Sierpiński curve Julia sets for quadratic rational maps

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Abstract

We investigate under which dynamical conditions the Julia set of a quadratic rational map is a Sierpiński curve.

1 Introduction

The Sierpiński carpet fractal shown in Figure 1 is one of the best known planar, compact, connected sets. On the one hand, it is a *universal plane continuum* in the sense that it contains a homeomorphic copy of any planar, one-dimensional, compact and connected set. On the other hand, there is a topological characterization of this set: It was shown by G. Whyburn ([13]), that any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closes curves is homeomorphic to the Sierpiński carpet. Sets with this property are known as *Sierpiński curves*.

In recent years, several authors have shown that Sierpiński curves can arise as the Julia sets of certain holomorphic funcions in a variety of ways. The first example of a map whose Julia set is a Sierpiński curve was found in 1992 by J. Milnor and T. Lei ([3]) in the family of quadratic rational maps given by $z \mapsto a(z + 1/z) + b$. More recently, other authors have shown that the Julia sets of a rational map of arbitrary degree can be a Sierpiński curve ([1, 12]). For example, in [1], Sierpiński curve Julia sets were shown to occur in the family $z \mapsto z^n + \lambda/z^d$ for some values of λ, and, in [12], for the rational map

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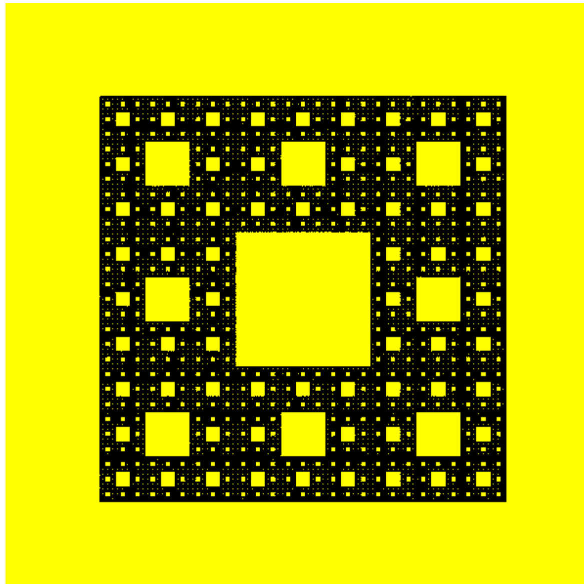


Figure 1: The Sierpiński carpet fractal. The black region correspond the the limit set by taking out the corresponding central white squares.

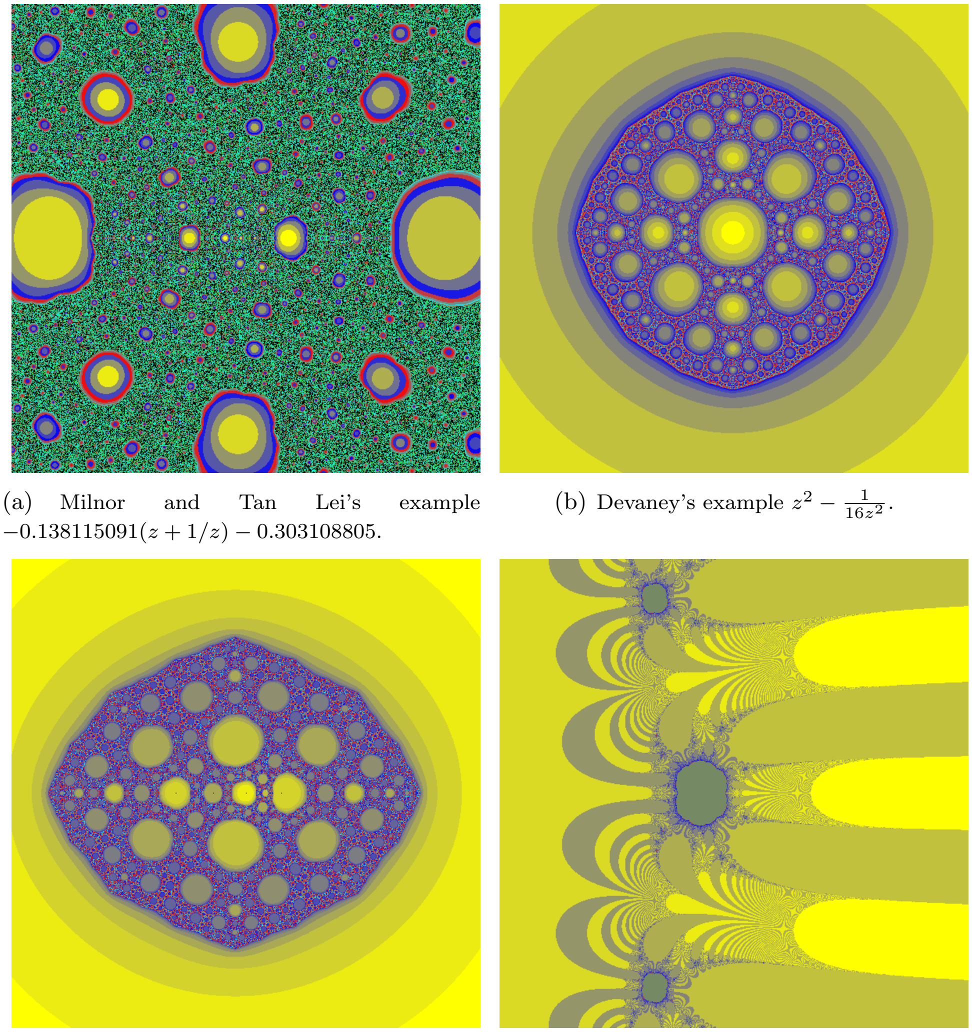
$z \mapsto t(1+(4/27)z^3/(1-z))$ also for some values of t. However, it is not just rational maps that can have Sierpiński curve Julia sets, as it was proven by S. Morosawa in [6] that entire transcendental maps in the family $z \mapsto ae^a(z-(1-a))e^z,\ a > 1$ have Sierpiński curve Julia sets. In Figure 2 we show four examples of Sierpiński curve Julia sets, one in each of the families above.

In this paper we present a more systematic approach to the problem of existence of Sierpiński curves as Julia sets of rational maps. In most of the cases mentioned above, the functions at hand have a superattracting basin of attraction, which captures all of the existing critical points. Our goal is to find dynamical conditions under which we can assure that the Julia set of a certain rational map is a Sierpiński curve.

To find completely general conditions for all rational maps is a long term program. In this paper we restrict to rational maps of degree two which have a (super)attracting periodic orbit, i.e., those which belong to $\mathrm{Per_n}(\lambda)$ for some $|\lambda| < 1$, the multiplier of the (super)attracting periodic orbit. We cannot even characterize all of those, but we cover mainly all the hyperbolic cases and most of the critically finite ones. The best results we obtain are for period $n \leq 3$ but we also give some topological conditions which are valid for higher periods.

Quadratic rational maps has been studied extensively by M.Rees [8, 9, 10] and J.Milnor [3], among others. The space of all quadratic rational maps from the Riemann sphere to itself can be parametrized using 5 complex parameters. However, the space consisting of all conformal conjugacy classes is biholomorphic to the space $\mathbb{C}^2$ [3] and will be denoted by $\mathcal{M}_2$.

Following [8], hyperbolic maps in $\mathcal{M}_2$ can be classified into four types *A, B, C and D,* according to the behaviour of their two critical points: *Adjacent* (type *A*), *Bitransitive* (type *B*), *Capture* (type *C*) and *Disjoint* (type D). In the *Adjacent* type, both critical points belong to the same Fatou component; in the *Bitransitive* case each critical point belongs to a different Fatou component, however these two Fatou components are part of the same immediate basin of an attracting cycle; in the *Capture* type each critical point belongs to a different Fatou component, however only one critical point belongs to the



(a) Milnor and Tan Lei's example $-0.138115091(z+1/z)-0.303108805$.

(b) Devaney's example $z^2-\frac{1}{16z^2}$.

(c) Steinmetz's example $1+(4/27)z^3/(1-z)$.

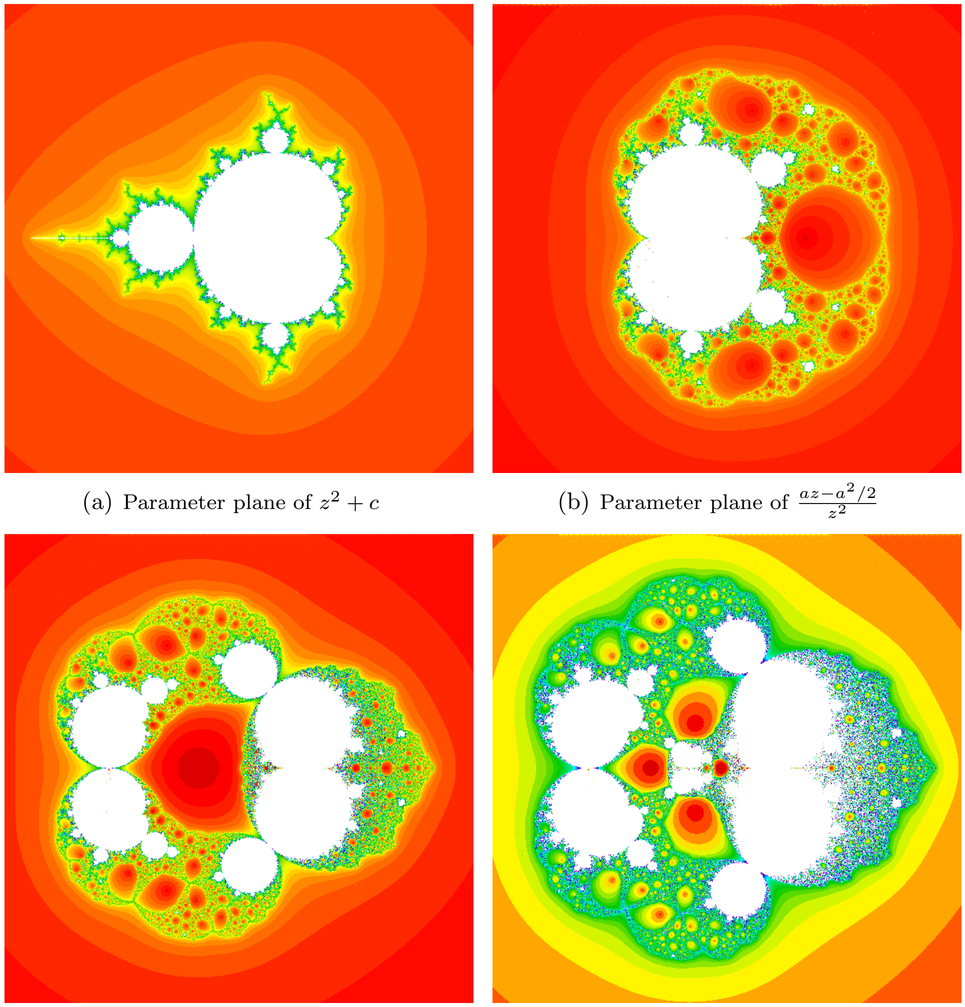
(d) Morosawa's example $1.1(e^z(z-1)+1)$.

Figure 2: Four examples of Sierpiński curve Julia sets.

immediate basin of a periodic point and the orbit of the other critical point eventually falls into this immediate basin; and finally, in the *Disjoint* type, the two critical points belong to the attracting basin of two disjoint attracting cycles.

In many of our statements we restrict to one-dimensional complex slices of $\mathcal{M}_2$ and in particular to $\mathrm{Per_n}(0)$, for $n \geq 1$. These slices contain all the conformal conjugacy classes of maps with a periodic critical orbit of period n. The first slice, $\mathrm{Per}_1(0)$, consists of all quadratic rational maps having a fixed critical point, which must be superattracting. By sending this point to infinity and the other critical point to 0, we see that all rational maps in this slice are conformally conjugate to a quadratic polynomial of the form $Q_c(z) = z^2+c$. Consequently, there are no Sierpiński curve Julia sets in this slice, since any Fatou component must share boundary points with the basin of infinity.

There are several reasons for which restricting in some cases to $\mathrm{Per_n}(0)$ is not too much of a loss of generality. Indeed, if f is a hyperbolic rational map of degree two not of type A (we will see later that this is not a relevant restriction), it follows from a Theorem of M. Rees (see Theorem 2.3) that the hyperbolic component $\mathcal{H}$ which contains f has a



(a) Parameter plane of $z^2 + c$

(b) Parameter plane of $\frac{az - a^2/2}{z^2}$

(c) Parameter plane of $\frac{(z-1)(z-a/(2-a))}{z^2}$

(d) Parameter plane of $\frac{(z-a)(z-(2a-1)/(a-1))}{z^2}$

Figure 3: The slices $\mathrm{Per}_1(0), \mathrm{Per}_2(0), \mathrm{Per}_3(0)$ and $\mathrm{Per}_4(0)$

unique *center* f_0, i.e., a map for which all attracting cycles are actually superattracting. In other words, $\mathcal{H}$ must intersect $\mathrm{Per_n}(0)$ for some $n \geq 1$, and this intersection is actually a disc. Moreover, by [2], all maps in $\mathcal{H}$ are conjugate to f_0 in a neighborhood of their Julia set. Hence the Julia set of $f_0 \in \mathrm{Per_n}(0)$ is a Sierpiński curve if and only if the Julia set of all maps $f \in \mathcal{H}$ are Sierpiński curves.

Something similar occurs for maps in $\mathrm{Per_n}(\lambda)$ with $|\lambda| < 1$. Such maps can be continuosly deformed to a map $f_0 \in \mathrm{Per_n}(0)$, by changing the multiplier of the periodic orbit to $t\lambda$ with $t \in [0, 1]$. In this path, all maps are quasiconformally conjugate to each other in a neighborhood of their Julia sets, so the same considerations as above apply in this case.

We now introduce some terminology, in order to state the main results in this paper. Suppose $f \in \mathrm{Per_n}(\lambda)$ for $|\lambda| < 1$ and $n \geq 1$. We denote by $U_0, U_1, \cdots U_{n-1}$ the Fatou components which form the immediate basin of the attracting cycle. We shall see that many of the topological properties which characterize Sierpiński curves are easy to obtain when the objects are Julia sets of rational maps. The remaining important object to study is the intersection between boundaries of Fatou components, and in particular, between those in the immediate basin. We denote by

$$\mathcal{K}_d = \{z \in \mathbb{C} \mid z \in \partial U_{i_0} \cap \partial U_{i_1} \cap \cdots \cap \partial U_{i_{d-1}},\ i_j \in \{0, 1, \cdots, n-1\},\ 0 \leq j \leq d-1\}.$$

The following theorem discards some classes of rational maps as candidates for having Sierpiński curves Julia sets. In other cases, it reduces the number of boundary intersections to check. Some of the items are straightforward while others are not.

Theorem A. *Let $f \in \mathcal{M}_2$.*

(a) If f is hyperbolic of type A then both critical orbits converge to an attracting fixed point and $J(f)$ is totally disconnected. Therefore $J(f)$ is not a Sierpiński curve.

(b) If $f \in \mathrm{Per}_1(\lambda)$ with $|\lambda| < 1$ then $J(f)$ is not a Sierpiński curve.

(c) If $f \in \mathrm{Per}_2(\lambda)$ with $|\lambda| < 1$ and f is hyperbolic or critically finite, then $J(f)$ is not a Sierpiński curve.

(d) If $f \in \mathrm{Per}_n(\lambda)$ with $|\lambda| < 1$, for $n = 3, 4$ and f is of type B (Bitransitive) then $J(f)$ is not a Sierpiński curve.

(e) If $f \in \mathrm{Per}_n(\lambda)$ with $|\lambda| < 1$, $n \geq 3$, and f is of type C then, $J(f)$ is a Siperpinski curve if and only if $\mathcal{K}_\ell = \emptyset$ for ℓ the smallest divisor of n greater than 1.

As an application of Theorem A we can make a fairly complete study of $\mathrm{Per}_3(0)$ (with its extensions mentioned above). According to Rees [11] it is possible to partition the one-dimensional slice into five pieces, each with different dynamics. In Figure 4 we display this partition, which we shall explain in detail in Section 4. Two and only two of the pieces, B_1 and B_∞, are hyperbolic components of type B. The regions Ω_1, Ω_2 and Ω_3 contain all hyperbolic components of type C (capture) and, of course, all non–hyperbolic parameters. We can prove the following.

Theorem B. *Let $f \in \mathrm{Per}_3(0)$. Then,*

(a) If $a \in (B_1 \cup B_\infty)$ then $J(f_a)$ is not a Sierpiński curve.

(b) If $a \in \Omega_2 \cup \Omega_3$ then $J(f_a)$ is not a Sierpiński curve.

(c) If $a \in \Omega_1$ and it is a capture parameter, then $J(f_a)$ is a Sierpiński curve.

As mentioned above, if f is hyperbolic, these properties extend to all maps in its hyperbolic component.

The outline of the paper is as follows: in Section 2 we give previous results concerning to the topology of the Julia set of quadratic hyperbolic rational maps. As a corollary of this we get Theorem A statements (a), (b) and (c). In Section 3 we prove the rest of Theorem A. In Section 4 we study the slice $Per_3(0)$ and prove Theorem B.

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2 Preliminary results and partial proof of Theorem A

In this section we collect some results related to the topology of Julia sets of rational maps, which we will use repeatedly. Afterwards, we prove statements (a), (b) and (c) of Theorem A, since they follow from these results in a pretty straightforward way.

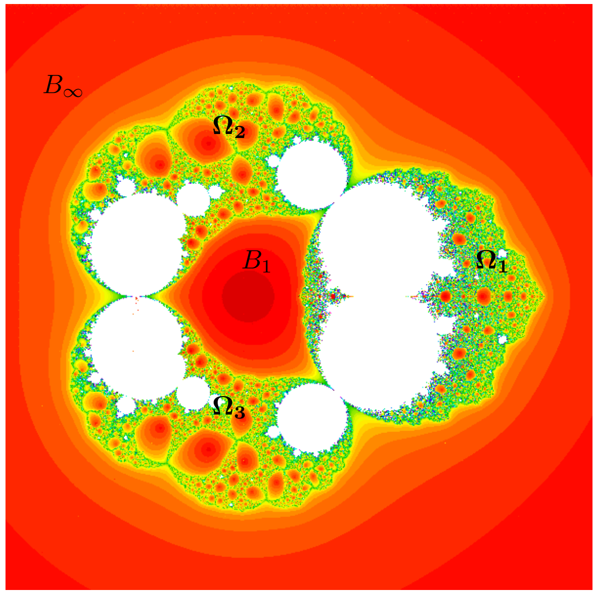


Figure 4: The slice $Per_3(0)$ and its pieces.

The first theorem states a dichotomy between the connectivity of the Julia set of a quadratic rational map and the dynamical behaviour of its critical points.

Theorem 2.1 (**Milnor, [3]**). *The Julia set $J(f)$ of a quadratic rational map f is either connected or totally disconnected (in which case the map is conjugate on the Julia set to the one-sided shift on two symbols). It is totally disconnected if and only if either:*

(a) both critical orbits converge to a common attracting fixed point, or

(b) both critical orbits converge to a common parabolic fixed point of multiplicity two but neither critical orbit actually lands on this point.

Theorem 2.2 (**[5], Thm. 19.2**). *If the Julia set of a hyperbolic rational map is connected, then it is locally connected.*

The next theorem states that any hyperbolic component of type B, C and D contains a critically finite rational map as its unique center.

Theorem 2.3 (**Rees, [8]**). *Let $\mathcal{H}$ be a hyperbolic component of type B,C or D of $\mathcal{M}_2$. Then, $\mathcal{H}$ contains a unique center f_0, i.e., f_0 is the unique critically finite map inside the hyperbolic component $\mathcal{H}$.*

Another important results gives conditions under which we can assure that all Fatou components are Jordan domains. Recall this was one of the conditions for having Sierpiński curve Julia sets.

Theorem 2.4 (**Pilgrim, [7]**). *Let f be a critically finite rational map with exactly two critical points, not counting multiplicity. Then exactly one of the following possibilities holds:*

(a) f is conjugate to z^d and the Julia set of f is a Jordan curve, or

(b) f is conjugate to a polynomial of the form $z^d + c$, $c \neq 0$, and the Fatou component corresponding to the basin of infinity under a conjugacy is the unique Fatou component which is not a Jordan domain, or

(c) *f is not conjugate to a polynomial, and every Fatou component is a Jordan domain.*

We shall combine the two results above to get the following corollary.

Corollary 2.5. *Let $f \in \mathcal{M}_2$ be a hyperbolic or critically finite map without (super)attracting fixed points. Then every Fatou component is a Jordan domain.*

Proof. Since, by hypothesis, f has no (super)attracting fixed points then f cannot be conjugate to a polynomial.

First assume that f is critically finite, not necessarily hyperbolic. Then, using Pilgrim's Theorem 2.4 the corollary follows. If f is hyperbolic, it belongs to a hyperbolic component $\mathcal{H}$. Let f_0 be its center, which exists and is unique by Rees's Theorem 2.3. Clearly, f_0 is critically finite and has no superattracting fixed points. Hence by Pilgrims's result all Fatou components of f_0 are Jordan domains. Since f and f_0 belong to the same hyperbolic component, they are conjugate on a neighborhood of the Julia set and therefore f has the same property. □

We are now ready to prove the first three statements in Theorem A.

Proof of Theorem A, statements (a), (b) and (c).

(a) Since f is of type A, both critical points belong to the same Fatou component U, which is of (super)attracting type. If the period of the cycle is 1, Milnor's Theorem 2.1 implies that $J(f)$ is totally disconnected. If the period is greater than one, the same theorem says that $J(f)$ is connected and therefore U is simply connected. In this case, $f : U \to f(U)$ is of degree three which is a contradiction since f has global degree 2.

(b) If $f \in \mathrm{Per}_1(\lambda)$, with $|\lambda| < 1$, there exists a Fatou component U containing an attracting fixed point and therefore a critical point. The degree of $f \,|\, U$ is 2, hence U is completely invariant. It follows that the boundary of U is the whole Julia set, hence it must intersect the boundary of any other Fatou component.

(c) It is easy to check that any rational map of degree two has exactly three fixed points and a unique 2-cycle. If $f \in \mathrm{Per}_2(\lambda)$, this cycle is (super)attracting. Without loss of generality the cycle is $\{0, \infty\}$. We also denote the Fatou components containing 0 and ∞ by U_0 and U_∞, respectively. If f has an additional attracting fixed point, then $f \in \mathrm{Per}_1(\mu)$ for some $|\mu| < 1$, and hence $J(f)$ is not a Sierpiński curve by (b). Otherwise, by Corollary 2.5 all Fatou components are Jordan domains. It follows that the map $f^2 : \overline{U_0} \mapsto \overline{U_0}$ is conjugate to either $z \to z^4$ or $z \to z^2$ on ∂U_0 depending on whether the free critical point lies in U_∞ or not. In particular, the map $f^2 \,|\, \partial U_0$ must have at least one fixed point, and this fixed point under f^2 is also a fixed point under f, since f has no other 2-cycle besides $\{0, \infty\}$. Hence $\partial U_0 \cap \partial U_\infty \neq \emptyset$, proving that $J(f)$ is not a Sierpiński curve.

□

3 Sufficient conditions for Sierpiński curve Julia sets. Proof of Theorem A

As we mentioned in the introduction a Sierpiński curve is any subset of the Riemann sphere that is homeomorphic to the Sierpiński carpet. Consequently, from [13] a Sierpiński curve Julia set is a Julia set with the following five properties: compact, connected, locally connected, nowhere dense, and any two complementary Fatou domains are bounded by disjoint simple closed curves. In what follows we shall see that most of these properties are satisfied simply because our sets are Julia sets of hyperbolic rational maps.

Proposition 3.1. *Let $f \in \mathrm{Per_n}(\lambda)$, with $n \geq 3$, be hyperbolic. Then, the Julia set $J(f)$ is compact, connected, locally connected and nowhere dense. Moreover, if f has no attracting fixed point (which is always the case for types B and C), then each Fatou component is a Jordan domain.*

Proof. The Julia set of a rational map is always compact and assuming hyperbolicity it is also nowhere dense. Using Theorem 2.1 we conclude that $J(f)$ is connected since we always have a (super)attracting periodic orbit of period $n > 1$. Because it is connected, it is also locally connected (Theorem 2.2).

Observe that f cannot be hyperbolic of type A because it has a cycle of period larger than 1 (Theorem A (a)). Hence $f \in \mathcal{H}$, a hyperbolic component of type B, C or D, without (super)attracting fixed points. We then know that all the Fatou components of f are Jordan domains (Corollary 2.5). $\square$

In view of this Proposition, for those maps in the hypothesis above, the Julia set is a Sierpiński curve if and only their Fatou components have disjoint closure.

In this section we assume that f is a quadratic rational map having a (super)attracting period n cycle with $n \geq 3$, or equivalently, $f \in \mathrm{Per_n}(\lambda)$.

We denote by U_i the Fatou components which form the immediate basin of attraction of the cycle $w_0, \ldots, w_{n-1}$. Fixing one of the two critical points at 0 and conjugating f with a Möbius transformation we may assume that $0 \in U_0$, and $\infty \in U_1$. So U_1 is the unique unbounded Fatou component of f. In the case where U_i is a Jordan domain (see Corollary 3.1) we denote by $\gamma_i : \mathbb{S}^1 \mapsto \partial U_i$ a homeomorphism that parametrizes the Jordan curve ∂U_i, with $i = 0, \ldots n-1$.

We denote by

$$\mathcal{K}_d = \{z \in \mathbb{C} \mid z \in \partial U_{i_0} \cap \partial U_{i_1} \cap \cdots \cap \partial U_{i_{d-1}},\ i_j \in \{0, 1, \cdots, n-1\},\ 0 \leq j \leq d-1\}.$$

To simplify notation we write $\mathcal{K} := \mathcal{K}_n$, i.e., the common intersection of $\overline{U_i},\ i = 0, 1, \ldots n-1$.

We start by proving some properties for $f \in \mathrm{Per_n}(0)$, to be used later in the more general proof.

Proposition 3.2. *Let $f \in \mathrm{Per_n}(0)$ with $n \geq 3$. Assume f is critically finite. Then*

(a) If f has no superattracting fixed points, then $\# \mathcal{K} \leq 2$.

(b) If $n = 3$ or $n = 4$, and f is of type B then $J(f)$ is not a Sierpiński curve.

(c) If f is of type C then either $\mathcal{K} = \emptyset$ or $\mathcal{K} = \{p\}$ with $f(p) = p$.

Proof.

(a) By assumption we know from Theorem 2.4 that all Fatou domains are Jordan domains, hence every point on the boundary is accessible from the interior by a unique internal ray. We prove statement (a) by contradiction. Suppose that $\mathcal{K}$ contains at least three points, denoted by p, q and r. Let $w_0 = 0$, $w_1 = \infty$ and w_2 be three points of the superattracting cycle. We start by building the quadrilateral formed by the internal rays joining w_0 with p and q and w_1 with p and q. This divides the Riemann sphere into two connected components, U and V only one of which, say U, contains w_2. Let us now add to this graph the two internal rays in U connecting w_2 to p and q. This addition has further subdivided U into two subsets U_1 and U_2. Notice that each of the three sets U_1, U_2 and V contains only two points of the cycle in its boundary.

Now the third intersection point r must belong to one of the sets in the partition. Therefore it cannot be accessed through internal rays by the three chosen points in the cycle w_0, w_1 and w_2.

(b) Since $f \in \mathrm{Per_n}(0)$ is of type B the free critical point must belong to U_i for some $i \neq 0$. Since f is of type B and $n \geq 3$, f has no superattracting fixed points and therefore U_i is a Jordan domain for all i (see Corollary 2.5). Observe that $f^n : \overline{U_i} \to \overline{U_i},\ i = 0, \ldots, n-1$ is a degree 4 map conjugate to $z \mapsto z^4$. Consequently $f^n \,|\, \partial U_i$ is conjugate to $\theta \mapsto 4\theta$ on the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Since the map is critically finite, every internal ray in U_i lands at a well–defined point on ∂U_i, $i = 0, \ldots n-1$. It follows that there are three fixed points of f^n on ∂U_i, namely $\gamma_i(0)$, $\gamma_i(1/3)$ and $\gamma_i(2/3)$, $i = 0, \ldots, n-1$. By construction each of these points is fixed under f^n, and so they are periodic points of period d for f with $d|n$. If one of them is periodic of period $d < n$ then such a point must belong to $\partial U_i \cap \partial U_j$ for some $i \neq j$ and so $J(f)$ cannot be a Sierpiński curve. So, let us assume $d = n$ and show that this is not possible. The $3n$ points involved in the construction form 3 different cycles of period n. Globally f has 4 period n cycles since each $f \in \mathrm{Per_n}(0)$ has one superattracting n cycle. This is a contradiction when $n = 3$ and $n = 4$ since any quadratic rational map has only 2 period 3 cycles, and 3 period 4 cycles respectively.

(c) We assume that f is of type C, i.e., the free critical point lands on the superattracting cycle (remember that f is critically finite) but it does not belong to the inmediate basin. By the same considerations as in part (b), all Fatou components are Jordan domains. In this case $f^n : \overline{U_i} \to \overline{U_i},\ i = 0, \ldots n-1$ is a map of degree two that is conformally conjugate to $z \mapsto z^2$ on the closed unit disk. Hence $f^n \,|\, \partial U_i$ is conjugated to $\theta \mapsto 2\theta$ on the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Thus, there is only one fixed point and two 2-cycles in each boundary of the U_i under f^n, namely $\gamma_i(0)$, $\gamma_i(1/3)$ and $\gamma_i(2/3)$, $i = 0, \ldots, n-1$. These are the landing points of the internal rays of angles $\theta = 0$ (fixed), $\theta = 1/3$ and $\theta = 2/3$ (period 2), for the map $\theta \to 2\theta$.

From statement (a), in order to prove statement (c) we only need to show that $\# \mathcal{K} \neq 2$ when f is of type C. Assume the converse, and denote by p and q the two points on the common boundary of all the U_i's. There are three possible dynamics for such points: either $f(p) = p$ and $f(q) = q$, or $f(p) = q$ and $f(q) = q$, or $f(p) = q$ and $f(q) = p$.

The first case (two fixed points for f on the common boundary) is not possible since there is a unique fixed point for f^n on the boundary of the U_i's, and, of course, fixed points of f are also fixed points of f^n.

The second case (one prefixed point and one fixed point on the common boundary) is not possible since $f : \overline{U_i} \to \overline{U_{i+1}}$ is a one–to–one map except when $i = 0$, while q as a point in ∂U_{i+1} would have two preimages (p and q) on ∂U_i.

The third case (two period 2 points) is also not possible. If n is even the argument follows as above since $f^n(p) = p$ and $f^n(q) = q$ and so we would have two fixed points for f^n on the common boundary of the U_i's, a contradiction. We now deal with the case of n odd.

On the one hand it is clear that $p = \gamma_i(1/3)$ and $q = \gamma_i(2/3)$, for $i = 0, 1, \ldots, n-1$. On the other hand, by the chain rule we have that $\mu := \left(f^2\right)'(p) = \left(f^2\right)'(q)$ satisfies $|\mu| > 1$, and so in a sufficiently small neighborhood of the points p and q the map f^2 is conformally conjugate to $z \to \mu z$ in a neighborhood of 0. Let $\alpha \in (-\pi, \pi]$ denote the argument of μ. Now we build a graph similar to the one in part (a), with vertices p and q and edges the internal rays from the points of the cycle to the points p and q. The topological picture in the Riemann sphere is homotopic to the one shown in Figure 3. It is then clear that the rotation induced by f^2 on a small neighborhood of p and q has opposite sign, hence, the multiplier cannot be the equal for both points.

□

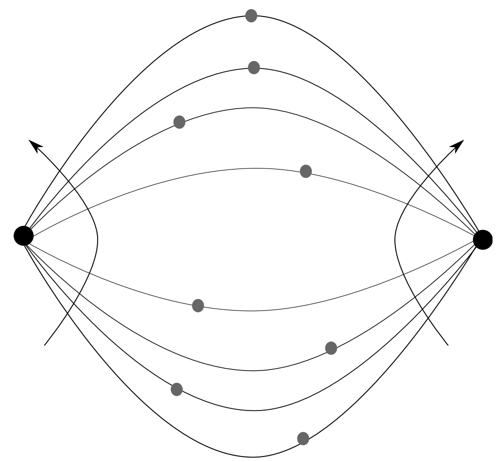


Figure 5: The topological graph given by the internal rays landing at the period two cycle $\{p, q\}$ in $\mathcal{K}$. The points p and q are plotted with small black circles, while the points w_j are plotted with small grey circles.

In the three pictures in Figure 6 we show three dynamical planes corresponding to the three possible behaviours stated in the previous lemma depending on the number of points in $\mathcal{K}$. Figure 6(a) corresponds to the dynamical plane of a Bitransitive parameter with two fixed points in $\mathcal{K}$, and Figures 6(b) and (c) correspond to Capture parameters with one fixed point in $\mathcal{K}$ and $\mathcal{K} = \emptyset$, respectively. It is immediate that, in the first two cases, the Julia set is not a Sierpiński curve, while, as we prove in the following proposition, it is in the third case.

Proposition 3.3. *Let $f \in \mathrm{Per_n}(0),\ \mathrm{n} \geq 3$. Assume f to be a critically finite, hyperbolic rational map of type C. Then, $J(f)$ is a Sierpiński curve if and only if $\mathcal{K}_d = \emptyset$ where d is the smallest divisor of n greater than 1.*

Most of the remainder of this section will be dedicated to the proof of this proposition. We will use several lemmas.

Since f is of type C the Fatou set is the union of the superattracting basin formed by $\{U_0, U_1, U_2, \dots U_{n-1}\}$ and all its preimages. Suppose we have named the components so that $f(U_i) = U_{i+1 \bmod (n)}$. We also denote by $\omega_i,\ i = \{0, \dots n-1\}$ the superattracting cycle, so $f(\omega_i) = \omega_{i+1 \bmod (n)}$. One of the implications of the proposition is trivial: If $J(f)$ is a Sierpiński curve, then $\mathcal{K}_d = \emptyset$ (in fact $\mathcal{K}_\ell = \emptyset$ for all $1 \leq \ell \leq n$). Hence, assuming $J(f)$ is not a Sierpiński curve, we must prove that $\mathcal{K}_d \neq \emptyset$ (where d is the smallest divisor of n greater than 1). It will suffice to show that $\mathcal{K}_m \neq \emptyset$, for some divisor m of n.

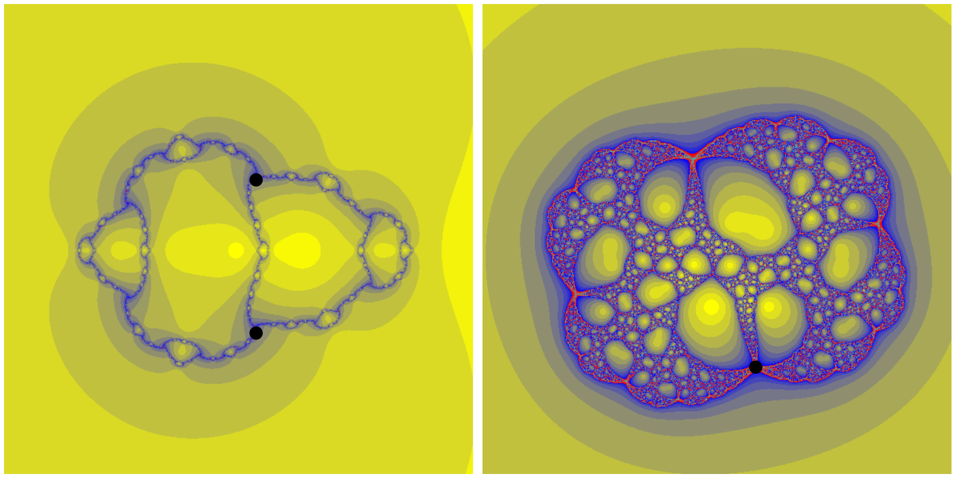
Lemma 3.4. *Let f be as in Proposition 3.3. Suppose there exists a point $s \in \partial U_0$ of period $m > 0$, where m is not a multiple of n. Let $m' := m \bmod (n)$. Then,*

1. *if $m' \mid n$ then $\mathcal{K}_{\frac{n}{m'}} \neq \emptyset$;*
2. *if $m' \nmid n$ then $\mathcal{K} \neq \emptyset$.*

Proof. First assume $m < n$. Given that $f^m(U_j) = U_{j+m}$ and $f^m(s) = s$ it follows that

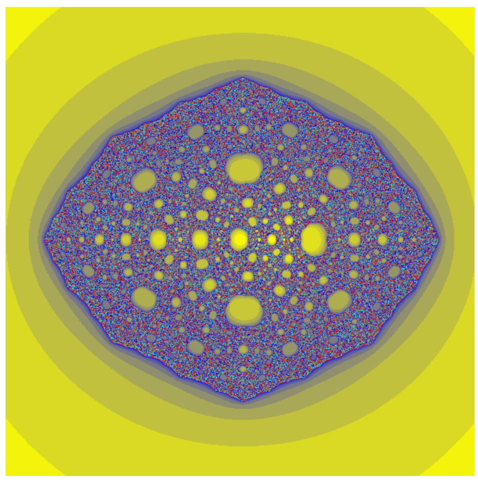
$$s \in \partial U_0 \cap \partial U_m \cap \partial U_{2m \bmod (n)} \cap \dots \cap \dots$$

If $m \mid n$ then $n = m \cdot \frac{n}{m}$ where $\frac{n}{m} \in \mathbb{N}$, and $s \in \partial U_0 \cap \partial U_m \cap \partial U_{2m} \cap \dots \cap U_{m(\frac{n}{m}-1)}$ which implies that $\mathcal{K}_{\frac{n}{m'}} \neq \emptyset$ (notice that if $m < n$ we have $m = m'$).



(a) Dynamical plane of $f_{0.75}$ of type *Bitransitive*.

(b) Dynamical plane of $f_{0.6+1.45i}$ of type *Capture*.



(c) Dynamical plane of $f_{3.2}$ of type *Capture* .

Figure 6: Three examples in $\mathrm{Per}_3(0)$. Points in $\mathcal{K}$ are plotted with a small black circle.

If $m \nmid n$ then the set $\{0, m, 2m \bmod (n), \dots, (n-1)m \bmod (n)\}$ is exactly a permutation of $\{0, 1, \dots, n-1\}$. Hence $\mathcal{K} := \mathcal{K}_n \neq \emptyset$.

Now observe that the case $m > n$, with m not being a multiple of n, can be argued exactly in the same way, just considering all indices modulo n. $\square$

From Corollary 3.1, in the hypothesis of Proposition 3.3, if $J(f)$ is not a Sierpiński curve, necessarily we must have at least two different Fatou components whose closures intersect. Since all Fatou components are preperiodic to the immediate basin of the superattracting cycle, and there are no critical points in the Julia set, it follows that if $J(f)$ is not a Sierpiński curve then two components of the immediate basin must have non disjoint closures. Hence let us assume without loss of generality that there exists a point p such that

$$p \in \partial U_0 \cap \partial U_i \ \text{ for some } 0 < i < n.$$

Let $g := f^n$. For any $j = 0, 1, \cdots, n-1$, $g|\overline{U_j}$ maps $\overline{U_j}$ onto itself and since the free critical point does not belong to any of the U_j's we conclude that it is conformally conjugate to the map $z \mapsto z^2$ defined on the closed unit disk $\overline{\mathbb{D}}$. It follows that $g \,|\, \partial U_j$ is conjugate to $\theta \mapsto 2\theta$ on the unit circle.

Let $k := n - i < n$ which means that $f^k(U_i) = U_0$. In particular, the point $q := f^k(p)$ must belong to ∂U_0, but it also must belong to the image under f^k of all components meeting at p, that is

$$q \in \partial U_0 \cap \partial U_k.$$

It might happen that $U_k = U_i$. In that case we have $n = 2k$ is even and hence $d = 2$ and we are done. Therefore we may assume from now on that $k \neq i$, i.e., that U_0, U_k and U_i are pairwise distinct.

We shall now see that we may reduce to the case when the orbits of p and q are totally disjoint from each other (in other words, if $J(f)$ is not a Sierpiński curve and the orbits of p and q are not totally disjoint from each other, it will follow from Lemma 3.4 that $\mathcal{K}_d \neq \emptyset$).

Lemma 3.5. *If there exist $N_p, N_q \in \mathbf{N} \cup \{0\}$ such that $g^{N_p}(p) = g^{N_q}(q)$ then there is a periodic point $s \in \partial U_0$ of period m, where*

$$m = \begin{cases} (N_q - N_p)n + k & \text{if } N_q \geq N_p, \text{ and} \\ (N_p - N_q)n - k & \text{if } N_p > N_q. \end{cases}$$

Proof. Since $q = f^k(p)$ and $g = f^n$, we have that $f^{N_p n}(p) = f^{N_q n + k}(p)$.

1. If $N_q \geq N_p$, define $s := f^{N_p n}(p)$. Then

$$f^m(s) = f^{(N_q - N_p)n + k}(s) = f^{(N_q - N_p)n + k + N_p n}(p) = f^{N_q n + k}(p) = f^{N_p n}(p) = s.$$

2. If $N_p \geq N_q$, define $s := f^{N_q n + k}(p)$. Then,

$$f^m(s) = f^{(N_p - N_q)n - k}(s) = f^{(N_p - N_q)n - k + N_q n + k}(p) = f^{N_p n}(p) = f^{N_q n + k}(p) = s.$$

□

Indeed, if the orbits of p and q under g eventually join to form one single orbit tail, the claim implies the existence of a periodic point of period m not multiple of n, since $0 < k < n$. Observe that $m' = m \bmod (n)$ equals either k or $n - k$, depending on which case we are in. We may then apply Lemma 3.4 to conclude that, either $\mathcal{K}_{\frac{n}{m'}} \neq \emptyset$ if m' divides n or $\mathcal{K}_n \neq \emptyset$ if m' does not divide n. In any case, $\mathcal{K}_d \neq \emptyset$ as we wanted to show. Hence we may assume from now on that q and p have distinct orbits under g.

Lemma 3.6. *If there exists $N \geq 0$ such that $g^N(p)$ is periodic of period m, then $g^N(q)$ is also periodic of period m. Conversely, if there exists $N \geq 0$ such that $g^N(q)$ is periodic of period m, then $g^{N+1}(p)$ is also periodic of period m.*

Proof. In the first case, we assume $g^{m+N}(p) = g^N(p)$. If we apply f^k to both sides of the equality and permute the order of the iterates we get $g^{m+N}(f^k(p)) = g^N(f^k(p))$. Since $q := f^k(p)$ we have $g^{m+N}(q) = g^N(q)$, i.e., $g^N(q)$ is also periodic of period m.

The second case is analogous. We assume $g^{m+N}(q) = g^N(q)$. If we apply f^{n-k} to both sides of the equality and permute the order of the iterates (and substitute $q = f^k(p)$) we get $g^m(g^{N+1}(p)) = g^{N+1}(p)$, as claimed. □

Observe that the orbits of p or q cannot be eventually fixed or 2-periodic. Indeed, by the claim we just proved, this would imply the existence of two fixed points or two 2-cycles under g on the boundary of U_0. But g is the doubling map on the boundary of U_0, and such map has exactly one fixed point and one 2-cycle, which leads to a contradiction.

Hence, by redefining p and q if necessary we have only two cases remaining. Either the orbits of p and q are periodic of period $m \geq 3$, or both orbits are infinite.

Let L_1 and L_2 denote the two disjoint arcs in ∂U_0 joining p and $g(p)$, whose union together with the endpoints form the whole Jordan curve ∂U_0.

We may then draw a graph with the following elements: the boundary of U_0 (which we may think as the unit circle), the two (different) points p and $g(p)$ and the internal rays joining the periodic point ω_i to p and $g(p)$, which live in the complement of U_0 and cannot intersect each other. Such a graph divides the complement of U_0 into two disjoint pieces P_1 and P_2, which contain L_1 and L_2 respectively in their boundary. See Figure 7.

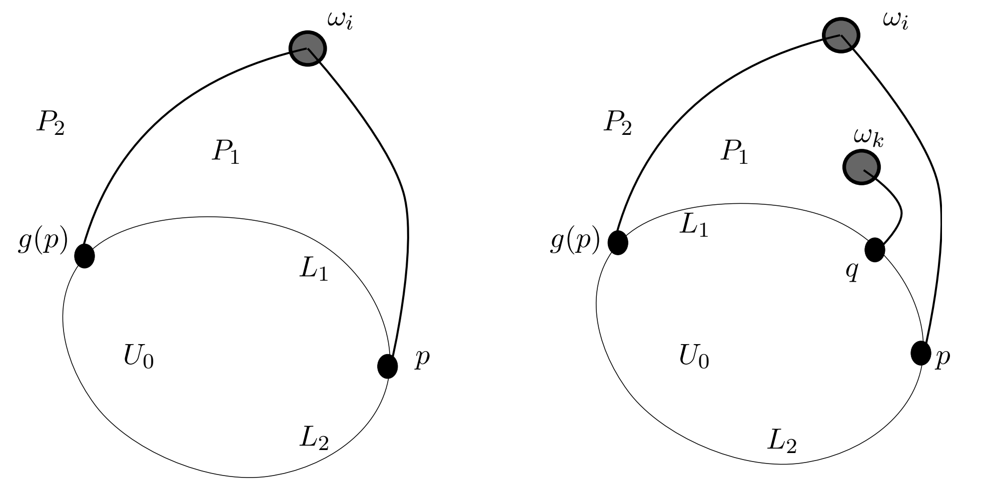


Figure 7: In the left we illustrate the graph formed by ω_i, the two internal rays joining p and $g(p)$ with ω_i and the boundary of U_0. This graph divides the complement of U_0 (in the Riemann sphere) in two pieces, namely, P_1 and P_2. Each piece contains the two disjoint arcs in ∂U_0 joining p and $g(p)$ which we denote, respectively, L_1 and L_2. The right picture illustrates the relative position of q in the graph. By construction q must be either in P_1 or in P_2 (P_1 in the figure). Consequently the whole orbit of q by g has to be contained in L_1.

We now place the point q in this picture. It clearly lies either in L_1 or in L_2. W.l.o.g. we assume it lies in L_1. It follows that w_k must belong to P_1, since q needs to be accessible from the interior of U_k, and the ray joining w_k to q cannot cross any other ray in the graph. But U_k is invariant under g, hence the whole orbit of q under g, namely $\mathcal{O}_g(q)$, must also be accessible from U_k. This forces the entire orbit of q to lie in L_1 and never enter L_2. An analogous argument forces the entire orbit of p to lie in one and only one of the analogous arcs, say L_1' and L_2', defined by the graph associated to q, $g(q)$, ∂U_0 and the internal arcs joining the periodic point ω_k with q and $g(q)$.

In summary, the orbits of p and q do not *mix*. This is an immediate contradiction if we assume the orbits to be periodic of the same period. Indeed, such orbits correspond, under angle doubling, to rational numbers of the form $\frac{2k\pi}{2^m-1}$, where m is the period, and $k \in \mathbb{Z}$. It is well known and easy to check that two such orbits cannot belong to disjoint connected arcs of $\mathbb{S}^1$. The same argument can be applied to the preperiodic scenario.

It remains to consider the case when both orbits are infinite. When transported to the unit circle, we can think of having two irrational angles φ_1 and φ_2 whose orbits under the doubling map are contained in disjoint arcs of $\mathbb{S}^1$. Let $s(\varphi_j) = s_0^j s_1^j \ldots,\ s_i^j \cdots \in \{0,1\}^{\mathbb{N}}$ be their associated binary sequences and let consider the four sectors of $\mathbb{S}^1$ given by the intervals $(0,1/4)$, $(1/4,1/2)$, $(1/2,3/4)$ and $(3/4,1)$. Points in these quadrants are exactly those with orbits starting by (respectively) $00\ldots$, $01\ldots$, $10\ldots$ and $11\ldots$.

Now observe that any orbit with a nonperiodic sequence, must have infinitely many changes from 0 to 1 and infintiely many from 1 to 0. This means that any such orbit must have infinitely many points in the second and third quadrant. If we assume that one orbit lies entirely in (θ_1,θ_2) and the other one in (θ_2,θ_1), understood as two complementary connected arcs of the circle, the observation above implies that the divisory angles θ_1 and θ_2 must belong to the second and third quadrant. But this implies that one of the orbits has *all* its points in the second and third quadrants, which only happens for the orbits $\overline{10}$ or $\overline{01}$, the 2-cycle.

This concludes the proof of Proposition 3.3 and we are now ready to prove the last two statements in Theorem A.

Proof of Theorem A, statements (d) and (e).

(d) The result follows from Proposition 3.2 and the fact that every hyperbolic component has a unique center, i.e., a unique critically finite map inside this hyperbolic component (Theorem 2.3).

(e) As before the result follows from Proposition 3.3 applied to the center of the hyperbolic component containing f.

□

4 The period three slice. Proof of Theorem B

In this section we restrict attention to rational maps in $Per_3(0)$. This slice contains all the conformal conjugacy classes of maps with a periodic critical orbit of period three. Using a suitable Möbius transformation we can assume that one critical point is located at the origin, and the critical cycle is $0 \mapsto \infty \mapsto 1 \mapsto 0$. Such maps can be written as $(z-1)(z-a)/z^2$, and using this expression the other critical point is now located at $2a/(a+1)$. However, we change this parametrization of $Per_3(0)$ so that the critical point is now located at a, obtaining the following expression

$$f_a(z) = \frac{(z-1)\left(z-\frac{a}{2-a}\right)}{z^2} \qquad \text{where } a \in \mathbb{C} \setminus \{0,2\}. \tag{1}$$

We exclude the values $a=0$ and $a=2$. In the first case the map f_0 has degree 1 and in the second case the map is not well defined. As we mentioned before, f_a, for $a \in \mathbb{C}\setminus\{0,2\}$, has a superattracting cycle $0 \mapsto \infty \mapsto 1 \mapsto 0$ and we denote by $U_0 = U_0(a), U_\infty = U_\infty(a), U_1 = U_1(a)$ the Fatou components containing the corresponding points of this superattracting cycle. This map has two critical points, located at $c_1 = 0$ and $c_2(a) = a$, and the corresponding critical values are $v_1 = \infty$ and $v_2(a) = -\frac{(1-a)^2}{a(2-a)}$. Thus, the dynamical behaviour of the map f_a is determined by the orbit of the free critical point $c_2(a) = a$. The parameter $a-$plane has been thoroughly studied by M. Rees ([11]) and we recall briefly some of its main properties. We parametrize the hyperbolic components of $Per_3(0)$ by the unit disc in the natural way. For the Bitransitive and Capture components we use the well defined Böttcher map in a small neighbourhood of each point of the critical cycle $\{0,\infty,1\}$ and for the Disjoint type components the multiplier of the attracting cycle different from $\{0,\infty,1\}$.

The first known result is the existence of only two Bitranstitve components ([11]) denoted by B_1 and B_∞ and defined by

$$B_1 = \{a \in \mathbb{C} \,|\, a \in U_1(a)\} \text{ and } B_\infty = \{a \in \mathbb{C} \,|\, a \in U_\infty(a)\}.$$

B_1 is open, bounded, connected and simply connected and B_∞ is open, unbounded, connected and simply connected in $\hat{\mathbb{C}}$. In the next result we collect these and other main properties of the parameter plane ([11]) (see Figure 8 and 9).

Proposition 4.1 (Rees, [11]). *For $f_a(z)$ with $a \in \mathbb{C} \setminus \{0, 2\}$, the following conditions hold:*

(a) The boundaries of B_1 and B_∞ meet at three parameters $0, x$ and $\bar{x}$ and the set $\mathbb{C} \setminus (B_1 \cup B_\infty \cup \{0, x, \bar{x}\})$ has exactly three connected components: Ω_1, Ω_2 and Ω_3.

(b) Each connected component Ω_i, for $i = 1, 2, 3$, contains a unique value a_i such that f_{a_i} is conformally conjugate to a polynomial map of degree 2. Moreover, each one of the three parameters a_i is the center of a hyperbolic component of period one.

(c) Each parameter value, $0, x$ and $\bar{x}$, is the landing point of two fixed parameter rays, one in B_1 and one in B_∞.

(d) The parameter values x and $\bar{x}$ correspond to parabolic maps having a fixed point with multiplier $e^{2\pi i/3}$ and $e^{-2\pi i/3}$, respectively.

In Figure 10 we plot the $a-$parameter plane. In this picture we label the two hyperbolic components B_1 and B_∞ of *Bitransitive* type and the cutting points $0, x$ and $\bar{x}$ that separate this parameter plane into three different zones: Ω_1, Ω_2 and Ω_3. Each zone contains a unique parameter a such that f_a is conformally conjugate to a quadratic polynomial. We will show that these three parameter values are a_1, a_2 and $\overline{a_2}$ (plotted with a small black circle), which correspond to the airplane, the rabbit and the co-rabbit, respectively. For this reason we call the different pieces the airplane, the rabbit and the co-rabbit piece, respectively.

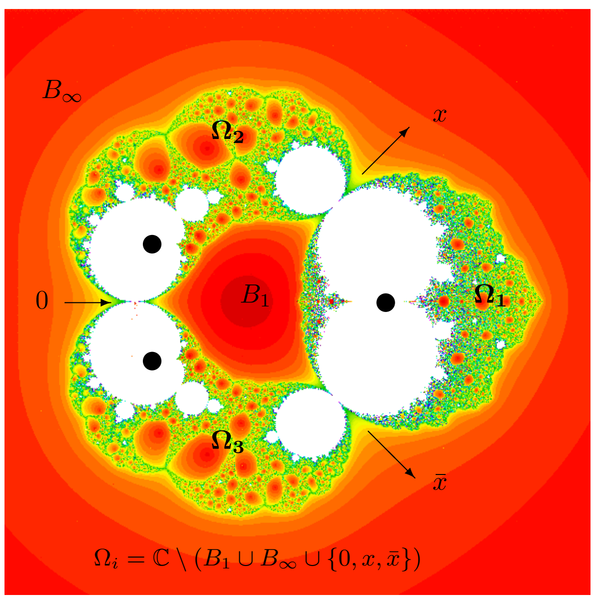
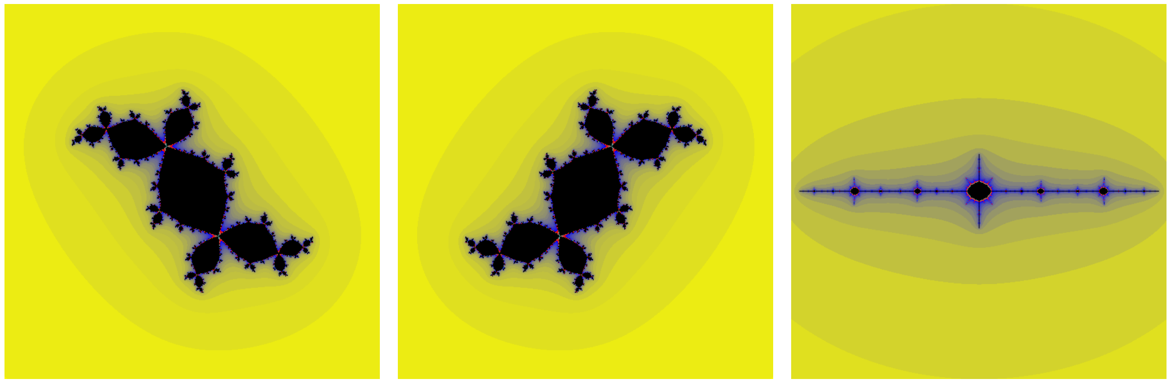


Figure 8: The slice $Per_3(0)$.

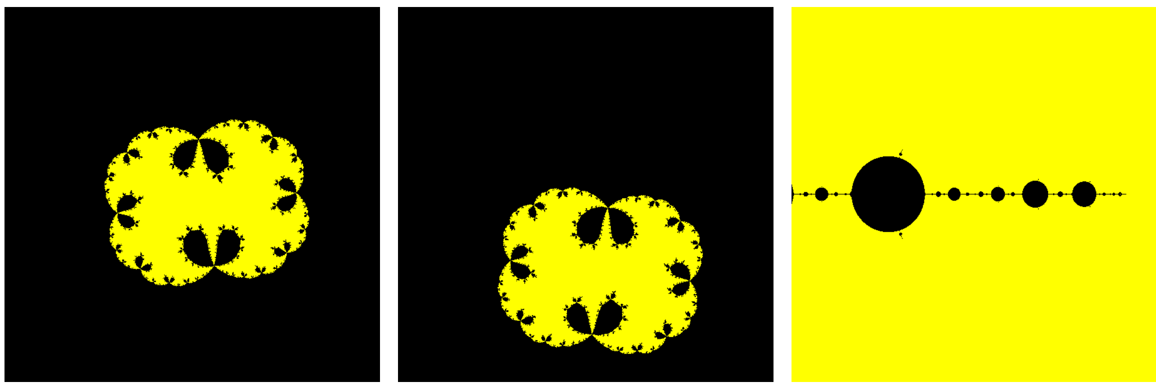
We can find explicitly the values of x and $\bar{x}$ and the quadratic polynomial f_{a_i}, for $i = 1, 2, 3$. First, we calculate the three parameters a_1, a_2 and a_3 such that the corresponding quadratic rational map f_{a_i} is conformally conjugate to a quadratic polynomial. This can happen if and only if the free critical point $c_2(a) = a$ is a superattracting fixed point. This superattracting fixed point plays the role of ∞ for the quadratic polynomial. This



(a) The Douady rabbit. The Julia set of $z^2 - 0.122561 + 0.744861i$.

(b) The Douady co–rabbit. The Julia set of $z^2 - 0.122561 - 0.744861i$.

(c) The airplane. The Julia set of $z^2 - 1.75488$.



(d) Julia set of $f_{0.33764+0.56228i}$, in $Per_3(0)$, conjugate to the Douady rabbit.

(e) Julia set of $f_{0.33764-0.56228i}$, in $Per_3(0)$, conjugate to the Douady co-rabbit.

(f) Julia set of $f_{2.32472}$, in $Per_3(0)$, conjugate to the airplane.

Figure 9: We plot the three unique monic, quadratic, centered polynomial having a superattracting 3-cycle: the rabbit, the co-rabbit and the airplane, and the three corresponding rational maps f_a that are conformally conjugate to a quadratic polynomial.

condition says that the corresponding critical value $v_2(a)$ coincides with the critical point $c_2(a)$, or equivalently

$$v_2(a) = -\frac{(1-a)^2}{a(2-a)} = a$$

which yields

$$a^3 - 3a^2 + 2a - 1 = 0.$$

The above equation has one real solution $a_1 \approx 2.32472$ and two complex conjugate solutions $a_2 \approx 0.33764 + 0.56228i$ and $a_3 \approx 0.33764 - 0.56228i$. We notice that there are only three monic and centered quadratic polynomials of the form $z^2 + c$ that exhibit a 3-critical cycle. These three polynomials are the airplane $z^2 - 1.7588$, the rabbit $z^2 - 0.122561 + 0.744861i$ and the co-rabbit $z^2 - 0.122561 - 0.744861i$. We claim that f_{a_1} is conformally conjugate to the airplane, f_{a_2} to the rabbit and $f_{\overline{a_2}}$ to the co-rabbit. To see this we define the map

$$\tau(z) = \frac{1}{z - a_i} + \frac{1}{a_i}$$

and then $P_i := \tau \circ f_{a_i} \circ \tau^{-1}$ is a centered quadratic polynomial, since ∞ is a superattracing fixed point and $z = 0$ is the unique finite critical point. Easy computations show that

$$P_i(z) = \frac{1}{a_i} - a_i^3(a_i - 2)z^2.$$

Finally, after conjugation with the affine map $\sigma(z) = -a_i^3(a_i - 2)z$, the corresponding quadratic polynomial $Q_i := \sigma \circ P_i \circ \sigma^{-1}$ is given by

$$Q_i(z) = z^2 - a_i^2(a_i - 2),$$

which coincides with the airplane for $i = 1$, the rabbit for $i = 2$ and the co-rabbit for $i = 3$. We call a_1 the *airplane parameter*, a_2 the *rabbit parameter* and $\overline{a_2}$ the *co-rabbit parameter*. Likewise, we call Ω_1 the *airplane piece* since it contains the airplane parameter a_1, Ω_2 the *rabbit piece* since it contains the rabbit parameter and Ω_3 the *co-rabbit piece* since it contains the co-rabbit parameter.

In the next lemma we show another property of the cutting parameter values x and $\bar{x}$, that will be important in order to determine their values.

Proposition 4.2. *Let Δ_i be the hyperbolic component containing a_i (so that $\Delta_i \subset \Omega_i$), $i = 1, 2, 3$. Then, the cutting parameter values x and $\bar{x}$ in Proposition 4.1 belong to the boundary of Δ_1, and not the boundary of Δ_2 and Δ_3.*

Proof. When a parameter a belongs to any of the Δ_i, $i = 1, 2, 3$, the corresponding dynamical plane exhibits an attracting fixed basin associated to an attracting fixed point denoted, in what follows, by $p(a)$. From Proposition 4.1 we know that f_x (respectively $f_{\bar{x}}$) has a parabolic fixed point, $p(x)$ (respectively $p(\bar{x})$), with multiplier $e^{2\pi i/3}$ (respectively $e^{-2\pi i/3}$). Thus x and $\bar{x}$ must belong to $\partial\Delta_1$, $\partial\Delta_2$, or $\partial\Delta_3$. Moreover since x and $\bar{x}$ also belong to ∂B_1 (and ∂B_∞), the dynamical plane for f_x and $f_{\bar{x}}$ are such that $p(x)$ and $p(\bar{x})$ must belong to $\partial U_0 \cap \partial U_1 \cap \partial U_\infty$. These are the two conditions defining the parameters x and $\bar{x}$ (see Figure 10).

When the parameter a, belonging to any of the Δ_i, $i = 1, 2, 3$, crosses the boundary of its hyperbolic component through its 1/3–bifurcation point, the attracting fixed point $p(a)$ becomes a parabolic fixed point of multiplier either $e^{2\pi i/3}$ or $e^{-2\pi i/3}$ since, at this precise parameter value, the attracting fixed point coalesces with a repelling periodic orbit of period three.

Since f_a, $a \in \mathbb{C}$, is a quadratic rational map, it has only two 3-cycles and, because we are in $Per_3(0)$, one of them is the critical cycle $\{0, \infty, 1\}$. So, the repelling periodic orbit which coalesces with $p(a)$ at the 1/3–bifurcation parameter must be the unique repelling 3-cycle existing for this parameter.

We investigate the location of this repelling 3-cycle for parameters in each of the hyperbolic components Δ_1, Δ_2 and Δ_3. To do so, we note that if a is any parameter in Δ_i, we have that $f_a^3 : \overline{U_0} \mapsto \overline{U_0}$ is conjugate to the map $z \mapsto z^2$ in the closed unit disc. Thus, there exists a unique point $z_0(a) \in \partial U_0$ such that $f_a^3(z_0(a)) = z_0(a)$. This fixed point could be either a (repelling) fixed point for f_a or a (repelling) 3-cycle of f_a.

It is clear that for $a = a_1$ the point $z_0(a_1)$ is a repelling 3-cycle, since, for the airplane, $\partial U_0 \cap \partial U_\infty \cap \partial U_1$ is empty. So, this configuration remains for all parameters in Δ_1 (the hyperbolic component containing the airplane parameter). At the 1/3–bifurcation points of Δ_1, the repelling periodic orbit $\{z_0(a), f(z_0(a)), f^2(z_0(a))\}$) coalesces with $p(a)$ (the attracting fixed point), and this collision must happen in $\partial U_0 \cap \partial U_\infty \cap \partial U_1$. So the 1/3–bifurcation parameters of Δ_1 are precisely the parameter values $a = x$ and $a = \bar{x}$, and so, $p(a)$ becomes either $p(x)$ or $p(\bar{x})$, respectively.

On the other hand for $a = a_i$, $i = 2, 3$ the point $z_0(a_i)$ is a fixed point (since for the rabbit and co-rabbit $\partial U_0 \cap \partial U_\infty \cap \partial U_1$ is precisely $z_0(a_i)$). As before this configuration remains for all parameters in Δ_i, $i = 2, 3$ (the hyperbolic components containing the

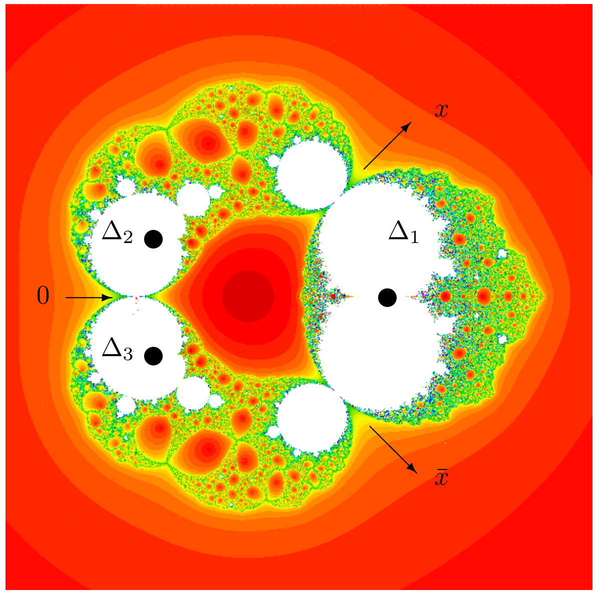


Figure 10: The slice $Per_3(0)$.

rabbit and co-rabbit, respectively). Therefore, at the 1/3–bifurcation point of Δ_i, $i = 2, 3$, the fixed point $p(a)$ coalesces with the repelling periodic orbit but this collision does not happen in $\partial U_0 \cup \partial U_\infty \cup \partial U_1$ since the repelling periodic orbit of period three does not belong to $\partial U_0 \cup \partial U_\infty \cup \partial U_1$. Consequently the resulting parabolic point is not in $\partial U_0 \cap \partial U_\infty \cap \partial U_1$ and the 1/3–bifurcation parameter can neither be x nor $\bar{x}$. $\square$

Doing easy numerical computations we get that there are five parameter values having a parabolic fixed point with multiplier $e^{2\pi i/3}$ or $e^{-2\pi i/3}$. These are

$$0, \quad \approx 1.84445 \pm 0.893455i, \quad \approx 0.441264 \pm 0.59116i.$$

It is easy to show that $x \approx 1.84445 + 0.893455i$ (and so, $\bar{x} \approx 1.84445 - 0.893455i$). Thus the parameters $0.441264 \pm 0.59116i$ corresponds to the 1/3–bifurcations of Δ_2 and Δ_3, respectively.

Now we are ready to state and prove Theorem B.

Proof of Theorem B.

(a) Assume $a \in (B_1 \cup B_\infty)$. From Theorem 2.3 we know that B_1 has a unique center at $a = 1$. Likewise, $a = \infty$ is the unique center of B_∞. In either case the corresponding map f_{a_0} is a critically finite hyperbolic map in $Per_3(0)$ of type B. Thus, from Theorem A (d) $J(f_{a_0})$ is not a Sierpiński curve. We conclude that $J(f_a)$ is not a Sierpiński curve either, since all Julia sets in the same hyperbolic component are homeomorphic.

(b) Assume $a \in \Omega_2$ (here we do not restrict to a hyperbolic parameter). From the previous proposition we know that there exists a fixed point $z_0(a) \in \partial U_0 \cap \partial U_\infty \cap \partial U_1$ and this fixed point is the natural continuation of $z_0(a_2)$ which cannot bifurcate until $a = x \in \Delta_1$. The case $a \in \Omega_3$ is similar.

(c) Finally we assume $a \in H$ where H is a hyperbolic component of type C in Ω_1. We know that Ω_1 contains the airplane polynomial for which $\partial U_0 \cap \partial U_\infty \cap \partial U_1 = \emptyset$.

This configuration cannot change unless the period 3 repelling cycle coalesces with a fixed point, which only happens at $a = x$ or $a = \bar{x}$. Hence the intersection is empty for all parameters in Ω_1. It follows from Theorem A (e) that this is the precise condition for $J(f_a)$ to be a Sierpiński curve.

□

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