$\mathbb{Z}_2$-SYMMETRIC PLANAR POLYNOMIAL HAMILTONIAN SYSTEMS OF DEGREE 3 WITH NILPOTENT CENTERS

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Abstract. We provide the normal forms and the global phase portraits in the Poincaré disk of all $\mathbb{Z}_2$-symmetric planar polynomial Hamiltonian systems of degree 3 having a nilpotent center at the origin.

1. Introduction and statement of the results

In this paper we study the global phase portrait of all $\mathbb{Z}_2$-symmetric planar polynomial Hamiltonian systems of degree 3 having a nilpotent center at the origin. Let $H(x, y)$ be a real polynomial in the variables x and y. Then a system of the form

$$x' = H_y \quad y' = -H_x$$

is called a *polynomial Hamiltonian system*. Here the prime denotes derivative with respect to the independent variable t.

Poincaré in [21] defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point. Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in $\mathbb{R}^2$, and assume that p is a center. Without loss of generality we can assume that p is at the origin of coordinates. Then after a linear change of variables and a rescaling of the time variable (if necessary), the system can be written in one of the following three forms

$$(1) \qquad x' = -y + P(x, y), \qquad y' = x + Q(x, y),$$

$$(2) \qquad x' = y + P(x, y), \qquad y' = Q(x, y),$$

$$(3) \qquad x' = P(x, y), \qquad y' = Q(x, y),$$

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where $P(x,y)$ and $Q(x,y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. In what follows a center of an analytic differential system in $\mathbb{R}^2$ is called *linear type*, *nilpotent* or *degenerate* if after an affine change of variables and a rescaling of the time it can be written as system (1), (2) or (3), respectively.

Without loss of generality we can assume that a Hamiltonian system of degree three with a nilpotent center at the origin is given by

$$
\begin{aligned}
x' &= y + a_2x^2 + 2a_3xy + 3a_4y^2 + a_6x^3 + 2a_7x^2y + 3a_8xy^2 + 4a_9y^3,\\
y' &= -3a_1x^2 - 2a_2xy - a_3y^2 - 4a_5x^3 - 3a_6x^2y - 2a_7xy^2 - a_8y^3.
\end{aligned} \tag{4}
$$

The classification of centers for real planar polynomial differential systems started with the classification of centers for quadratic polynomial differential systems, and these results go back mainly to Dulac [10], Kapteyn [13, 14] and Bautin [2]. In [22] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. There are many partial results for the centers of planar polynomial differential systems of degree larger than two. For instance for polynomial differential systems of the form linear plus homogeneous nonlinearities of degree greater than three the centers at the origin are not characterized, but there are partial results for degree four and five for the linear type centers, see for instance Chavarriga and Giné [3, 4]. Some results for higher degree are known see for instance [12]. Recently Colak, Llibre and Valls [5, 6, 7, 8] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. More recently, Dias, Llibre and Valls [9] classified the global phase portraits of all Hamiltonian planar polynomial vector fields of degree three symmetric with respect to the $x-$axis having a nilpotent center at the origin.

In this work we classify and provide the global phase portraits of all $\mathbb{Z}_2$-symmetric planar polynomial Hamiltonian systems of degree 3 having a nilpotent center at the origin.

Let $\mathcal{X}\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field associated to system (4). We define the matrix

$$
M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Then it is $\mathbb{Z}_2$–*equivariant* if

(5) either $M\mathcal{X}(x,y) = \mathcal{X}(-x,y)$,

(6) or $-M\mathcal{X}(x,y) = \mathcal{X}(x,-y)$,

(7) or $-\mathcal{X}(x,y) = \mathcal{X}(-x,-y)$,

and $\mathcal{X}$ is $\mathbb{Z}_2$–*reversible* if either

(8) either $M\mathcal{X}(x,y) = -\mathcal{X}(-x,y)$,

(9) or $-M\mathcal{X}(x,y) = -\mathcal{X}(x,-y)$,

(10) or $\mathcal{X}(x,y) = \mathcal{X}(-x,-y)$.

Other classes of polynomial vector fields in $\mathbb{R}^2$ with a $\mathbb{Z}_2$-symmetry have been studied by several authors, see for instance [15, 16, 17, 23].

Systems (4) satisfying equation (7) and (9) were studied in the article [5, 6] and [9], respectively. Systems (4) does not satisfy equations (5), (6) and (10). Hence it remains only to study the nilpotent centers of the $\mathbb{Z}_2$-symmetric planar polynomial Hamiltonian systems of degree 3 satisfying (8).

The classification will be done using the Poincaré compactification of polynomial vector fields, see section 2. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow.

Our main results are the following ones.

Theorem 1. *All planar polynomial Hamiltonian systems of degree* 3 *with a nilpotent center at the origin satisfying* (8)*, after a linear change of variables and a rescaling of its independent variable t, can be written as one of the following seven systems:*

(I) $x' = y$, $y' = -x^3$;
(II) $x' = y + x^2$, $y' = -2xy - x^3/a^2$, *with* $a \neq 0$;
(III) $x' = y + ax^2 + y^3$, $y' = -2axy - x^3$;
(IV) $x' = y + ax^2 - y^3$, $y' = -2axy - x^3$;
(V) $x' = y + ax^2 + y^2 + by^3$, $y' = -2axy - x^3$;
(VI) $x' = y + ax^2 + by^2 - x^2y + cy^3$, $y' = -2axy - x^3 + xy^2$;
(VII) $x' = y + ax^2 + by^2 + x^2y + cy^3$, $y' = -2axy - x^3 - xy^2$,

where $a \in (-1/\sqrt{2}, 1/\sqrt{2})$ *and* $b, c \in \mathbb{R}$.

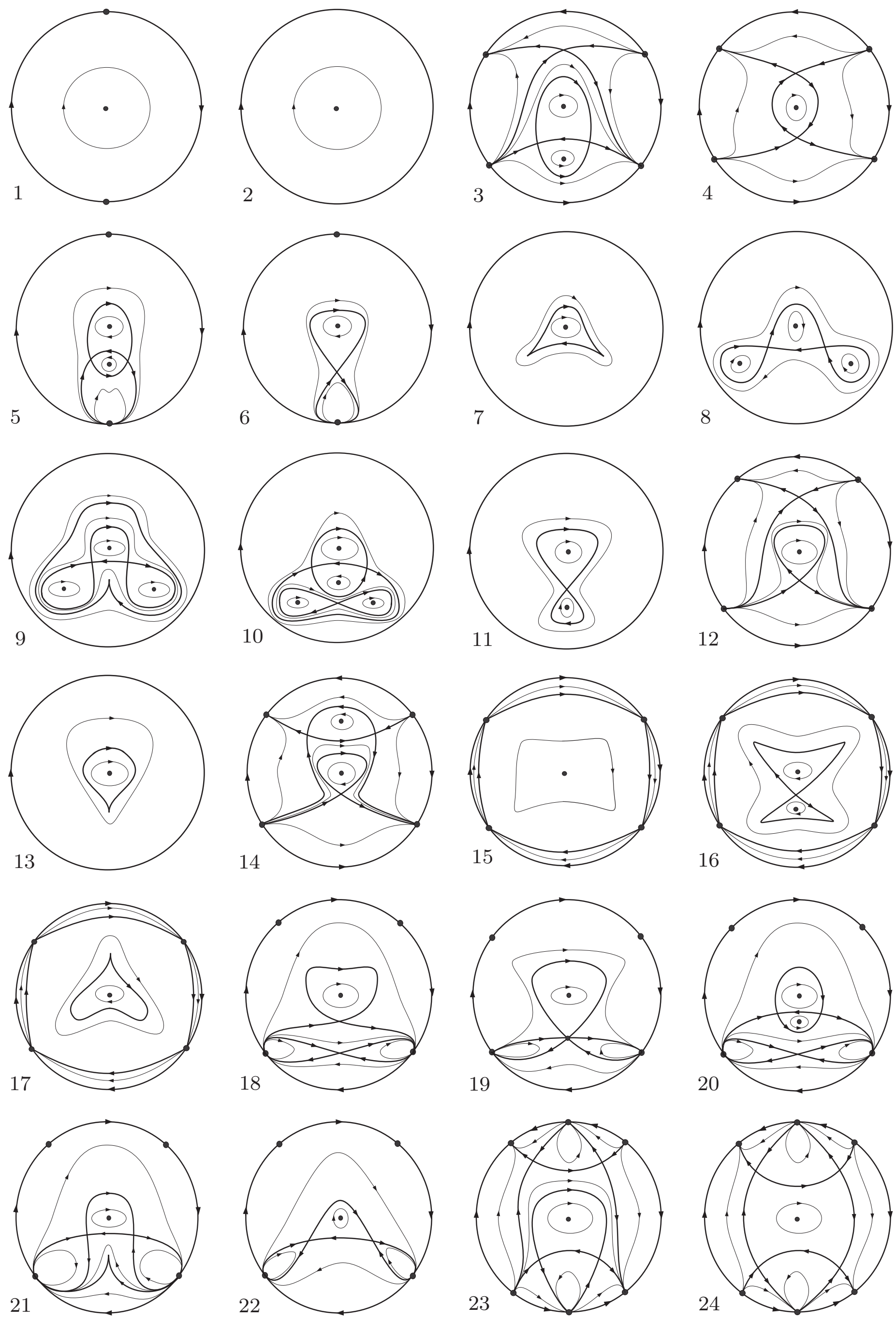


FIGURE 1. Global phase portraits of the vector fields in Theorem 2.

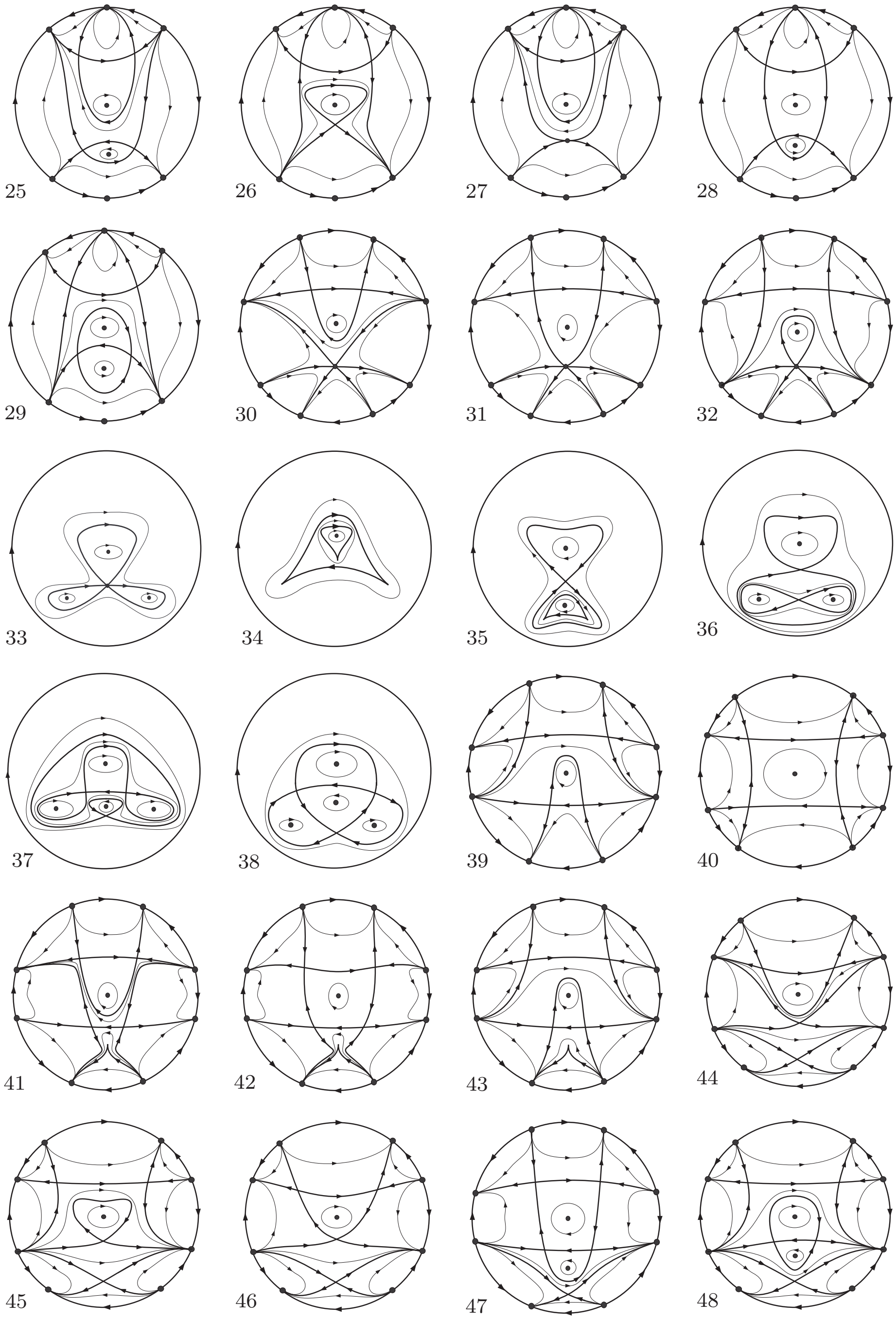


Figure 2. Continuation of Figure 1.

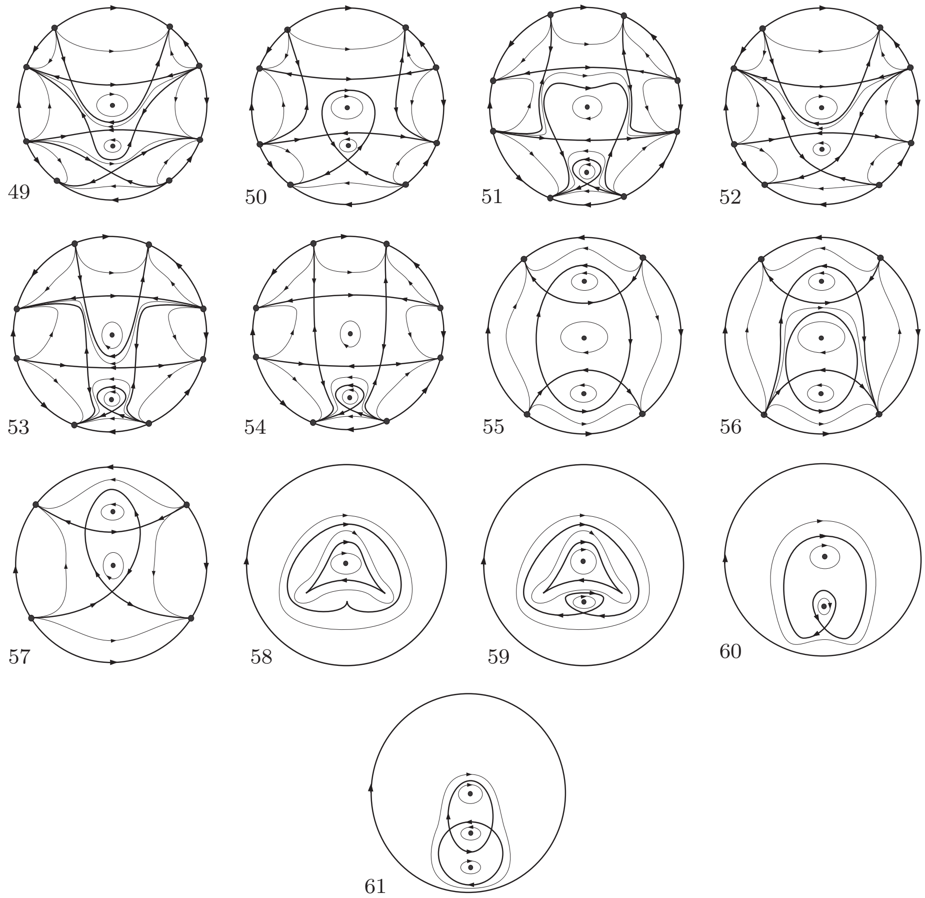


FIGURE 3. Continuation of Figure 1.

Theorem 2. *The global phase portraits of the seven families* $(I)-(VII)$ *in Theorem 1 are topologically equivalent to the phase portraits of Figure 1:*

(a) 1 *for systems* (I) *and* (II)*;*
(b) 2 *for systems* (III)*;*
(c) 3 *and* 4 *for systems* (IV)*;*
(d) 2*,* 3*,* $5-14$ *for systems* (V)*;*
(e) 2*,* 3*,* $7-11$*,* $13-57$ *for systems* (VI)*;*
(f) $1-14$*,* 37*,* 38*,* $57-61$ *for systems* (VII)*.*

2. Preliminary results

In this section we summarize the Poincaré compactification that we shall use for describing the global phase portrait of our Hamiltonian systems. For more details on the Poincaré compactification see Chapter 5 of [11]. Let $\mathbb{S}^2$ be the sphere of points $(s_1, s_2, s_3) \in \mathbb{R}^3$ such that $s_1^2+s_2^2+s_3^2 = 1$, called the *Poincaré sphere*. Given a polynomial vector field

$$X(x, y) = (x', y') = (P(x, y), Q(x, y))$$

in $\mathbb{R}^2$ of degree d (where d is the maximum of the degrees of the polynomials P and Q) it can be extended analytically to the Poincaré sphere by projecting each point $x \in \mathbb{R}^2$ identified with the point $(x_1, x_2, 1) \in \mathbb{R}^3$ in the Poincaré sphere using the straight line through x and the origin of $\mathbb{R}^3$. The equator $S^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. In this way we obtain a vector field $\bar{X}$ in $\mathbb{S}^2 \setminus \mathbb{S}^1$. This vector field $\bar{X}$ is formed by two copies of X: one on the northern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ and another on the southern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 < 0\}$. The local charts needed for doing the calculations on the Poincaré sphere are

$$U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}, \quad V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, with the corresponding local maps

$$\varphi_i(s) : U_i \to \mathbb{R}^2, \quad \psi_i(s) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i(s) = -\psi_i(s) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$.

We extend $\bar{X}$ to a vector field $p(X)$ at the whole sphere $\mathbb{S}^2$ by taking $p(X) = v^d\bar{X}$. The expression for the corresponding vector field on $\mathbb{S}^2$ in the local chart U_1 is given by

$$(11) \quad u' = v^d\left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad v' = -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right);$$

the expression for U_2 is

$$(12) \quad u' = v^d\left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad v' = -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and the expression for U_3 is $u' = P(u, v)$, $v' = -Q(u, v)$. The expressions for the charts V_i are those for the charts U_i multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence for studying the vector field X it is enough to study its Poincaré compactification restricted to the northern hemisphere plus $\mathbb{S}^1$. To draw the phase portraits we consider the projection, by $\pi(s_1, s_2, s_3) = (s_1, s_2)$, of the closed northern hemisphere into the local disk $\mathbb{D} = \{(s_1, s_2) : s_1^2+s_2^2 \leq 1\}$, called the *Poincaré disk*.

Finite singular points of X are the singular points $D\pi \circ p(X)$ in the interior of $\mathbb{D}$, and they can be studied using U_3. Infinite singular points of X are the singular points of $D\pi \circ p(X)$ contained in $\mathbb{S}^1$. Note that if

$s \in \mathbb{S}^1$ is an infinite singular point, then $-s$ is also an infinite singular point. Hence to study the infinite singular points it suffices to look for them only at $U_{1|v=0}$ and at the origin of U_2.

Now we see how to characterize the global phase portraits in the Poincaré disc of all $\mathbb{Z}_2$-symmetric planar polynomial Hamiltonian systems of degree 3 having a nilpotent center at the origin.

We recall that a separatrix of $p(X)$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point. It was proved by Neumann [19] that the set of all separatrices of $p(X)$ is closed. We will denote it by $S(p(X))$. The canonical regions of $p(X)$ are the open connected components of $\mathbb{D}\setminus S(p(X))$. The union of $S(p(X))$ with one solution chosen from each canonical region will be called a separatrix configuration. We say that two separatrix configurations $S(p(X))$ and $S(p(Y))$ are topologically equivalent if there is an orientation preserving (or reversing) homeomorphism which maps trajectories of $S(p(X))$ into trajectories of $S(p(Y))$. The following result is due to Markus [18], Neumann [19] and Peixoto [20].

Theorem 3. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations $S(p(X))$ and $S(p(Y))$ are topologically equivalent.*

Finally we mention without getting into too much detail an important result that classifies the finite singular points of Hamiltonian planar polynomial differential systems. For a detailed definition of the (topological) index of a singular point see for instance Chapter 6 of [11], it can be computed easily using the *Poincaré formula* which takes into account the *parabolic sector*, *hyperbolic sector*, and *elliptic sectors* at a singular point, for details see page 18 of [11]. A vector field is said to have the *finite sectorial decomposition property* at a singular point q if either q is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors. We note that all the isolated singular points of a polynomial differential system satisfy the finite vectorial decomposition property, see [11].

Theorem 4 (Poincaré Formula)**.** *Let q be an isolated singular point having the finite sectorial decomposition property. Let e, h an p denote the number of elliptic, hyperbolic and parabolic sectors of q, respectively. Then the index of q is $(e-h)/2+1$.*

From Theorem 4 the following result follows easily.

Corollary 5. *The indices of a saddle, a center and a cusp are* $-1, 1$ *and* 0*, respectively.*

To determine the possible number and local phase portraits of the finite singular points of the systems we will use the *Poincaré-Hopf Theorem* for vector fields in the $2-$dimensional sphere.

Theorem 6. *For every vector field on the sphere* $\mathbb{S}^2$ *with a finite number of singular points, the sum of the indices of these singular points is* 2.

We note that singular points with index 0 are more difficult to detect because they do not contribute to the total index of the singular points of the vector fields on the Poincaré sphere. To overcome this difficulty we present the following proposition, but first we make a remark and give some definitions.

If a singular point p of an analytic vector field X has the two real parts of the eigenvalues of $DX(p)$ non-zero then p is *hyperbolic*. If the eigenvalues of $DX(p)$ are purely imaginary, then p is either a center or a focus. If only one eigenvalue of $DX(p)$ is 0, then p is *semi-hyperbolic*. The hyperbolic and semi-hyperbolic singular points are called *elementary*. If both eigenvalues of $DX(p)$ are 0 but $DX(p)$ is not identically zero, then p is *nilpotent*. Finally, if $DX(p)$ is identically zero then p is *linearly zero*. The local phase portraits of hyperbolic, semi-hyperbolic and nilpotent singular points can be studied using, for instance, Theorems 2.15, 2.19 and 3.5 of [11], respectively. The linearly zero singular points must be studied using the changes of variables known as blow-ups, see for instance [1] and [11].

Remark 7. *Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see Theorem 3.5 of* [11] *and taking into account that Hamiltonian systems cannot have foci).*

3. Proof of Theorem 1

Without loss of generality we can assume that a Hamiltonian system of degree three with a nilpotent center at the origin is given by

$$x' = H_y, \quad y' = -H_x,$$

where

$$\begin{aligned} H(x,y) = {} & y^2/2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3 + a_8x^4 + a_9x^3y \\ & + a_{10}x^2y^2 + a_{11}xy^3 + a_{12}y^4. \end{aligned}$$

Therefore we have the following Hamiltonian system

$$\begin{aligned} x' &= y + a_5x^2 + 2a_6xy + 3a_7y^2 + a_9x^3 + 2a_{10}x^2y + 3a_{11}xy^2 + 4a_{12}y^3, \\ y' &= -(3a_4x^2 + 2a_5xy + a_6y^2 + 4a_8x^3 + 3a_9x^2y + 2a_{10}xy^2 + a_{11}y^3). \end{aligned} \tag{13}$$

In order that systems (13) be $\mathbb{Z}_2$-reversible as in (8) it must be invariant under $(x, y, t) \to (-x, y, -t)$ and so we have that $a_4 = a_6 = a_9 = a_{11} = 0$. Hence systems (13) become

$$\begin{aligned} x' &= y + a_5x^2 + 3a_7y^2 + 2a_{10}x^2y + 4a_{12}y^3, \\ y' &= -(2a_5xy + 4a_8x^3 + 2a_{10}xy^2). \end{aligned} \tag{14}$$

Since systems (14) must have a center at the origin, by Theorem 3.5 of [11] we must have $2a_8 > a_5^2$.

Case 1. Assume $a_{10} > 0$. By the change of coordinates and reparametrization of the time of the form

$$x \to \alpha X, \quad y \to \beta Y, \quad t \to \gamma\tau, \tag{15}$$

with $\alpha = -1/\sqrt{2a_{10}}$, $\beta = -\sqrt{a_8}/a_{10}$ and $\gamma = \sqrt{a_{10}/2a_8}$, systems (14) can be written as

$$\begin{aligned} X' &= Y - \frac{a_5}{2\sqrt{a_8}}X^2 - \frac{3a_7\sqrt{a_8}}{a_{10}}Y^2 + X^2Y + \frac{4a_{12}a_8}{a_{10}^2}Y^3, \\ Y' &= \frac{a_5}{\sqrt{a_8}}XY - X^3 - XY^2. \end{aligned}$$

We obtain the normal form (VII). Note that, since $2a_8 > a_5^2$, we have

$$\left| a = \frac{-a_5}{2\sqrt{a_8}} \right| < \frac{1}{\sqrt{2}}.$$

Case 2. Assume $a_{10} < 0$. By the change of coordinates and reparametrization of the time of the form as in (15) with $\alpha = -1/\sqrt{-2a_{10}}$, $\beta = \sqrt{a_8}/a_{10}$ and $\gamma = \sqrt{-a_{10}/2a_8}$, systems (14) can be written as

$$\begin{aligned} X' &= Y - \frac{a_5}{2\sqrt{a_8}}X^2 + \frac{3a_7\sqrt{a_8}}{a_{10}}Y^2 - X^2Y + \frac{4a_{12}a_8}{a_{10}^2}Y^3, \\ Y' &= \frac{a_5}{\sqrt{a_8}}XY - X^3 + XY^2. \end{aligned}$$

We obtain the normal form (VI).

Case 3. Assume $a_{10} = 0$ and $a_7 \neq 0$. By the change of coordinates and reparametrization of the time as in (15) with $\alpha = -1/(\sqrt{6a_7}a_8^{1/4})$, $\beta = 1/3a_7$, $\gamma = -\sqrt{3a_7}/(\sqrt{2}a_8^{1/4})$ if $a_7 > 0$, and $\alpha = 1/(\sqrt{-6a_7}a_8^{1/4})$, $\beta =$

$1/3a_7$, $\gamma = -\sqrt{-3a_7}/(\sqrt{2}\ a_8^{1/4})$ if $a_7 < 0$, systems (14) can be written as

$$\begin{aligned} X' &= Y \pm \frac{a_5}{2\sqrt{a_8}} X^2 + Y^2 + \frac{4a_{12}}{9a_7^2} Y^3, \\ Y' &= \mp \frac{a_5}{\sqrt{a_8}} XY - X^3. \end{aligned}$$

We obtain the normal form (V).

Case 4. Assume $a_7 = a_{10} = 0$ and $a_{12} \neq 0$. By the change of coordinates and reparametrization of the time as in (15) with $\alpha = -1/(2(|a_{12}|a_8)^{1/4}), \beta = -1/(2\sqrt{|a_{12}|})$, $\gamma = (|a_{12}|/a_8)^{1/4}$, systems (14) can be written as

$$\begin{aligned} X' &= Y - \frac{a_5}{2\sqrt{a_8}} X^2 \pm Y^3, \\ Y' &= \frac{a_5}{\sqrt{a_8}} XY - X^3. \end{aligned}$$

We obtain the normal forms (III) and (IV).

Case 5. Assume $a_7 = a_{10} = a_{12} = 0$ and $a_5 \neq 0$. By the change of coordinates and reparametrization of the time as in (15) with $\alpha = \beta = 1/a_5$ and $\gamma = 1$ systems (14) can be written as

$$\begin{aligned} X' &= Y + X^2, \\ Y' &= -2XY - \frac{4a_8}{a_5^2} X^3. \end{aligned}$$

We obtain the normal form (II).

Case 6. Assume $a_7 = a_{10} = a_{12} = a_5 = 0$. By the change of coordinates and reparametrization of the time as in (15) with $\alpha = 1$, $\beta = -2\sqrt{a_8}$ and $\gamma = -1/(2\sqrt{a_8})$ systems (14) can be written as obtain the normal form (I).

In short we have proved Proposition 1.

4. Proof of Theorem 2

4.1. Global phase portrait of system (I). The phase portrait of this system is topologically equivalent to the phase portrait 1 of Figure 1, and this is proved in [9].

4.2. Global phase portrait of system (II). Since $a \in (-1/\sqrt{2}, 1/\sqrt{2})$, $a \neq 0$, we have that $(0, 0)$ is the unique finite singular point of systems (II).

We now investigate the infinite singular points of systems (II). In the local chart U_1 from (11) system (II) is

$$u' = -\frac{1}{a^2} - 3uv - u^2v^2, \quad v' = -v^2(uv+1).$$

When $v = 0$ there are no infinite singular points on U_1.

In U_2 from (12) systems (II) becomes

$$u' = v^2 + 3vu^2 + \frac{u^4}{a^2}, \quad v' = \frac{uv(u^2+2a^2v)}{a^2}. \tag{16}$$

The origin is an infinite singular point of the system, whose linear part is zero. So we need to do blow-ups to describe the local dynamics at this point. We perform the directional blow-up $(u,v) \mapsto (u,w)$ with $w = v/u^2$ and we get

$$u' = u^3\Big(\frac{1}{a^2}u + 3uw + uw^2\Big), \quad w' = -u^3w\Big(\frac{1}{a^2} + 4w + 2w^2\Big). \tag{17}$$

Now we eliminate the common factor u^3 between u' and w' and we get the vector field

$$u' = \frac{1}{a^2}u + 3uw + uw^2, \quad w' = -\frac{1}{a^2}w - 4w^2 - 2w^3. \tag{18}$$

System (18) has the origin as its unique singular point. The eigenvalues of the linear part at the origin are $1/a^2$ and $-1/a^2$, so it is a hyperbolic saddle.

Going back through the changes of variables until system (16) as shown in Figure 4, we have that the global phase portrait of system (II) is topologically equivalent to the phase portrait 1 of Figure 1.

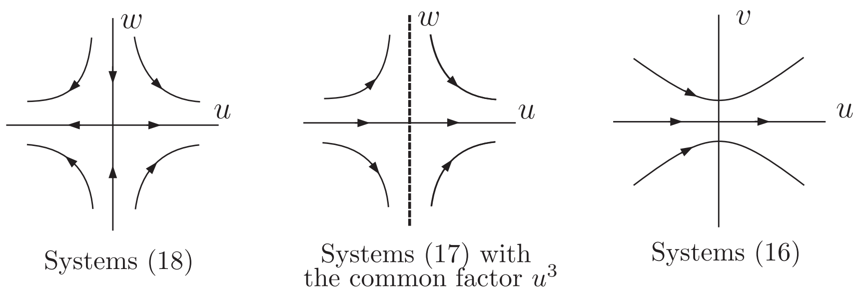


FIGURE 4. Blow-up of the origin of U_2 of system (II).

4.3. **Global phase portrait of systems (III).** Again the origin $(0,0)$ is the unique finite singular point of these systems since $a \in (-1/\sqrt{2}, 1/\sqrt{2})$. We will now investigate the infinite singular points of systems (III).

On the local chart U_1 systems (III) become

$$u' = -1 - 3auv - u^2v^2 - u^4, \quad v' = -v(av + uv^2 + u^3).$$

When $v = 0$ there are no infinite singular points on U_1.

In U_2, systems (III) can be written as

$$u' = 1 + v^2 + 3au^2v + u^4, \quad v' = uv(2av + u^2).$$

Again when $v = 0$ there are no infinite singular points on U_2. Therefore the global phase portrait of systems (III) are topologically equivalent to the phase portrait 2 of Figure 1.

4.4. **Global phase portrait of systems (IV).** When $a < 0$ the singular points are the origin,

$$(0, \pm 1) \quad \text{and} \quad (\pm\sqrt{-2a}(1-2a^2)^{1/4}, \sqrt{1-2a^2}).$$

In this case we have that $(0,1)$ is a center and the other singular points are saddles. When $a > 0$ the singular points are the origin,

$$(0, \pm 1) \quad \text{and} \quad (\pm\sqrt{2a}(1-2a^2)^{1/4}, -\sqrt{1-2a^2}).$$

Here, we have that $(0,-1)$ is a center and the other singular points are saddles. Finally, when $a = 0$ the finite singular points are $E_0 = (0,0)$ and $E_\pm = (0, \pm 1)$. We will study only the singular point E_+ because the study of the other singular point is analogous. Since the singular point E_+ is nilpotent, using Theorem 3.5 of [11] we obtain that $(0,1)$ is a saddle.

We will now investigate the infinite singular points of systems (IV).

On the local chart U_1 systems (IV) become

$$u' = -1 - 3auv + u^4 - u^2v^2, \quad v' = -v(av - u^3 + uv^2).$$

The infinite singular points are $P_\pm = (\pm 1, 0)$. The eigenvalues of the linear part at P_+ are $(4,1)$. Hence it is an repelling hyperbolic node. On the other hand, the eigenvalues of the linear part of the systems at P_- are $(-4,-1)$ Hence it is an attracting hyperbolic node.

In U_2 systems (IV) are given by

$$u' = -1 + v^2 + 3au^2v + u^4, \quad v' = uv(2av + u^2).$$

Hence the origin is not a singular point in U_2.

Taking into account the local information on the finite and infinite singular points together with the fact that the system is Hamiltonian (and so the saddles $(\pm\sqrt{2a}(1-2a^2)^{1/4}, -\sqrt{1-2a^2})$ are connected but are not connected with the saddle at $(0,-1)$) we get that when $a \neq 0$ the global phase portrait of systems (IV) are topologically equivalent to the phase portrait 3 of Figure 1. On the other hand, when $a = 0$ the global phase portrait of systems (IV) are topologically equivalent to the phase portrait 4 of Figure 1.

4.5. **Global phase portrait of systems (V).** We study the infinite singular points of systems (V).

4.5.1. *Infinite singular points.* In U_1 systems (V) become

$$\begin{aligned} u' &= -1 - 3auv - bu^4 - u^3v - u^2v^2, \\ v' &= -v(av + bu^3 + u^2v + uv^2). \end{aligned}$$

When $v = 0$ the candidates for singular points of systems (V) are the roots of the polynomial $1 + bu^4$. Therefore, if $b \geq 0$ there are no infinite singular points on the local chart U_1. If $b < 0$ the points $(|b|^{-1/4}, 0)$ are repelling hyperbolic nodes and $(-|b|^{-1/4}, 0)$ are attracting hyperbolic nodes. Now we study the origin of U_2 of systems (V) which in U_2 write as

$$u' = b + v + v^2 + 3au^2v + u^4, \quad v' = uv(2av + u^2). \tag{19}$$

If $b \neq 0$ the origin is not singular. If $b = 0$ the origin is singular. Since the origin is nilpotent, using Theorem 3.5 of [11] together with blow-up techniques we obtain that the phase portrait of the origin consists of one hyperbolic, one elliptic and two parabolic sectors, see Figure 5.

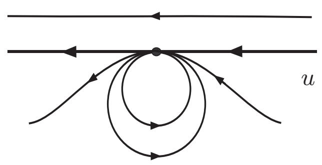


FIGURE 5. Local phase portrait at the origin of system (19) for $b = 0$.

In short we have Table 1

Parameters	Infinite singular points in chart U_1 and U_2
$b>0$	There are no infinite singular points
$b<0$	In the chart U_1 there are one attracting node and one repelling node. In the chart U_2 there are no infinite singular points
$b=0$	In the chart U_1 there are no infinite singular points. In the chart U_2 the origin has a one hyperbolic, one elliptic and two parabolic sectors

TABLE 1. Infinite singular points in the local charts U_1 and U_2.

4.5.2. *Finite singular points.* If $b=0$ the finite singular points are

$$p_0=(0,-1) \text{ and } p_\pm=(\pm\sqrt{2a(1-2a^2)},2a^2-1) \text{ (whenever } a>0).$$

Computing the eigenvalues of the Jacobian matrix at these points we get that $(0,-1)$ is a center if $a>0$ and a saddle if $a<0$. If $a=0$ it is nilpotent, using Theorem 3.5 in [11] we get that it is a saddle. On the other hand, the points $p_\pm$ exist if and only if $a>0$. In this case, computing the eigenvalues of the Jacobian matrix at these points we get that they are both saddles.

Assume now that $b\neq 0$. In this case the candidates for finite singular points of systems (V) other than the origin are

$$E_{1,2}=\left(0,\frac{-1\pm\sqrt{1-4b}}{2b}\right),$$

$$E_{3,4}=\left(\pm\sqrt{\frac{a(1-\sqrt{A})}{b}},-\frac{1-\sqrt{A}}{2b}\right),$$

and

$$E_{5,6}=\left(\pm\sqrt{\frac{a(1+\sqrt{A})}{b}},-\frac{1+\sqrt{A}}{2b}\right)$$

where $A=4b(2a^2-1)+1$. Note that $E_{1,2}$ exist whenever $b^2-4c\geq 0$. On the other hand, $E_{3,4}$ exist whenever $A=0$ and $0<a<1/\sqrt{2}$ with $b\neq 0$, or $A>0$ and $0\leq a<1/\sqrt{2}$ with $b\neq 0$ and $E_{5,6}$ exist whenever $A=0$ and $0<a<1/\sqrt{2}$ with $b\neq 0$, or $A>0$ and $0\leq a<1/\sqrt{2}$ with $b>0$, or $A>0$, $-1/\sqrt{2}<a<0$ and $b<0$. Computing the eigenvalues of the Jacobian matrix at these points we get that all the points are either hyperbolic or nilpotent except the case $A>0$, $a=0$ and $b=1/4$ in which case the point is linearly zero. In this last case, using blow-up techniques we get that the point is the union of two hyperbolic sectors. Finally, using Theorem 3.5 in [11] for the nilpotent points we get Table 2 for the finite singular points. We recall that when we write $a > 0$ means that $0 < a < 1/\sqrt{2}$ and when we write $a < 0$ we mean $-1/\sqrt{2} < a < 0$.

Conditions		Equilibria different from $(0,0)$
$A < 0$		There are no singular points
$b = 0$	$a > 0$	p_0 center, $p_\pm$ saddles
	$a \leq 0$	p_0 saddle
$A \geq 0$, $a < 0$, $b > 1/4$		There are no singular points
$A = 0$, $a > 0$		E_3, E_4 cusps
$A > 0$, $a > 0$	$b > 1/4$	E_3, E_4 saddles, E_5, E_6 centers
$A > 0$, $a > 0$	$b = 1/4$	E_1 cusp, E_3, E_4 saddles, E_5, E_6 centers
$A > 0$, $a > 0$	$0 < b < 1/4$	E_1, E_5, E_6 centers, E_2, E_3, E_4 saddles
$A > 0$, $a > 0$	$b < 0$	E_1 center, E_2, E_3, E_4 saddles
$A > 0$, $a = 0$, $0 < b < 1/4$		E_1 saddle, E_2 center
$A > 0$, $a = 0$, $b < 0$		E_1, E_2 saddles
$A = 0$, $a = 0$, $b = 1/4$		E_1 union of 2 hyperbolic sectors
$A > 0$, $a < 0$	$b = 1/4$	E_1 cusp
$A > 0$, $a < 0$	$0 < b < 1/4$	E_1 saddle, E_2 center
$A > 0$, $a < 0$	$b < 0$	E_1, E_5, E_6 saddles, E_2 center

TABLE 2. Finite singular points of systems V.

Now taking into account the local information on the finite and infinite singular points together with the fact that our system is Hamiltonian (used whenever convenient to obtain possible saddle connections), we have Table 3 where we have listed the phase portraits that systems (V) can be topologically equivalent with, taking into account the above mentioned regions.

4.6. **Global phase portrait of systems (VI).** Without loss of generality we can assume that $b \geq 0$ (otherwise changing $(x, y, a, b, c) \to (-x, -y, -a, -b, c)$ we get the same system with $b \geq 0$). We recall that the Hamiltonian of system (VI) is

$$H = \frac{1}{2}y^2 + ax^2y + \frac{b}{3}y^3 + \frac{1}{4}x^4 - \frac{1}{2}x^2y^2 + \frac{c}{4}y^4.$$

We now study the infinite singular points of these systems. We distinguish between the cases $c \geq 1$, $0 \leq c < 1$ and $c < 0$.

Conditions		Phase portraits
$A < 0$		Figure 2
$b = 0$	$a > 0$	Figure 5
$b = 0$	$a \leq 0$	Figure 6
$A \geq 0$, $a < 0$, $b > 1/4$		Figure 2
$A = 0$, $a > 0$		Figure 7
$A > 0$, $a > 0$	$b > 1/4$	Figure 8
$A > 0$, $a > 0$	$b = 1/4$	Figure 9
$A > 0$, $a > 0$	$0 < b < 1/4$	Figure 10
$A > 0$, $a > 0$	$b < 0$	Figure 3
$A > 0$, $a = 0$, $0 < b < 1/4$		Figure 11
$A > 0$, $a = 0$, $b < 0$		Figure 12
$A = 0$, $a = 0$, $b = 1/4$		Figure 13
$A > 0$, $a < 0$	$b = 1/4$	Figure 13
$A > 0$, $a < 0$	$0 < b < 1/4$	Figure 11
$A > 0$, $a < 0$	$b < 0$	Figure 14

TABLE 3. Topologically equivalent phase portraits of systems V.

4.6.1. *Infinite singular points.* In the local chart U_1 systems (VI) can be written as

$$\begin{aligned} u' &= -1 + 2u^2 - cu^4 - v(3au + u^2v + bu^3), \\ v' &= -v(av - u + uv^2 + bu^2v + cu^3). \end{aligned}$$

When $v = 0$, the candidates for singular points are the roots of the polynomial $-1 + 2u^2 - cu^4$, i.e.

$$\pm\frac{\sqrt{c(1+\sqrt{1-c})}}{c} \quad \text{and} \quad \pm\frac{\sqrt{c(1-\sqrt{1-c})}}{c}, \quad c \neq 0.$$

In the local chart U_2 systems (VI) can be written as

$$u' = v^2 + 3au^2v + bv - 2u^2 + c + u^4, \quad v' = uv(2av + u^2 - 1). \tag{20}$$

4.6.2. *Case $c \geq 1$.* When $c > 1$ there are no infinite singular points on the local chart U_1. If $c = 1$ the points $(\pm 1, 0)$ are infinite singular points on U_1. We will study only the singular point $(1, 0)$ because the study of the other singular point is analogous. When $b \neq -3a$ the point $(1, 0)$ is nilpotent. First we translate $(1, 0)$ the origin. Applying Theorem 3.5 of [11] we obtain that the phase portrait of the singular point $(1, 0)$ consists of one hyperbolic, one elliptic and two parabolic sectors, see Figure 5.

When $b = -3a$, on the other hand, the point $(1, 0)$ is an infinite singular point, whose linear part is zero, hence we need to do a blow-up to characterize the local dynamics at this point. First we translate $(1, 0)$ to the origin. Doing the blow-up $(u, v) \to (u, w)$ with $w = v/u$ and eliminating the common factor u we get the system

$$
\begin{aligned}
u' &= -4u - 4u^2 + 6auw - u^3 + 9au^2w - uw^2 + 3au^3w - 2u^2w^2\\
v' &= 2w + uw - 4w^2a - 3w^2au + w^3 + w^3u.
\end{aligned}
\tag{21}
$$

When $u = 0$, since $2a^2 - 1 < 0$, the only singular point of system (21) is the origin, and it is a saddle. The blow-up of the origin gives the same information as in the case of systems (II), hence the point $(1, 0)$ of U_1 has two hyperbolic sectors, see Figure 4. It is easy to check that the origin of chart U_2 is not a singular point.

4.6.3. *Case* $0 \leq c < 1$. When $0 < c < 1$ there are 4 infinite singular points on the local chart U_1. It is easy to see that all of them are nodes (two attracting nodes and two repelling nodes). On the other hand, if $c = 0$ there are only two singular points $(\pm\sqrt{2}/2, 0)$ on U_1 and both are nodes.

In U_2 systems (VI) are given by (20). Therefore if $0 < c < 1$ the origin is not a singular point in U_2. When $c = 0$ and $b \neq 0$ the origin of the chart U_2 is a nilpotent singular point, using Theorem 3.5 of [11] we see that locally the origin of U_2 consists of one hyperbolic, one elliptic and two parabolic sectors. On the other hand, when $b = c = 0$ the origin of the chart U_2 is an infinite singular point of the system, whose linear part is zero, hence we need to do a blow-up to characterize the local dynamics at this point. Doing the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminating the common factor u we get the system

$$
u' = -2u + u(u^2 + 3auw + w^2), \quad w' = w(1 - auw - w^2). \tag{22}
$$

When $u = 0$ the singular points of system (22) are the roots of the polynomial $w(1 - w)(1 + w)$. Therefore we have three singular points. In this case the origin is a saddle and the other singular points are nodes. Consequently the origin of the local chart U_2 has two elliptic and two parabolic sectors, see Figure 6.

4.6.4. *Case* $c < 0$. In this case the systems (VI) have only two singular points

$$
\pm\frac{\sqrt{c(1 - \sqrt{1 - c})}}{c}.
$$

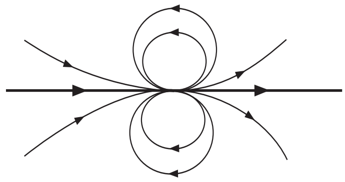


FIGURE 6. Local phase portrait at the origin of system (22).

These points are nodes. Finally, the origin of chart U_2 is not a singular point.

In short we have Tables 4 and 5.

Parameters		Infinite singular points in chart U_1
$c > 1$		There are no infinite singular points
$c = 1$	$b \neq -3a$	Two singular points with one hyperbolic one elliptic and two parabolic sectors
	$b = -3a$	Two singular points with two hyperbolic sectors
$0 < c < 1$		Two attracting nodes and two repelling nodes
$c = 0$	$b \neq 0$	One attracting node and one repelling node
	$b = 0$	
$c < 0$		

TABLE 4. Infinite singular points in chart U_1.

Parameters		The origin of the chart U_2
$c > 1$		There are no singular points
$c = 1$	$b \neq -3a$	
	$b = -3a$	
$0 < c < 1$		
$c = 0$	$b \neq 0$	One hyperbolic, one elliptic and two parabolic sectors
	$b = 0$	Two elliptic and two parabolic sectors
$c < 0$		There are no singular points

TABLE 5. The origin of the chart U_2.

4.6.5. *Finite singular points.* The candidates for singular points of systems (VI) other than the origin are

$$E_{1,2} = \left(0, \frac{-b \pm \sqrt{b^2 - 4c}}{2c}\right), \quad c \neq 0$$

$$E_{3,4} = \left(\pm\sqrt{y_3(y_3 - 2a)}, y_3\right), \quad E_{5,6} = \left(\pm\sqrt{y_5(y_5 - 2a)}, y_5\right), \quad c \neq 1$$

where

$$y_{3,5} = \frac{-(b+3a) \pm \sqrt{(b+3a)^2 + 4(1-c)(1-2a^2)}}{2(c-1)}.$$

As we shall see, we need to study only the singularities $E_{1,2}$ which are the singularities that are on the y-axis. These singularities occur only when $b^2 - 4c \geq 0$.

Consider $b^2 - 4c \geq 0$ and $c \neq 0$. The calculations in this case are very similar to the previous systems. So we will only present the final result in the Table 6.

When $b^2 - 4c > 0$ and $c = 0$. The singular points, on the y-axis, of systems (VI) are the origin and $(0, -1/b)$. The singular point $(0, -1/b)$ is a saddle if $ab < -1/2$ and a center if $ab > -1/2$. When $ab = -1/2$ this point is nilpotent, using Theorem 3.5 of [11] we obtain that $(0, -1/b)$ is a saddle.

When $b^2 - 4c = 0$ and $b \neq 0$. Besides the origin, system (VI) have, on the y-axis, one singular point, namely $(0, -2/b)$. When $ab \neq -1$ the singular point $(0, -2/b)$ is nilpotent, using Theorem 3.5 of [11] we obtain that $(0, -2/b)$ is a cusp. When $ab = -1$, on the other hand, the point $(0, -2/b)$ is a finite singular point, whose linear part is zero, hence we need to do a blow-up to characterize the local dynamics at this point. First we translate $(0, -2/b)$ to the origin. The systems (VI) become

$$\begin{aligned} x' &= \frac{1}{b}x^2 - \frac{b}{2}y^2 - x^2y + \frac{b^2}{4}y^3 \\ y' &= -\frac{2}{b}xy - x^3 + xy^2. \end{aligned} \tag{23}$$

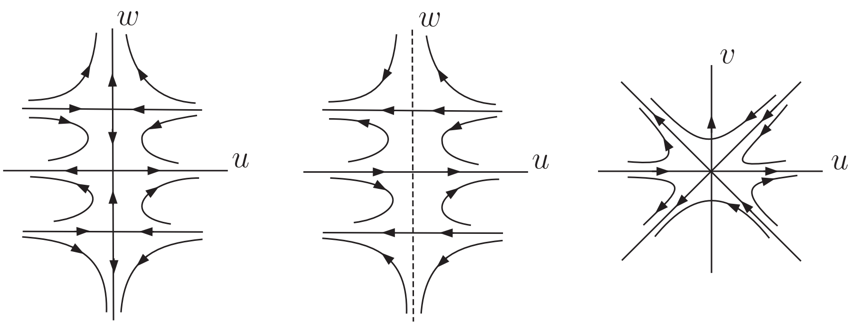
We perform the directional blow-up $(u, v) \to (u, w)$ with $w = v/u$ and we have

$$\begin{aligned} u' &= \frac{1}{b}u^2 - u^2w(u + \frac{b}{2}w - \frac{b^2}{4}uw^2) \\ w' &= -u^2 - \frac{3}{b}uw + uw^2(2u + \frac{b}{2}w - \frac{b^2}{4}uw^2). \end{aligned} \tag{24}$$

We eliminate the common factor u between u' and w', and get vector field

$$
(25) \qquad \begin{aligned}
u' &= \frac{1}{b}u - uw(u + \frac{b}{2}w - \frac{b^2}{4}uw^2) \\
w' &= -u - \frac{3}{b}w + w^2(2u + \frac{b}{2}w - \frac{b^2}{4}uw^2).
\end{aligned}
$$

When $u = 0$ the singular points of system (25) are $(0, 0)$ and $(0, \pm\sqrt{6}/b)$ and all of them are saddles. Consequently the local phase portrait at the point $(0, -2/b)$ of systems (VI), with $ab = -1$, has six hyperbolic, see Figure 7.



Systems (25) Systems (24) with the common factor u Systems (23)

FIGURE 7. Local phase portrait at the point $(0, -2/b)$ of systems (VI), with $ab = -1$.

When $b^2 - 4c = 0$ and $b = 0$, that is, $b = c = 0$ it is easy to check that the origin is the only finite singular on the y-axis.

In short we have Table 6.

Again the next step is to count the indices of the finite and infinite singular points of systems (VI) on the Poincaré sphere. First we need the following proposition. We denote

$$
c^* = -\frac{4 + a^2 + 6ab + b^2}{4(2a^2 - 1)}.
$$

Proposition 8. *System (VI) has non-elementary points with $x \neq 0$ if and only if $c = c^*$. They exist when $3a + b \neq 0$, $1 + a^2 + ab > 0$ and they are two cusps.*

Proof. We compute the Groebner basis between x', y' and the determinant of the Jacobian matrix. The first component of the Groebner basis is

$$
(2a^2 - 1)^2(b^2 - 4c)(1 + 2ab + 4a^2c)^2(4 + a^2 + 6ab + b^2 - 4c + 8a^2c)y.
$$

Parameters			Finite singular points on y–axis except the origin.
$b^2-4c<0$			There are no singular points
$b^2-4c>0$	$c>0$	$a^*<\overline{a}<a$ $a<a^*<\overline{a}$	One center and one saddle
		$a^*\le a\le\overline{a}$	Two saddles
	$c<0$	$\overline{a}<a^*\le a$ $a\le\overline{a}<a^*$	One center and one saddle
		$\overline{a}<a<a^*$	Two centers
	$c=0$	$ab\le -1/2$	One saddle
		$ab>-1/2$	One center
$b^2-4c=0$	$b\neq 0$	$ab\neq -1$	One cusp
		$ab=-1$	One singular point with six hyperbolic sectors
	$b=0$		There are no singular points

Table 6. Finite singular points, on the y–axis, of systems (VI). In this table $\overline{a}=\frac{-b+\sqrt{b^2-4c}}{4c}$ and $a^*=\frac{-b-\sqrt{b^2-4c}}{4c}$.

Note that if $y=0$ the unique singular point is the origin. Moreover, since $a\in(-1/\sqrt{2},1/\sqrt{2})$ we have that $2a^2-1\neq 0$. When $c=b^2/4$ the singular point which is not hyperbolic is $(0,-2/b)$ and belongs to the y-axis. If $1+2ab+4a^2c=0$, we obtain the values $\overline{a}$ and a^* given in Table 6 and the non-hyperbolic singular point is $(0,2a)$ which again lyes in the y-axis.

Finally, when

$$4+a^2+6ab+b^2-4c+8a^2c=0,\ \text{i.e } c=c^*$$

we get that the non-hyperbolic finite singular points are, whenever they exist,

$$\Big(\pm\frac{2\sqrt{(1-2a^2)(1+a^2+ab)}}{3a+b},\frac{2(2a^2-1)}{3a+b}\Big).$$

They exist when $3a+b\neq 0$ and $1+a^2+ab\geq 0$. Note that when $1+a^2+ab=0$ then $c^*=\frac{5a^2-b^2+2}{4(2a^2-1)}$ and both points lye on the y-axis. Hence, in order that they exist and are outside the y-axis we must have $3a+b\neq 0$ and $1+a^2+ab>0$. Computing the eigenvalues of the Jacobian matrix at these points we see that they are nilpotent and using Theorem 3.5 in [11] we conclude that they are cusps. This concludes the proof of the proposition. $\square$

We will study the different global phase portraits on the Poincaré sphere as follows: $c = 1$ (that we will distinguish between $b + 3a = 0$ and $b+3a \neq 0$); $c = 0$ (that we will distinguish between the cases $b = 0$ and $b \neq 0$); $b^2 = 4c$ with $ab = -1$; $c = c^*$ with $a + 3b \neq 0$ (otherwise $c^* = 1$) and $1 + a^2 + ab > 0$ (otherwise there are no non-hyperbolic singular points outside the y-axis), will be studied in full detail. We recall that $c^* > 1$ and $c = b^2/4 > 0$. Later, we will study the cases $c > 1$ (so that if $c = c^*$ then $1 + a^2 + ab \leq 0$ and if $b^2 - 4c = 0$ then $ab \neq 1$), $c \in (0, 1)$ (with $c \neq c^*$ and if $b^2 - 4c = 0$ then $ab \neq -1$) and $c < 0$. These last three cases will be studied using the information on the index taking into account the detailed information on the infinite singular points and on the finite singular points lying in the y-axis done above.

In the following subsections we will determine the different local phase portraits in the different cases mentioned above.

4.6.6. *Case $c = 1$ and $b + 3a = 0$.* If $a \in (-2/3, 0) \cup (0, 2/3)$ there are no finite singular points among the origin. Taking into account the local information on the infinite singular points given in Tables 4 and 5 we conclude that the global phase portrait is topologically equivalent to 15 in Figure 1.

If $a \in (-1/\sqrt{2}, -2/3) \cup (2/3, 1/\sqrt{2})$ there are two finite singular points (both of them in the y-axis) which are a saddle and a center. Taking into account the local information on the infinite singular points given in Table 5 we conclude that the global phase portrait is topologically equivalent to 16 in Figure 1.

Finally, if $a = \pm 2/3$ among the origin, there is a cusp. Again, taking into account the local information on the infinite singular points, we conclude that the global phase portrait is topologically equivalent to 17 in Figure 1.

4.6.7. *Case $c = 1$ and $b + 3a \neq 0$.* If $1 + 4a^2 + 2ab \leq 0$ and $b > 2$, among the origin there are two saddles (both on the y-axis). Taking into account the local information on the infinite singular points given in Table 5 we conclude that the global phase portrait is topologically equivalent to 18 in Figure 1 (it is attained for example when $b = 3$ and $a = -1/2$).

If $1 + 4a^2 + 2ab = 0$ and $b = 2$ among the origin, there is a finite singular point in the y-axis formed by six hyperbolic sectors. Hence, the global phase portrait is topologically equivalent to 19 in Figure 1.

If $1+4a^2+2ab>0$ and $b>2$ among the origin, there is a center and a saddle on the y-axis which are are a center and a saddle and two saddles outside the y-axis. Taking into account the information on the infinite singular points given in Table 5 together with using the first integral H to obtain possible saddle connections between the saddle at the y-axis with the other two saddles (always connected by symmetry), we conclude that the global phase portraits are topologically equivalent to 20 in Figure 1 (attained, for instance, when $b=3$ and $a=1/2$).

If $1+4a^2+2ab>0$ and $b=2$ then among the origin there are a cusp in the y-axis and two saddles outside the y-axis. The global phase portrait is topologically equivalent to 21 in Figure 1 (attained for example when $a=1/2$).

If $1+4a^2+2ab>0$ and $b<2$ then among the origin there are two saddles outside the y-axis. The global phase portrait is topologically equivalent to 22 in Figure 1 (attained, for instance, when $b=1$ and $a=1/2$).

4.6.8. *Case $c=0$ and $b=0$.* In this case there are four saddles outside the y-axis. Taking into account the information on the infinite singular points given in Table 5 and using the first integral H to obtain all the saddle connections, we conclude that the global phase portraits are topologically equivalent to 23 in Figure 1 (it is attained, for example, when $a=1/2$) and to 24 in Figure 1 (attained when $a=0$).

4.6.9. *Case $c=0$ and $b\neq 0$.* In this case, among the origin, there exists always the finite singular point $(0,-1/b)$ studied in Table 6.

If $b<-1/(2a)$ then there are four saddles outside the y-axis and if $b\geq -1/(2a)$ there are two saddles outside the y-axis. Taking this into account, the information on the infinite singular points given in Table 5 and using the first integral H to obtain possible saddle connections, we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1 (we have included in parenthesis the possible values of a and b for which these global phase portraits are attained): 25 ($a=-1/2$, $b=1/2$); 26 ($a=-1/2$, $b=2$); 27 ($a=-1/2$, $b=1$); 28 ($a=-1/3$, $b=1$); and 29 ($a=-7/30$, $b=1$).

4.6.10. *Case $b^2-4c=0$ and $ab=-1$.* In this case among the origin there always exists the finite singular point $(0,-2/b)$ (see Table 6). If $b\leq\sqrt{2}$ or $b=2$ there are no more finite singular points.

On the other hand, if $b>\sqrt{2}$ with $b\neq 2$, there are two finite singular points outside the y-axis which are: two saddles if $b\in(\sqrt{2},2)$ and two centers if $b > 2$. Taking this into account, the information on the infinite singular points given in Table 5 and using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1 (again, we have included in parenthesis the possible values of a and b for which these global phase portraits are attained): 30 ($b = 3/2$); 31 ($b = \sqrt{3}$); 32 ($b = 18/10$); 19 ($b = 2$); 33 ($b = 3$).

4.6.11. *Case $c = c^*$ with $b+3a \neq 0$ and $1+a^2+ab > 0$.* Note that $c^* > 0$ and there are two cusps outside the y-axis. If $4 + a(a + 6b + 2ab^2) > 0$ there are no more finite singular points. If $4 + a(a + 6b + 2ab^2) = 0$ there is a cusp in the y-axis while if $4 + a(a + 6b + 2ab^2) < 0$ there is a saddle and a center. Taking this into account, the information on the infinite singular points given in Table 5, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1 (we have included in parenthesis the possible values of a and b for which these global phase portraits are attained): 7 ($a = -1/2$, $b = 2$); 34 ($a = -1/2$, $b = 3 - 1/\sqrt{2}$) and 35 ($a = -1/2$, $b = 24/10$).

4.6.12. *Case $c > 1$ so that if $c = c^*$ then $1 + a^2 + ab \leq 0$ and if $b^2 - 4c = 0$ then $ab \neq 1$.* We recall that in this case there are no infinite singular points. First we consider the case in which $b^2 - 4c < 0$. Among the finite singular points we only know that the origin is a center. Hence the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. Since systems (VI) have at most four finite singular points in addition to the origin outside the y-axis, and since $c \neq c^*$, none of them are non-elementary (see Proposition 8), we have the following possibilities: (i) no more finite singular points, or (ii) two saddles and two centers. In case (i) the global phase portrait is topologically equivalent to 2 in Figure 1 and is attained for instance for $c = 2$, $b = 1$ and $a = -1/2$. In case (ii) the global phase portrait is topologically equivalent to 8 in Figure 1 and is attained for instance for $c = 2$, $b = 1$ and $a = 1/2$.

Now we consider the case in which $b^2 - 4c = 0$. Among the finite singular points we only know that the origin is a center and there is a cusp in the y-axis. Hence the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. Since systems (VI) have at most four finite singular points in addition to the origin, and none of them are non-elementary, we have the following possibilities: (i) no more finite singular points, or (ii) two saddles and two centers. In case (i) the global phase portrait is topologically equivalent to 9 in Figure 1 and is attained for instance for $c = 4$, $b = 4$ and $a = 1/2$. In case (ii) the global phase portrait is topologically equivalent to 13 in Figure 1 and is attained for instance for $c = 4$, $b = 4$ and $a = -1/2$.

Finally, we consider the case in which $b^2-4c > 0$. If $a^* \leq a \leq \overline{a}$ there are two saddles in the y-axis. Hence the known singular points have total index -2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 4. Since the systems (VI) have at most four finite singular points in addition to the origin, and non of them are non-elementary, we have that they must be two centers. The global phase portrait is topologically equivalent to 36 in Figure 1 and it is attained, for instance, when $c = 4$, $b = 5$ and $a = -1/4$. If $a > a^*$ or $a < \overline{a}$, there is one saddle and one center in the y-axis. So, the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. Since systems (VI) have at most four finite singular points in addition to the origin, and none of them are non-elementary, we have the following possibilities: (i) no more finite singular points, or (ii) two saddles and two centers. In case (i) the global phase portraits are topologically equivalent to 11 in Figure 1 for instance when $a = -1/2$, $b = 3$, $c = 2$. In case (ii) using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1 (we have included in parenthesis the possible values of a, b and c for which these global phase portraits are attained): 37 ($a = 1/2$, $b = 35/10$, $c = 3$); 38 ($a = 1/2$, $b = 3.66965166..$, $c = 3$ here b is a root of the polynomial

$$\begin{array}{l}-1411344 - 256608\lambda + 2485107\lambda^2 + 181412\lambda^3 - 1427428\lambda^4 \\ -883552\lambda^5 - 21744\lambda^6 + 59200\lambda^7 + 8512\lambda^8);\end{array}$$

and 10 ($a = 1/2$, $b = 4$ and $c = 3$).

4.6.13. *Case $c \in (0,1)$ with $c \neq c^*$ and if $b^2 - 4c = 0$ then $ab \neq 1$.* We recall that in this case there are four nodes in the local chart U_1. First we consider the case in which $b^2 - 4c < 0$. Among the finite singular points we only know that the origin is a center. Hence the known singular points have total index 10 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -8. Since systems (VI) have at most four finite singular points in addition to the origin outside the y-axis, and none of them are non-elementary, they must be four saddles. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of

Figure 1: 39 ($a = 1/2$, $b = 1$, $c = 1/2$) and 40 ($a = -1/3$, $b = 1$, $c = 1/2$).

Now assume that $b^2 - 4c = 0$. Among the finite singular points we only know that the origin is a center and there is a cusp in the y-axis. Hence the known singular points have total index 10 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -8. Since the systems (VI) have at most four finite singular points in addition to the origin outside the y-axis, and none of them are non-elementary, they must be four saddles. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 41 ($a = -1/2$, $b = \sqrt{2}$, $c = 1/2$); 42 ($a = -\sqrt{2}/3$, $b = \sqrt{2}$, $c = 1/2$) and 43 ($a = -1/3$, $b = \sqrt{2}$, $c = 1/2$).

Finally, we consider the case in which $b^2 - 4c > 0$. If $a^* \leq a \leq \overline{a}$ there are two saddles in the y-axis. Hence the known singular points have total index 6 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -4. Since systems (VI) have at most four finite singular points in addition to the origin, and none of them are non-elementary, they must be two saddles. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 44 ($a = -63/100$, $b = 2$, $c = 1/2$); 45 ($a = -1/3$, $b = 2$, $c = 1/2$); 46 ($a = -0.61646555..$, $b = 2$, $c = 1/2$ here a is a root of the polynomial

$$\begin{aligned}&54\sqrt{2} - 59 + (142\sqrt{2} - 148)\lambda + (132\sqrt{2} - 168)\lambda^2 - (20\sqrt{2} + 32)\lambda^3 \\ &-(6\sqrt{2} + 17)\lambda^4 + (18\sqrt{2} + 12)\lambda^5 + 18\lambda^6).\end{aligned}$$

If $a < a^* < \overline{a}$ or $a^* < \overline{a} > a$ there is one saddle and one center in the y-axis. So, the known singular points have total index 10 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -8. Since systems (VI) have at most four finite singular points in addition to the origin, and none of them are non-elementary, they must be four saddles. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 47 ($a = -7/30$, $b = 7/10$, $c = 1/10$); 48 ($a = -1/5$, $b = 7/10$, $c = 1/10$); 49 ($a = -4/15$, $b = 7/10$, $c = 1/10$); 50 ($a = -1/4$, $b = 1$, $c = 0.23213904..$ here c is a root of the polynomial

$$\begin{aligned}&-2064384 + 28744704\lambda - 148561920\lambda^2 + 344888192\lambda^3 \\ &-34882836\lambda^4 + 115789500\lambda^5 + 4100625\lambda^6);\end{aligned}$$

51 ($a=-1/4$, $b=1$, $c=6/25$); 52 ($a=-1/2$, $b=1/2$, $c=0.05730830..$ here c is a root of the polynomial

$$\begin{aligned}&333-12564\lambda+132318\lambda^2-266296\lambda^3+245881\lambda^4\\&-121500\lambda^5+26244\lambda^6);\end{aligned}$$

53 ($a=-1/2$, $b=1/2$, $c=3/50$); 54 ($a=-71/150$, $b=71/50$, $c=1/2$).

4.6.14. *Case* $c<0$. We note that there are two nodes in the local chart U_1 and that $b^2-4c>0$. If $\overline{a}<a<a^*$ then among the origin there are two centers. So, the known singular points have total index 10 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -8. Since systems (VI) have at most four finite singular points in addition to the origin, and none of them are non-elementary, they must be four saddles. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 55 ($a=-1/4$, $b=3/4$, $c=-1$); 56 ($a=-1/4$, $b=17/20$, $c=-1$).

If $a<\overline{a}$ or $a>a^*$ then among the origin there is a center and a saddle. So, the known singular points have total index 6 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -4. Since systems (VI) have at most four finite singular points in addition to the origin, and none of them are non-elementary, they must be two saddles. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 3 ($a=-40/100$, $b=1$, $c=-1$); 57 ($a=-1/3$, $b=1$, $c=-1$); 14 ($a=-48/150$, $b=1$, $c=-1$).

4.7. **Global phase portrait of systems (VII).** Without loss of generality we can assume that $b\geq 0$ (otherwise changing $(x,y,a,b,c)\to(-x,-y,-a,-b,c)$ we get the same system with $b\geq 0$). We recall that the Hamiltonian of system (VII) is

$$H=\frac{1}{2}y^2+ax^2y+\frac{b}{3}y^3+\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{c}{4}y^4.$$

We now study the infinite singular points of these systems. We distinguish between the cases $c>0$, $c=0$ and $c<0$.

4.7.1. *Infinite singular points.* In the local chart U_1 systems (VII) can be written as

$$\begin{aligned} u' &= -1 - 2u^2 - cu^4 - v(3au + u^2v + bu^3), \\ v' &= -v(u + av + uv^2 + bu^2v + cu^3). \end{aligned}$$

When $v = 0$, the candidates for singular points are the roots of the polynomial $-1 - 2u^2 - cu^4$, i.e.

$$\pm\frac{\sqrt{c(-1+\sqrt{1-c})}}{c} \quad \text{and} \quad \pm\frac{\sqrt{c(-1-\sqrt{1-c})}}{c}, \quad c \neq 0.$$

In the local chart U_2 systems (VII) can be written as

$$u' = c + bv + v^2 + 3au^2v + 2u^2 + u^4, \quad v' = uv(2av + u^2 + 1).$$

4.7.2. *Case $c \geq 0$.* When $c > 0$ there are no infinite singular points on the local charts U_1 and U_2. When $c = 0$ there are no infinite singular points on the local chart U_1. Hence, the origin of U_2 is a singular point. When $b \neq 0$ the origin is nilpotent, using Theorem 3.5 of [11] together with blow-up techniques we obtain that the phase portrait of the origin consists of one hyperbolic, one elliptic and two parabolic sectors, see Figure 5. When $b = 0$, on the other hand, the origin is an infinite singular point, whose linear part is zero, hence we need to do a blow-up to characterize the local dynamics at this point. Doing the blow-up $(u, v) \to (u, w)$ with $w = v/u$ and eliminating the common factor u we get the system

$$\begin{aligned} u' &= 2u + u(u^2 + 3auw + w^2) \\ w' &= -w - w(auw + w^2). \end{aligned} \tag{26}$$

When $u = 0$ the only singular point of system (26) is the origin, and it is a saddle. The blow-up of the origin gives the same information as in the case of systems (II), hence the origin of U_2 has two hyperbolic sectors, see Figure 4.

4.7.3. *Case $c < 0$.* In this case the systems (VII) have only two singular points

$$\pm\frac{\sqrt{c(-1-\sqrt{1-c})}}{c}.$$

These points are nodes. Finally, the origin of chart U_2 is not a singular point.

In short we have Table 7.

Parameters		Infinite singular points in chart U_1 and U_2
$c>0$		There are no infinite singular points
$c<0$		In the chart U_1 there are one attracting node and one repelling node. In the chart U_2 there are no infinite singular points
$c=0$	$b \neq 0$	In the chart U_1 there are no infinite singular points. In the chart U_2 the origin has a one hyperbolic, one elliptic and two parabolic sectors
	$b=0$	In the chart U_1 there are no infinite singular points. In the chart U_2 the origin has two hyperbolic sectors

TABLE 7. Infinite singular points in the charts U_1 and U_2.

4.7.4. *Finite singular points.* The candidates for singular points of systems (VII) other than its origin are

$$E_{1,2} = \left(0, \frac{-b \pm \sqrt{b^2-4c}}{2c}\right), \quad c \neq 0$$

$$E_{3,4} = \left(\pm\sqrt{-y_3(y_3+2a)}, y_3\right), \quad E_{5,6} = \left(\pm\sqrt{-y_5(y_5+2a)}, y_5\right), \quad c \neq 1$$

where

$$y_{3,5} = \frac{(-b+3a) \pm \sqrt{(b-3a)^2+4(1-c)(1-2a^2)}}{2(c-1)}.$$

As in previous cases we need to study only the singularities $E_{1,2}$ which are the singularities that are on the y-axis. These singularities occur only when $b^2-4c \geq 0$. The calculations in this case are very similar to the systems (V) and systems (VI). So we will only present the final result in the Table 8.

Lemma 9. *We recall that when $c<0$ there are at most two singular points are outside the y-axis.*

Proof. Since $c<0$ and $a \in (-1/\sqrt{2}, 1/\sqrt{2})$ we have that $y_3y_5<0$. In order that the four points $E_{3,4}$ and $E_{5,6}$ exist we must have $y_3(y_3+2a)<0$ and $y_5(y_5+2a)<0$.

If $y_3>0$ then in order that $E_{3,4}$ exist we must have $y_3+2a<0$ which implies $a<0$. But then since $y_3y_5<0$ we have that $y_5<0$ and since $a<0$ then also $y_5+2a<0$. This yields $y_5(y_5+2a)>0$ and so $E_{5,6}$ do not exist.

On the other hand if $y_3 < 0$ then in order that $E_{3,4}$ exist we must have $y_3 + 2a > 0$ which implies $a > 0$. But then since $y_3 y_5 < 0$ we have that $y_5 > 0$ and since $a > 0$ then also $y_5 + 2a > 0$. This yields $y_5(y_5 + 2a) > 0$ and so $E_{5,6}$ do not exist. This concludes the proof of the lemma. $\square$

Parameters			Finite singular points on y–axis except the origin.
$b^2 - 4c < 0$			There are no singular points
$b^2 - 4c > 0$	$c > 0$	$a_1^* < \overline{a}_1 < a$ $a < a_1^* < \overline{a}_1$	One center and one saddle
		$a_1^* \le a \le \overline{a}_1$	Two centers
	$c < 0$	$\overline{a}_1 < a_1^* < a$ $a < \overline{a}_1 < a_1^*$	One center and one saddle
		$\overline{a}_1 \le a \le a_1^*$	Two saddles
	$c = 0$	$ab \le 1/2$	One saddle
		$ab > 1/2$	One center
$b^2 - 4c = 0$	$b \neq 0$	$ab \neq 1$	One cusp
		$ab = 1$	One singular point with 2 hyperbolic sectors
	$b = 0$		There are no singular points

Table 8. Finite singular points, on the y–axis, of systems (VII). In this table $a_1^* = \frac{b-\sqrt{b^2-4c}}{4c}$ and $\overline{a}_1 = \frac{b+\sqrt{b^2-4c}}{4c}$.

Again the next step is to count the indices of the finite and infinite singular points of systems (VII) on the Poincaré sphere. First we need the following proposition. We take the notation

$$\overline{c} = -\frac{4 + a^2 - 6ab + b^2}{4(2a^2 - 1)}.$$

Proposition 10. *System (VII) has non-elementary points with $x \neq 0$ if and only if $c = \overline{c}$. They exist when $b - 3a \neq 0$, $1 + a^2 - ab < 0$ and they are two cusps.*

Proof. We compute the Groebner basis between x', y' and the determinant of the Jacobian matrix. The first component of the Groebner basis is

$$(2a^2 - 1)^2(b^2 - 4c)(-1 + 2ab - 4a^2c)^2(4 + a^2 - 6ab + b^2 - 4c + 8a^2c)y.$$

Note that if $y = 0$ the unique singular point is the origin. Moreover, since $a \in (-1/\sqrt{2}, 1/\sqrt{2})$ we have that $2a^2 - 1 \neq 0$. When $c = b^2/4$ the singular point which is not hyperbolic is $(0, -2/b)$ and belongs to the y-axis. If $-1 + 2ab - 4a^2c = 0$, that is, $c = (1 - 2ab)/(4a^2)$ we obtain the values $\overline{a}_1$ and a_1^* given in Table 8 and the non-hyperbolic singular point is $(0, -2a)$ which again lyes in the y-axis.

Finally, when

$$4 + a^2 - 6ab + b^2 - 4c + 8a^2c = 0, \text{ i.e } c = \overline{c}$$

we get that the non-hyperbolic finite singular points are, whenever they exist,

$$\Big(\pm \frac{2\sqrt{(2a^2 - 1)(1 + a^2 - ab)}}{3a - b}, \frac{2(2a^2 - 1)}{b - 3a} \Big).$$

They exist when $b - 3a \neq 0$ and $1 + a^2 - ab \leq 0$. Note that when $1 + a^2 + ab = 0$ then $\overline{c} = \frac{5a^2 - b^2 + 2}{4(2a^2 - 1)}$ and both points lye on the y-axis.

Hence, in order that they exist and are outside the y-axis we must have $b - 3a \neq 0$ and $1 + a^2 - ab < 0$. Computing the eigenvalues of the Jacobian matrix at these points we see that they are nilpotent and using Theorem 3.5 in [11] we conclude that they are cusps. This concludes the proof of the proposition. $\square$

We will study the different global phase portraits on the Poincaré sphere as follows: $c = 0$ (that we will distinguish between the cases $b = 0$ and $b \neq 0$); $b^2 = 4c$ with $ab = 1$; $c = \overline{c}$ with $b - 3a \neq 0$ and $1 + a^2 - ab < 0$ (otherwise there are no non-hyperbolic singular points outside the y-axis), will be studied in full detail. We recall that $\overline{c} > 0$ and $c = b^2/4 > 0$. Later, we will study the cases $c > 0$ (so that if $c = \overline{c}$ then either $b - 3a = 0$ or $1 + a^2 - ab \geq 0$ and if $b^2 - 4c = 0$ then $ab \neq 1$) and $c < 0$. These last two cases will be studied using the information on the index taking into account the detailed information on the infinite singular points and on the finite singular points lying in the y-axis done above.

In the following subsections we will determine the different local phase portraits in the different cases mentioned above.

4.7.5. *Case $c = 0$.* If $b = 0$ the only finite singular point is the origin. Taking into account the information on the infinite singular points given in Table 7 we get that the global phase portrait is topologically equivalent to 1 in Figure 1.

On the other hand, if $b \neq 0$ then among the origin there always exist the finite singular point $(0, -1/b)$ studied in Table 8. If $a \leq 1/(2b)$

there are no finite singular points outside the y-axis, and if $a > 1/(2b)$ there are two saddles outside the y-axis. Taking this into account, the information on the infinite singular points given in Table 7 and the fact that the saddles outside the y-axis are connected by symmetry, we conclude that the global phase portraits are topologically equivalent to the following ones of Figure 1 (we have included in parenthesis the possible values of a and b for which these global phase portraits are attained): 6 ($a = 1/2$, $b = 1$) and 5 ($a = 1/2$, $b = 2$).

4.7.6. *Case $b^2 - 4c = 0$ and $ab = 1$.* In this case among the origin there only exists the finite singular point $(0, -2/b)$ which is formed by two hyperbolic sectors. Since there are no infinite singular points the global phaser portrait is topologically equivalent to 13 of Figure 1.

4.7.7. *Case $c = \overline{c}$ with $b - 3a \neq 0$ and $1 + a^2 - ab < 0$.* Note that $\overline{c} > 0$ and so there are two cusps outside the y-axis. Moreover there are no infinite singular points. If $4 + a(a - 6b + 2ab^2) > 0$ there are no more finite singular points. If $4 + a(a - 6b + 2ab^2) = 0$ there is a cusp in the y-axis while if $4 + a(a - 6b + 2ab^2) < 0$ there is a saddle and a center. Taking this into account, the information on the infinite singular points and the symmetry, we conclude that the global phase portraits are topologically equivalent to the ones of Figures 7 (when $a = 1/2$ and $b = 4$), 58 (when $a = 1/2$ and $b = 3 + 1/\sqrt{2}$); and 59 (when $a = 7/10$ and $b = 2.22438$).

4.7.8. *Case $c > 0$ so that if $c = \overline{c}$ then either $b - 3a = 0$ or $1 + a^2 - ab \geq 0$ and if $b^2 - 4c = 0$ then $ab \neq 1$ or $c \neq \overline{c}$.* We recall that in this case there are no infinite singular points. First we consider the case in which $b^2 - 4c < 0$. Among the finite singular points we only know that the origin is a center. Hence the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. Since systems (VII) have at most four finite singular points in addition to the origin outside the y-axis (see Proposition 10), none of them are non-elementary. Note that in this case we have exactly the same infinite and finite singular points as in the case $c > 1$ for system (VI) and so we get the same topologically equivalent global phase portraits: 2 (when $a = 1/2$ and $b = c = 1$); and 8 (when $a = 1/2$, $b = 5$ and $c = 7$).

Now we consider the case in which $b^2 - 4c = 0$. Among the finite singular points we only know that the origin is a center and there is a cusp in the y-axis. Hence the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. Since systems (VII) have at most four finite singular points in addition to the origin, and none of them are non-elementary, we have exactly the same infinite and finite singular points as in the case $c > 1$ for system (VI) and so we get the same topologically equivalent global phase portraits: 9 (when $a = 1/2$ and $b = c = 4$); and 13 (when $a = 1/2$, $b = 3$ and $c = 9/4$).

Finally, we consider the case in which $b^2 - 4c > 0$. If $a > a_1^*$ or $a < \overline{a}_1$, there is one saddle and one center in the y-axis. So, the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. Since systems (VII) have at most four finite singular points in addition to the origin, and none of them are non-elementary, we have the same finite singular points as in the case $c > 1$ for system (VI) and so we get the same topologically equivalent global phase portraits: 11 (when $a = -7/10$, $b = 1$ and $c = 1/5$); 60 (when $a = 1/2$, $b = 2.9$ and $c = 2$); 10 (when $a = 1/5$, $b = 7$ and $c = 11.5$); 38 (when $a = 1/5$, $b = 7$ and $c = 11.99722984$.. here c is the root of the polynomial

$$-3333909112734375 + 10575417764843750\lambda - 11748870462834375\lambda^2 \\ +2482717923483000\lambda^3 - 192679718006000\lambda^4 + 5112897914880\lambda^5 \\ +429981696\lambda^6);$$

and 37 (when $a = 1/5$, $b = 7$ and $c = 12$).

On the other hand, if $a_1^* \leq a \leq \overline{a}_1$ there are two centers in the y-axis. Hence the known singular points have total index 6 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -4. Since the systems (VII) have at most four finite singular points in addition to the origin, and non of them are non-elementary, we have that they must be two saddles. Hence, the possible global phase portraits are topologically equivalent to 61 of Figure 1 and it is attained for example when $a = 1/2$, $b = 3$ and $c = 1$.

4.7.9. *Case* $c < 0$. We note that there are two nodes in the local chart U_1 and that $b^2 - 4c > 0$. If $\overline{a}_1 \leq a \leq a_1^*$ then among the origin there are two saddles. So, the known singular points have total index 2 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index 0. In view of Lemma 9 and Proposition 10 we know that system (VII) has at most two finite singular points outside the y-axis, and none of them are non-elementary. By symmetry they cannot exist. Using the first integral H to obtain possible saddle connections, we conclude that the possible global phase portraits are topologically equivalent to the following ones of Figure 1: 12 ($a = 0$, $b = 1$, $c = -1$); 4 ($a = 0$, $b = 0$, $c = -1$).

On the other hand, if $a < a_1^* < \overline{a}_1$ or $a_1^* < \overline{a}_1 < a$ then among the origin there is a center and a saddle. So, the known singular points have total index 6 on the Poincaré sphere. By Theorem 6, the remaining finite singular points must have total index -4. Since systems (VII) have at most four finite singular points in addition to the origin, and none of them are non-elementary, we have the same finite singular points as in the case $c < 0$ for system (VI) and so we get the same topologically equivalent global phase portraits: 3 (when $a = -1/2$, $b = 1$ and $c = -25$); 57 (when $a = -1/2$, $b = 1$ and $c = -17.86185044..$ here c is the root of the polynomial

$$-7200 + 106560\lambda - 619776\lambda^2 + 1436966\lambda^3 - 691358\lambda^4 \\ +73872\lambda^5 + 6561\lambda^6);$$

and 14 (when $a = -1/2$, $b = 1$ and $c = -14$).

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