International Journal of Bifurcation and Chaos, Vol. 15, No. 10 (2005) 3281–3293

INDECOMPOSABLE CONTINUA AND MISIUREWICZ POINTS IN EXPONENTIAL DYNAMICS

ROBERT L. DEVANEY
Mathematics Department, Boston University,
Boston, MA 02215, USA

XAVIER JARQUE
Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona,
Gran Via 585, 08007 Barcelona, Spain

MÓNICA MORENO ROCHA
Mathematics Department, Tufts University,
Medford, MA 02155, USA

Received January 22, 2005; Revised February 4, 2005

In this paper we describe several new types of invariant sets that appear in the Julia sets of the complex exponential functions $E_\lambda(z) = \lambda e^z$ where $\lambda \in \mathbb{C}$ in the special case when λ is a Misiurewicz parameter, so that the Julia set of these maps is the entire complex plane. These invariant sets consist of points that share the same *itinerary* under iteration of E_λ. Previously, the only known types of such invariant sets were either simple hairs that extend from a definite endpoint to ∞ in the right half plane or else indecomposable continua for which a single hair accumulates everywhere upon itself. One new type of invariant set that we construct in this paper is an indecomposable continuum in which a pair of hairs accumulate upon each other, rather than a single hair having this property. The second type consists of an indecomposable continuum together with a completely separate hair that accumulates on this continuum.

Keywords: Complex exponential map, Julia set; indecomposable continuum; Misiurewicz point.

1. Introduction

In this paper we describe several new types of invariant sets that appear in the Julia sets of complex exponential functions $E_\lambda(z) = \lambda e^z$ where $\lambda \in \mathbb{C}$. These invariant sets consist of points that share the same *itinerary* under iteration of E_λ. Since these exponential functions are $2\pi i$ periodic, there are several "natural" ways (described below) to decompose the plane into countably many strips of vertical height 2π which are then indexed in the natural way by the integers according to the increasing imaginary parts of the strips. The itinerary of a point is then the sequence of integers that describes how the orbit of that point passes through these various strips. Thus we investigate the sets of points whose orbits make the transit through these strips in the same fashion.

For complex analytic maps, the Julia set consists of all points at which the family of iterates of the map fails to form a normal family in the sense of Montel. Equivalently, the Julia set may be described as either the closure of the set of repelling periodic points or else as the set of points on which the map behaves chaotically. For E_λ, the Julia set is also the closure of the set of points whose orbits tend to ∞ [Goldberg & Keen, 1986]. We denote the Julia set of the exponential map by $J(E_\lambda)$. It is well known that, if $J(E_\lambda)$ contains an open set, then in fact the Julia set must be the entire plane. Otherwise, $J(E_\lambda)$ is a nowhere dense subset of the plane.