_+(\varepsilon, a, b, \mu_1, \mu_2, \mu_3)$  that evaluates to 0 at  $\varepsilon = 0$  and such that system (22) has canard cycles around  $(x, y) = (+1, 0)$  when*

$$a + \mu_1 + \mu_2 = \lambda_+, \quad b + \mu_3 < 0. \quad (24)$$

*There exists a smooth function  $\lambda_-(\varepsilon, a, b, \mu_1, \mu_2, \mu_3)$  that evaluates to 0 at  $\varepsilon = 0$  and such that system (22) has canard cycles around  $(x, y) = (-1, 0)$  when*

$$a + \mu_1 - \mu_2 = \lambda_-, \quad b - \mu_3 < 0. \quad (25)$$

*Proof.* We write  $(x, y) = (1 + \tilde{y}, \tilde{x})$  and consider

$$\begin{cases} \dot{\tilde{x}} &= 2\tilde{y} + \tilde{x}^2 + \tilde{y}^2, \\ \dot{\tilde{y}} &= \varepsilon((a + \mu_1 + \mu_2) + (b + \mu_3)\tilde{x} + (\mu_2 + 2a + b\tilde{x})\tilde{y} + a\tilde{y}^2). \end{cases}$$

A well-known trick is to simply replace  $a + \mu_1 + \mu_2$  by a new parameter which we call  $\lambda$ :

$$\begin{cases} \dot{\tilde{x}} &= 2\tilde{y} + \tilde{x}^2 + \tilde{y}^2, \\ \dot{\tilde{y}} &= \varepsilon(\lambda + (b + \mu_3)\tilde{x} + (\mu_2 + 2a + b\tilde{x})\tilde{y} + a\tilde{y}^2). \end{cases}$$

The dynamics in the new system will be found back in the original system when we restrict to  $\lambda = a + \mu_1 + \mu_2$ . In this form we can readily apply Theorem 4 of [7]. It is easy to verify that under the conditions of Lemma 15, all conditions of this theorem are verified. It implies the existence of canard cycles. Similarly near  $(x, y) = (-1, 0)$ .  $\square$

Using the implicit function theorem, we can for example explicitly write  $\mu_1$  in terms of the other parameters so that (24) is satisfied, thereby showing the presence of canard cycles around  $(x, y) = (+1, 0)$ . Similarly for the left contact point. Also, considering (24 – 25) as a system of equations, the same implicit function theorem allows to find an implicit solution writing  $(\mu_1, \mu_2)$  in terms of the remaining parameters  $(a, b, \mu_3, \varepsilon)$ .

Lemma 15 shows the presence of small-amplitude canard cycle near the contact point. They consist of (a perturbation of) a fast vertical path connecting  $(x, y) = (\cos \theta_0, \sin \theta_0)$  and  $(x, y) = (\cos \theta_0, -\sin \theta_0)$ , (for  $\theta_0 \approx 0$  or  $\theta_0 \approx \pi$ ) together with a small arc on the circle. The stability of the obtained periodic orbits can be computed using a slow divergence integral, see [6]. This integral is the divergence of the fast vector field, integrated along the slow arcs, and it is well-known that it can be computed in arbitrary coordinate systems. In polar coordinates the divergence is given by  $2y + O(\varepsilon) = 2\sin \theta + O(\varepsilon)$ , so we define

$$I_+(\theta_0) = - \int_{-\theta_0}^{\theta_0} \frac{2 \sin^2 \theta}{a \cos^2 \theta + b \cos \theta \sin \theta + \mu_1 + \mu_2 \cos \theta + \mu_3 \sin \theta} d\theta.$$

as the slow divergence integrals of slow-fast cycles around the contact point  $(x, y) = (+1, 0)$ , and

$$I_-(\theta_0) = - \int_{2\pi-\theta_0}^{\theta_0} \frac{2 \sin^2 \theta}{a \cos^2 \theta + b \cos \theta \sin \theta + \mu_1 + \mu_2 \cos \theta + \mu_3 \sin \theta} d\theta.$$

for slow divergence integrals of slow-fast cycles around  $(-1, 0)$ . It is clear that these expressions only make sense when the integrand is well-defined for  $\theta$  in the integration interval. We can now prove:

**Proposition 16.** *Coexistence of canard cycles around slow-fast Hopf points in a  $(1 : 1)$  configuration in (22) occurs for any choice of  $(a, b, \mu_3)$  with  $b < 0$  and  $|\mu_3| < |b|$ , conveniently choosing  $(\mu_1, \mu_2)$  in terms of  $(\varepsilon, a, b, \mu_3)$  and of the size of the cycles. In case of coexisting canard cycles, one always has a  $(1 : 1)$  configuration when  $a \neq 0$ .*

*Proof.* From Lemma 15, we know that  $b < 0$  and  $|\mu_3| < -b$  for coexisting canard cycles to appear. Letting  $\theta_+ \in (0, \pi)$  be an angle uniquely defining a canard cycle passing near the contact point  $(x, y) = (+1, 0)$ , and letting  $\theta_- \in (0, \pi)$  be an angle uniquely defining a canard cycle passing near  $(x, y) = (-1, 0)$ . Then Lemma 15 states that  $\mu_1$  and  $\mu_2$  are determined in terms of  $\theta_+$  and  $\theta_-$  (and the other parameters), and that  $(\mu_1, \mu_2) = (-a, 0) + o(1)$ . The slow divergence is given by

$$I_+(\theta_+) = - \int_{-\theta_+}^{\theta_+} \frac{2 \sin \theta}{\mu_3 + b \cos \theta - a \sin \theta} d\theta,$$

(with a similar expression for  $I_-(\theta_-)$ ). Observe that  $\frac{\partial}{\partial a} I_+(\theta_+)$  is strictly positive and that  $I_+(\theta_+) = 0$  at  $a = 0$ , which implies that the slow divergence integral has a fixed sign along  $\theta_+$ . It follows, for  $a \neq 0$ , from results in [8] and [10] that the number of canard cycles around the contact point  $(x, y) = (+1, 0)$  is one, i.e. we cannot have more than one canard cycle in each nest. Similarly for the contact point at  $(x, y) = (-1, 0)$ .  $\square$

In case  $a = 0$  the coexisting canard cycles appear along  $\mu_1 = \mu_2 = 0$ ; in that case we have a symmetric vector field, corresponding to the singularly perturbed center case. While the slow-fast center case has been studied before and the tools from slow divergence integral can be used, the degenerate situation that appears in this context also needs to be studied when there are extra singularities in the slow dynamics (i.e.  $\mu_3 + b \cos \theta_+ - a \sin \theta_+ = 0$ ). Though known results could be extended to such situations, there is no direct result to fall back on. We have therefore decided to exclude the case  $a = 0$  from the slow-fast study, while it is clear that there is not to be expected any phenomenon different from the case  $a \neq 0$ .

In the remainder we check the presence of a nest of  $N \geq 2$  canard cycles around one of the contact points. In short we show that  $N = 2$  appears and that it is the most logical upper bound to expect, though a full proof stays out of reach. Clearly, it suffices to check the cycles around  $(x, y) = (+1, 0)$ , so we assume that  $a + \mu_1 + \mu_2 \approx 0$  and  $b + \mu_3 < 0$ . The slow divergence integral is expressed by

$$I(\theta_0) = \int_{-\theta_0}^{\theta_0} \frac{2 \sin^2 \theta}{a \sin^2 \theta - b \cos \theta \sin \theta + \mu_2(1 - \cos \theta) - \mu_3 \sin \theta} d\theta.$$

**Proposition 17.** *Around a slow-fast Hopf point at  $(x, y) = (+1, 0)$ , there are at most two canard cycles.*

*Proof.* Clearly  $I(0) = 0$ , so any solution of  $I = 0$  leads by Rolle to the existence of an intermediary solution of  $I'(\theta_0) = 0$ . If furthermore, for any solution of  $I' = 0$ , we find  $I''(\theta_0)$  has a fixed sign not depending on  $\theta_0$ , then clearly there is only one such point and hence also at most only one solution of  $I = 0$  (besides  $\theta = 0$ ). By the results of [10], we know that this translates to the presence of at most two canard cycles. Let  $N(\theta)$  be the denominator appearing in the integrand of  $I$ , i.e.  $I(\theta) = \int_{-\theta_0}^{\theta_0} 2 \sin^2 \theta / N(\theta) d\theta$ . We find

$$\begin{aligned} I'(\theta) &= \frac{2 \sin^2 \theta_0}{N(\theta_0)} + \frac{2 \sin^2 \theta_0}{N(-\theta_0)} = \frac{2 \sin^2 \theta_0}{N(\theta_0)N(-\theta_0)}(N(\theta_0) + N(-\theta_0)) \\ &= \frac{4 \sin^2 \theta_0}{N(\theta_0)N(-\theta_0)}(1 - \cos \theta)(\mu_2 + a + a \cos \theta). \end{aligned}$$

A zero is found at  $\theta = \theta_* := \arccos \frac{\mu_2 + a}{-a}$ , only when  $-2a < \mu_2 < 0$  or  $0 < \mu_2 < -2a$ . A lengthy computation shows that

$$I''(\theta_*) = \sin \theta_* \frac{4a^2(\mu_2 + 2a)}{(b\mu_2 + ba - \mu_3a)^2},$$

which has a fixed sign. This proves the proposition.  $\square$

Besides the canard cycles involved in this study, there can be also canard cycles at more degenerate contact points (not slow-fast Hopf): we distinguish slow-fast Bogdanov–Takens points of codimension 2, or more in general slow-fast codimension  $n$  contact points.

**Lemma 18.** *At  $(x, y) = (+1, 0)$ , a slow-fast Bogdanov–Takens contact point appears when  $(\mu_1, \mu_3) = (-a - \mu_2, -b) + o(1)$  and  $\mu_2 \neq -2a$ . A slow-fast contact point of codimension 3 appears when  $(\mu_1, \mu_2, \mu_3) = (a, -2a, -b) + o(1)$  and  $b \neq 0$ . A slow-fast contact point of codimension 4 appears when  $(\mu_1, \mu_2, \mu_3, b) = (a, -2a, 0, 0) + o(1)$ . There are no slow-fast contact points of codimension higher than 4.*

*Proof.* Recall expression (23). Expanding it further in  $\theta$ , a contact point of codimension  $n$  is found (for a definition, see [9]) when the first  $n$  coefficients of this expansion are zero. The lemma hence follows after an easy computation.  $\square$

The cyclicity near slow-fast contact points of codimension  $n$  is known up to  $n = 3$ . For  $n = 1$ , it is elaborated in [12], for  $n = 2$  in [9], for  $n = 3$  in the PhD thesis of R. Huzak (first part of the proof published in [18]). Using these results, it is possible to prove that the cyclicity near slow-fast contact points is at most 2. For slow-fast contact points of codimension 4, there are no results that can be applied though.

## 6. The cyclicity of the reversible families

The quadratic family introduced in this paper has, basically, two symmetric subfamilies: (2) or (18). The objective of this section is to study the existence, nonexistence, uniqueness, and maximum number of limit cycles of the first subfamily respect to the plane  $(a, b)$ , because the cyclicity of the second one is zero since it has no limit cycles (see Proposition 14). First, see Lemma 19, we show this duality, after we find the maximum number of limit cycles of (2), see Theorem 20. Then we restrict our analysis to  $(a, b) \in \Omega$ , where we prove that there are several regions in the parameter space  $(a, b, \mu)$  such that (2) has no limit cycles, and finally, we prove that there is a region in the  $(a, b)$ -plane such that (2) has two limit cycles for suitable values of  $\mu$ , always in configuration  $(1 : 1)$ . These values correspond with Hopf and Bogdanov–Takens bifurcations.

**Lemma 19.** *System (6) which is invariant with respect to a point or a straight line can be transformed to (2) or (18), respectively.*

*Proof.* After a translation and a rotation of (6), the proof follows imposing the invariance of the transformed vector field with respect to  $(x, y, t) \rightarrow (-x, -y, -t)$  or  $(x, y, t) \rightarrow (x, -y, -t)$ .  $\square$

**Theorem 20.** *System (2) has at most two limit cycles, and if it has limit cycles, then the unique possible configuration is  $(1 : 1)$ .*

*Proof.* By using the change of variables  $u = x, v = ax^2 + bxy + \mu$  system (2) becomes

$$\begin{cases} u' &= buv, \\ v' &= f(u, v)u^2 + g(u, v), \end{cases} \quad (26)$$

where

$$f(u, v) = (a^2 + b^2)u^2 + a(b - 2)v + 2\mu a - b^2$$

and

$$g(u, v) = (bv + v - \mu)(v - \mu).$$

Now, by applying the transformation  $\{x = u^2, y = v\}$ , the previous system reduces to

$$\begin{cases} x' &= 2bxy, \\ y' &= ((a^2 + b^2)x + a(b - 2)y + 2\mu a - b^2)x + (by + y - \mu)(y - \mu), \end{cases}$$

which is a quadratic system with an invariant straight line. Therefore, last system has at most one limit cycle. This implies that system (26) has at most 2 limit cycles, and if it has limit cycles the unique possible configuration is  $(1 : 1)$ . This ends the proof of the first part of the statement because systems (26) and (2) are equivalent.  $\square$

As announced in the beginning of this section, we will study in detail system

$$\begin{cases} \dot{x} &= ax^2 + bxy + \mu, \\ \dot{y} &= x^2 + y^2 - 1. \end{cases} \quad (27)$$



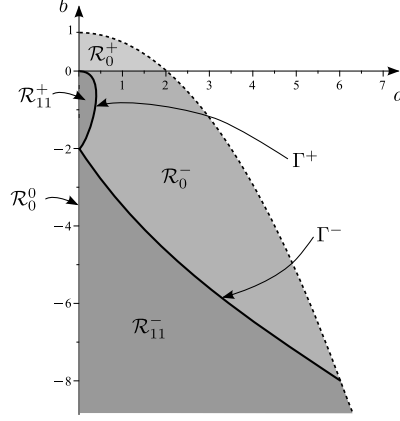


Figure 6: Decomposition of  $\Omega$  in the disjoint regions.

when  $(a, b) \in \Omega$ . We remark that some of the conclusions about non existence can also be extended to the full space  $(a, b, \mu)$ , but we are only interested in system (27) inside the class of system (6) as a perturbation of a fake saddle singularity, i.e.  $(a, b) \in \Omega$ .

The zero locus of

$$G(a, b) = 8a^2 + b^3 + 4b^2 + 4b \quad (28)$$

define the points, in the  $(a, b)$ -plane, for which system (27) can exhibit cusp points, see the proof of Proposition 21. Hence, for analyzing the limit cycles of this system, we will split  $\Omega$  in disjoint regions:

$$\begin{aligned} \mathcal{R}_0^0 &:= \{(a, b) \in \Omega \mid a = 0\}, \\ \mathcal{R}_0^+ &:= \{(a, b) \in \Omega \mid a \neq 0, 0 < b < 1, G(a, b) > 0\}, \\ \mathcal{R}_0^- &:= \{(a, b) \in \Omega \mid a \neq 0, b \leq 0, G(a, b) > 0\}, \\ \mathcal{R}_{11}^\pm &:= \{(a, b) \in \Omega \mid a \neq 0, \pm(2 + b) > 0, G(a, b) < 0\}, \\ \Gamma^\pm &:= \{(a, b) \in \Omega \mid a \neq 0, \pm(2 + b) > 0, G(a, b) = 0\}, \end{aligned} \quad (29)$$

see Figure 6.

After the results and simulations of this section we can state that system (27) can only have limit cycles, always in configuration  $(1 : 1)$ , when the values of  $(a, b, \mu)$  are in  $\mathcal{R}_{11}^+$  and  $\mu < 0$  or  $\mathcal{R}_{11}^-$  and  $\mu > 0$ . In fact when  $(a, b) \in \mathcal{R}_{11}^\pm$  the two limit cycles bifurcate simultaneously from two weak foci.

### 6.1. Local behavior of the singularities

This subsection deals with the number and local phase portrait of the singularities of system (27).

**Proposition 21.** Consider  $(a, b) \in \Omega$ . If  $(a, b) = (0, 0)$  then system (27) has no finite singularities when  $\mu \neq 0$  or the unit circle is filled with singularities otherwise. On the other hand, when  $(a, b) \neq (0, 0)$ , consider  $\mu^\pm = (-a \pm \sqrt{a^2 + b^2})/2$ , then the next properties hold.

(i) If  $\mu \notin [\mu^-, \mu^+]$ , then system (27) has no finite singularities.

(ii) If  $\mu = \mu^\pm$ , then system (27) has exactly two singularities. Moreover,

- When  $(a, b) \in \Gamma^+$  and  $\mu = \mu^-$ , or  $(a, b) \in \Gamma^-$  and  $\mu = \mu^+$ , both singularities are regular cusps;
- when  $(a, b) = (0, -2)$ , they are degenerate nilpotent singularities;
- in the remaining cases, they are saddle-nodes.

(iii) If  $\mu \in (\mu^-, \mu^+)$ , then system (27) has exactly four finite singularities: a pair of saddles and a pair of anti-saddles, located in the unit circle, and the points of each pair are antipodals.

*Proof.* The statement for  $(a, b) = (0, 0)$  can be deduced directly from the structure of (27). From now on we assume that  $(a, b) \neq (0, 0)$ .

The number of singularities follows directly studying the intersection points of the zero-locus of the components of the vector field:  $ax^2 + bxy + \mu$  and the circle  $x^2 + y^2 - 1$ , and using the symmetry of system (27). A simple computation shows that they only intersect when  $\mu \in [\mu^-, \mu^+]$ . Moreover, when  $\mu = \mu^\pm$  both curves are tangent in two antipodal symmetric points and when  $\mu \in (\mu^-, \mu^+)$  there are two pairs of two antipodal points.

Since all the singularities lie on the unit circle, we can write each of them as

$$(x_0, y_0) = \left( \frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right),$$

with  $t$  a solution of  $S = (a + \mu)t^4 - 2bt^3 + (-2a + 2\mu)t^2 + 2bt + a + \mu = 0$ . Then we have

$$\begin{aligned} \text{tr } DX(x_0, y_0) &= -2 \frac{at^2 - (b + 2)t - a}{t^2 + 1}, \\ \det DX(x_0, y_0) &= -2 \frac{bt^4 + 4at^3 - 6bt^2 - 4at + b}{(t^2 + 1)^2}. \end{aligned}$$

The resultant, with respect to the variable  $t$ , of the numerators of the above trace and determinant with  $S$  are  $16T_{ab}^2$  and  $4096(a^2 + b^2)^2 D_{ab}^2$ , respectively, where  $T_{ab} = (4a^2 + (b + 2)^2)\mu - ab^2 + 4a$  and  $D_{ab} = 4\mu^2 + 4a\mu - b^2$ . Finally, the resultant, with respect to  $\mu$ , of  $T_{ab}$  and  $D_{ab}$ , is  $-G(a, b)^2$ , see (28).

From the above resultants, we see that when  $\mu \in (\mu^-, \mu^+)$ , the determinant  $\det DX(x_0, y_0)$  never vanishes. Consequently, the local behavior of all singularities is given by Hartman–Grobmann Theorem because its hyperbolicity except in the weak focus case. But all the singularities are saddles or anti-saddles. The

location on the circle follows from the work of Berlinskii, see [2]. This concludes statements (i) and (iii).

When  $\mu = \mu^\pm$ , using the previous resultants, the determinant  $\det DX(x_0, y_0)$  vanishes and the trace  $\text{tr} DX(x_0, y_0)$  only vanishes when  $G(a, b) = 0$ . Hence, when  $G(a, b) \neq 0$  the points are semi-hyperbolic, in fact they are of saddle-node type, see [11].

The remaining cases are  $\mu = \mu^\pm$  and  $G(a, b) = 0$ . We focus on  $\mu = \mu^+$ , the other case being completely similar. The set  $\{G(a, b) = 0\} \cap \Omega = \Gamma^+ \cup \Gamma^- \cup \{(0, -2)\}$  is parameterized by  $(a, b) = (\pm s(1 - s^2), -2s^2)$ , taking the  $+$  sign and  $s \in (0, 1)$  for  $\Gamma^+$ , taking the  $-$  sign and  $s > 1$  along  $\Gamma^-$ , and  $s = 1$  for  $(a, b) = (0, -2)$ . Solving  $\{\dot{x} = 0, \dot{y} = 0, \text{tr} DX = 0\}$  along  $\Gamma^+$  with respect to  $(x, y, s)$  shows that solutions are only present when  $\mu = -s$  (and along the solution also  $\det DX = 0$ ). Evaluating the expression  $\mu^+ = (-a + \sqrt{a^2 + b^2})/2$  in this parameterized form along  $\Gamma^+$  yields  $\mu^+ = s^3 \neq -s = \mu$ . On the other hand, solving  $\{\dot{x} = 0, \dot{y} = 0, \text{tr} DX = 0\}$  along  $\Gamma^-$  shows solutions along  $\mu = s$ , which corresponds to  $\mu^+$  when  $s > 1$ .

When  $\mu = \mu^\pm$  and the trace is nonzero, the singularities are clearly semi-hyperbolic of saddle-node type. On the other hand, in the nilpotent case, one can apply Proposition 9, compute  $r_2$  and  $s_2$  along  $\Gamma^\pm$  and one finds that  $r_2 \cdot s_2$  is always nonzero, except when  $(a, b, \mu) = (0, -2, \pm 1)$  (in that case,  $s_2 = 0$  but  $r_2 \neq 0$  since  $\theta = \pm\pi/4$ ). Proposition 13 can be applied to see that it is a degenerate nilpotent singularity.  $\square$

## 6.2. Nonexistence of limit cycles

Next result provides conditions on the parameter space where system (27) does not have limit cycles.

**Theorem 22.** *System (27) has no limit cycles in the cases:*

- (i)  $0 < b < 1$ .
- (ii)  $a = 0$  or  $b = 0$ .
- (iii)  $\mu \notin (\mu^-, \mu^+)$ .
- (iv)  $-2 \leq b < 0$  and  $\mu \in [0, \mu^+)$ , or  $b \leq -2$  and  $\mu \in (\mu^-, 0]$ .
- (v)  $(a, b) \in \mathcal{R}_0^- \cap \{-2 \leq b < 0\}$  and  $\mu \in (\mu^-, \hat{\mu}]$ , or  $(a, b) \in \mathcal{R}_0^- \cap \{b \leq -2\}$  and  $\mu \in [\hat{\mu}, \mu^+)$ , where

$$\hat{\mu} = \frac{a(b^2 - 4)}{4a^2 + (b + 2)^2}.$$

*Proof.* (i) First, it is well-known that any limit cycle of a quadratic system has only one singularity of index one inside of it, and such singularity is a focus. Hence, for proving this statement, it is enough to demonstrate that each singularity of (27), with  $0 < b < 1$ , is a saddle, or a node, or a saddle-node. We will split the proof of this case in two parts:  $\mu = 0$  and  $\mu \neq 0$ .

Suppose now  $\mu = 0$ . Then, see Proposition 21, system (27) has four singularities, two of them on the line  $\{x = 0\}$ , which are  $(0, 1)$  and  $(0, -1)$ , and the other two singularities are the intersections of the line  $\{ax + by = 0\}$  and the circle  $\{x^2 + y^2 = 1\}$ , which we denote by  $p_1$  and  $p_2$ . A straightforward computation of the trace and the determinant at these singularities, by using (8) and (9), and the Hartman–Grobmann Theorem imply that  $(0, 1)$  and  $(0, -1)$  are nodes, and that  $p_1$  and  $p_2$  are saddles.

Next suppose that  $\mu \neq 0$ . Let  $(x_0, y_0)$  be a singularity of (27). Since  $\mu \neq 0$ ,  $x_0 \neq 0$ . Hence we have  $y_0 = -(ax_0^2 + \mu)/bx_0$ , and a simple computation shows that

$$(\text{tr}^2 DX - 4 \det DX)(x_0, y_0) = \frac{(a(b+2)x_0^2 - \mu(b-2))^2 + 8b^3x_0^4}{b^2x_0^2}.$$

Thus,  $(\text{tr}^2 DX - 4 \det DX)(x_0, y_0) > 0$  for  $0 < b < 1$ , which implies that  $(x_0, y_0)$  is a node if  $\det DX(x_0, y_0) > 0$ , and it is a saddle if  $\det DX(x_0, y_0) < 0$ . The case  $\det DX(x_0, y_0) = 0$  follows from Proposition 21 and the point  $(x_0, y_0)$  is a saddle-node singularity. Thus, we have proved statement (i).

(ii) First, suppose  $b = 0$ . If  $a = 0$ , then (27) has no singularities; and if  $a > 0$ , then (27) has two invariant straight lines which contain all the singularities of the system. These properties implies that (27) has no limit cycles.

Suppose that  $a = 0$ . From previous cases we can assume  $b < 0$ . If  $\mu = 0$ , then the resulting system is a particular case of (18), which does not have limit cycles. If  $\mu \neq 0$  and  $f(x) = x^{-\frac{b+2}{b}}$ , then we obtain

$$\text{div}(fX) = -\frac{\mu(b+2)x^{-\frac{2(b+1)}{b}}}{b},$$

where  $X = (P, Q)$  be the vector field associated with (27). Thus,  $\text{div}(fX)$  does not change sign for  $x > 0$ . Hence, the classical Bendixson–Dulac Theorem implies the assertion.

(iii) It follows from (i) and (ii) of Proposition 21 because in the first case system (27) has no singularities, and in the second one the type of singularities implies the nonexistence of limit cycles.

(iv) We can assume  $b < 0$  and  $a > 0$ . By using the function  $f(x) = x^{-\frac{b+2}{b}}$  we obtain

$$\text{div}(fX) = -\frac{x^{-\frac{2(b+1)}{b}}}{b} (a(2-b)x^2 + \mu(b+2)).$$

Hence, if  $-2 \leq b < 0$  and  $\mu \geq 0$ , then  $\text{div}(fX) \geq 0$  for  $x > 0$ ; and if  $b \leq -2$  and  $\mu \leq 0$ , then  $\text{div}(fX) \geq 0$  for  $x > 0$ . Thus, the Bendixson–Dulac Theorem implies that system (27) has no limit cycles.

(v) As  $\mu \neq 0$  the  $y$ -axis is without contact with respect to the vector field. By symmetry we can restrict our analysis to the half plane  $x > 0$ . In this region, system (27) has one saddle and one anti-saddle, see Proposition 21. The straight line,  $L_{0S}$ , joining the origin and the saddle point also passes trough the other saddle. Consequently, as the vector field is quadratic,  $L_{0S}$  has no tangent

points except the saddle point and the limit cycle can not cross it. Hence the limit cycle remains, if it exists, in the angular region defined by  $L_{0S}$  and the  $y$ -axis that contains the anti-saddle point. The proof follows because, for the values of the parameters, the line where the trace vanishes does not intersect this angular region.  $\square$

### 6.3. Hopf and Bogdanov–Takens bifurcations

From Theorem 22 it follows that system (27) can only exhibit limit cycles when  $\mu \in (\mu^-, \mu^+)$ . Next we prove the existence of a Bogdanov–Takens bifurcation curve and a Hopf bifurcation surface.

**Theorem 23.** *For each  $(a_0, b_0) \neq (0, -2)$  in  $\Gamma^+(\Gamma^-)$  there exists  $\mu_0 < 0$  ( $\mu_0 > 0$ ) such that system (27) undergoes two simultaneous Bogdanov–Takens bifurcations on  $\mathcal{R}_{11}^+$  ( $\mathcal{R}_{11}^-$ ).*

**Theorem 24.** *System (27) undergoes two simultaneous non-degenerate Hopf bifurcations on the surface*

$$\mu = \frac{a(b^2 - 4)}{4a^2 + (b + 2)^2} \quad (30)$$

*if and only if  $(a, b) \in \mathcal{R}_{11}^\pm$ .*

A direct consequence of the above two theorems is the next result.

**Corollary 25.** *For each  $(a, b)$  in  $\mathcal{R}_{11}^+$  ( $\mathcal{R}_{11}^-$ ), there are values of  $\mu < 0$  ( $\mu > 0$ ) such that system (27) has two limit cycles in configuration  $(1 : 1)$ .*

*Proof of Theorem 23.* We focus on  $\Gamma^+$ , where we write

$$(a, b) = (s(1 - s^2), -2s^2), \quad 0 < s < 1$$

(see the proof of Proposition 21), and find singularities of nilpotent type when  $\mu = -s$  at  $(x, y) = (\cos \theta, \sin \theta)$  when  $\tan \theta = -s$ . A lengthy computation of  $r_2$  and  $s_2$  from Proposition 9 shows that  $r_2 s_2 \neq 0$  whenever  $(a, b) \neq (0, -2)$  (i.e. whenever  $s \neq 1$ ). Using Remark 10, we know that regular cusps appearing in the family (27) unfold completely according to a Bogdanov–Takens diagram inside the family (27). This proves the theorem.  $\square$

*Proof of Theorem 24.* Let  $X = (P, Q)$  be the vector field associated with system (27). This system has critical points with vanishing trace when  $\{(x, y) \in \mathbb{R}^2 \mid P(x, y) = Q(x, y) = \text{tr} DX(x, y) = 0\}$ . A necessary condition so that the previous set is not empty is

$$\text{Res}(\text{Res}(P, \text{tr} DX, y), \text{Res}(\text{Res}(Q, \text{tr} DX, y), x)) = 0.$$

The surface (30) follows from the above equation when  $b \neq -2$ . We choose  $\mu^*$  such that  $(a, b, \mu^*)$  is in this surface, then the change

$$(a, b) = \left( \frac{r(1 - t^2)}{2(t^2 + 1)}, \frac{2rt}{t^2 + 1} - 2 \right)$$

with  $r > 0$  and  $-1 < t < 1$ , writes the singularities of (27) as

$$p_t^\pm = \pm \left( \frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right),$$

$$q_t^\pm = \pm \left( \frac{2(-2t^2 + rt - 2)(t^2 - 1)}{(t^2 + 1)\sqrt{F}}, \frac{rt^4 - 8t^3 + 6rt^2 - 8t + r}{(t^2 + 1)\sqrt{F}} \right),$$

with  $F = r^2t^4 + 14r^2t^2 - 32rt^3 + 16t^4 + r^2 - 32rt + 32t^2 + 16$ . Straightforward computations shows that, when  $a \neq 0$ ,  $F > 0$ . Moreover, the function  $G(a, b)$ , defined in (28), moves to

$$G(r, t) = \frac{2r^2(t^6 + 4rt^3 - 5t^4 - 5t^2 + 1)}{(t^2 + 1)^3}.$$

The trace and the determinant at the singularities, see (8) and (9), are

$$\begin{aligned} \text{tr}(p_t^+) &= 0, & \det(p_t^+) &= -\frac{2G(r, t)}{r^2}, \\ \text{tr}(q_t^+) &= \pm \frac{2(t^2 + 1)^2 G(r, t)}{r\sqrt{F(r, t)}}, & \det(q_t^+) &= \frac{2G(r, t)}{r^2}. \end{aligned}$$

Their sign is given in the next table:

	If $G(r, t) < 0$		If $G(r, t) > 0$	
	tr	det	tr	det
$p_t^+$	0	$> 0$	0	$< 0$
$p_t^-$	0	$> 0$	0	$< 0$
$q_t^+$	$< 0$	$< 0$	$> 0$	$> 0$
$q_t^-$	$> 0$	$< 0$	$< 0$	$> 0$

If  $G(r, t) < 0$  and  $a \neq 0$  then Theorem 6(i) asserts that  $p_t^+$  is not a center. Consequently, its stability is determined and  $p_t^\pm$  are weak foci. If  $G(r, t) > 0$  there are no weak foci because  $p_t^\pm$  are saddles. Thus, there is no Hopf bifurcations if  $(a, b) \in \mathcal{R}_0^\pm$ .

Finally, we show that two Hopf bifurcations exist when  $G(r, t) < 0$ , that is, when  $(a, b) \in \mathcal{R}_{11}^\pm$ . Let  $\varepsilon$  be a small parameter. Thus, system (27) with  $\mu = \mu^* + \varepsilon$  has a critical point,  $(x_\varepsilon, y_\varepsilon)$ , close to  $p_t^+$ , and a straightforward computation shows that

$$x_\varepsilon = \frac{2t}{t^2+1} + \varepsilon \frac{(t^2-1)(t^4+2t^2+1)}{2(t^6+4rt^3-5t^4-5t^2+1)} + O(\varepsilon^2),$$

$$y_\varepsilon = \frac{t^2-1}{t^2+1} - \varepsilon \frac{t(t^4+2t^2+1)}{t^6+4rt^3-5t^4-5t^2+1} + O(\varepsilon^2).$$

This implies that  $\text{tr}(DX)(x_\varepsilon, y_\varepsilon) = -(r^3/G(r, t))\varepsilon + O(\varepsilon^2)$ . Therefore, when we cross the surface (30), a critical point of focus type of system (27) changes the stability and, as it is not a center on the surface, a limit cycle bifurcates from  $p_t^+$ . At the same time, and with opposite stability, another one bifurcates from  $p_t^-$ .  $\square$

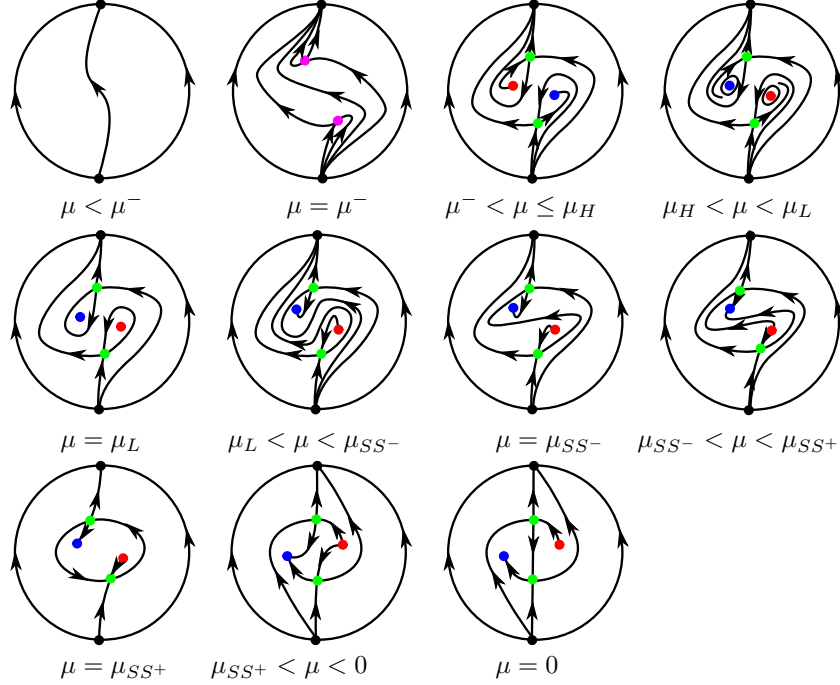


Figure 7: Phase portraits evolution, for  $\mu \leq 0$  when  $(a, b) \in \mathcal{R}_{11}^+$ .

#### 6.4. Phase portraits

From the previous sections, system (27) exhibits limit cycles when  $(a, b) \in \mathcal{R}_{11}^+$  and  $\mu < 0$  or when  $(a, b) \in \mathcal{R}_{11}^-$  and  $\mu > 0$ . Figures 7 and 8 shows the phase portraits on these parameter regions. We have only shown these transitions because they are the only ones that, with our results and simulations, exhibit limit cycles. We think that there are no other bifurcations nor phase portraits in these parameter regions.

We explain these transitions following the results of the previous sections and some simulations using the software P4, see [11].

Assume that  $(a, b) \in \mathcal{R}_{11}^+$ . When  $\mu < \mu^-$  there are no critical points, see Proposition 21. For  $\mu < \mu^-$  two symmetric saddle-nodes appear that bifurcate in a pair of symmetric saddle and an anti-saddle points. For  $\mu = \mu_H$ , see Theorem 24, the system has two symmetric weak foci and two symmetric small limit cycle bifurcate from them that remains for  $\mu_H < \mu < \mu_L$ . The limit cycles grow in amplitude and disappear in two finite symmetric homoclinic connections for  $\mu = \mu_L$ . Then two types of finite saddle connections appear for  $\mu = \mu_{SS-}$  and  $\mu = \mu_{SS+}$ . When  $\mu = 0$ , see Theorem 22,  $x = 0$  is an invariant straight line, the points over this line are saddles and there are no limit cycles.

Assume that  $(a, b) \in \mathcal{R}_{11}^-$ . As in the previous evolution, when  $\mu = 0$ , see

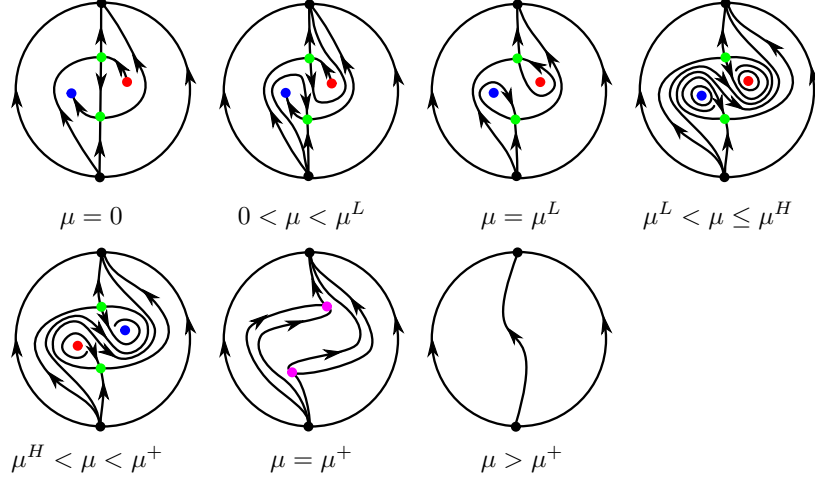


Figure 8: Phase portraits evolution, for  $\mu \geq 0$  when  $(a, b) \in \mathcal{R}_{11}^-$ .

Theorem 22,  $x = 0$  is an invariant straight line, the points over this line are saddles and there are no limit cycles. Then for  $\mu = \mu^L$  the system has two symmetric finite homoclinic connections. When they break, two symmetric limit cycles appear and shrink, to disappear in two simultaneous Hopf bifurcations for  $\mu = \mu^H$ , see Theorem 24. Then the symmetric pair of saddle and anti-saddle points collide in two symmetric saddle points when  $\mu = \mu^+$ . Afterwards,  $\mu > \mu^+$ , the system has no finite singular points.

Theorem 24 provides existence of  $\mu_H < 0$  ( $\mu^H > 0$ ) for any  $(a, b)$  in  $\mathcal{R}_{11}^+$  ( $\mathcal{R}_{11}^-$ ). Consequently the Hopf surface projects the full region  $\mathcal{R}_{11}^\pm$ . Theorem 23 provides existence of  $\mu_L < 0$  ( $\mu^L > 0$ ) for values  $(a, b)$ , close to  $\Gamma^+$  ( $\Gamma^-$ ) in  $\mathcal{R}_{11}^+$  ( $\mathcal{R}_{11}^-$ ). We have continued numerically these homoclinic connection values to obtain a homoclinic connection surface that also projects in the full region  $\mathcal{R}_{11}^\pm$ . Theorem 22 proves the nonexistence of limit cycles in  $\mathcal{R}_0^+$  and all our simulations never provide them in  $\mathcal{R}_0^-$ . All the results and numerical simulations given in this paper give us to think that system (27) never exhibit limit cycles in  $\mathcal{R}_0 = \mathcal{R}_0^+ \cup \mathcal{R}_0^-$ . They only exist, always in configuration (1 : 1), in  $\mathcal{R}_{11} = \mathcal{R}_{11}^+ \cup \mathcal{R}_{11}^-$ .

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