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MISIUREWICZ POINTS FOR COMPLEX EXPONENTIALS

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In this paper we examine the structure of the chaotic regime or Julia set of certain complex exponential maps $E_\lambda(z) = \lambda e^z$. In the case where λ is a Misiurewicz point (i.e. the singular value 0 is eventually periodic), it is known that the Julia set for the map is the entire plane. In this case the Julia set also possesses certain curves or "hairs" that are permuted by the map. We examine the dynamics on these hairs in detail. We describe a certain extended symbolic dynamics by which the topological structure of the hairs may be determined completely.

1. Introduction

Our goal in this paper is to describe in detail the dynamics of the complex analytic map $E_\lambda(z) = \lambda e^z$ in the case where the complex parameter value λ is a *Misiurewicz point.* This means that the singular value 0 has orbit that is strictly preperiodic. For example, the parameter values $2k\pi i$, $k \in \mathbf{Z}$, are Misiurewicz points, since 0 is mapped directly to $2k\pi i$, which is a repelling fixed point. Similarly, $(2k+1)\pi i$ also yields Misiurewicz points, since 0 maps in this case onto fixed points of the second iteration. We will discuss these important special cases in detail in the sequel.

In complex dynamics, all of the "chaotic" dynamics occurs on the so-called Julia set of the map. We will define this set in more detail in the following section. Here we simply note that the Julia set for E_λ has several equivalent formulations. The Julia set is the closure of the set of repelling periodic points and it is, at the same time, the closure of the set of points that escape to infinity. Since repelling periodic points are dense in the Julia set, it follows that the dynamics are sensitive to initial conditions on the Julia set.

In the Misiurewicz case, it follows from a theorem of Sullivan [1985], as extended to the case of entire transcendental functions by Goldberg and Keen [1986], that the Julia set of E_λ is the entire complex plane. Moreover, as was shown in [Devaney & Krych, 1984], the dynamics of these maps may be analyzed in detail using symbolic dynamics. In particular, given any sequence $s = (s_0 s_1 s_2 \ldots)$ of nonzero integers for which the s_i are bounded, we can find a unique point z_s whose orbit is bounded for E_λ and for which the orbit of z_s moves about certain natural fundamental domains in the manner prescribed by the s_i. We will describe this construction in more detail below.

One of the important dynamical properties of the exponential map is that each of the points z_s comes equipped with at least one "hair" $\gamma(z_s)$ attached. The hair is a smooth curve that extends from z_s to infinity in the right half plane. The map E_λ permutes these curves, and the orbit of any point on $\gamma(z_s)$ (except z_s) tends to infinity in the right half plane. This fact was proved in [Devaney & Tangerman, 1986]. Related questions can be found in [da Silva Viana, 1988] or [Gelfreich *et al.*, 1992].

One of the important properties of Misiurewicz points is the fact that certain z_s may have more than one hair attached. Indeed, it is often the case that infinitely many z_s come equipped with multiple