THE EXTENDED 16-TH HILBERT PROBLEM FOR DISCONTINUOUS PIECEWISE LINEAR CENTERS SEPARATED BY A NON-REGULAR LINE

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Abstract. We study the discontinuous piecewise differential systems formed by two linear centers separated by a non-regular straight line. We provide upper bounds for the maximum number of limit cycles that these discontinuous piecewise differential systems can exhibit and we show that these upper bounds are reached.

1. Introduction and statement of the main result

One of the main interesting objects in the study of differential systems are limit cycles. A *limit cycle* is a periodic orbit of the differential system isolated in the set of all periodic orbits of the system.

Limit cycles have played and are playing an important role for explaining phsyical phenomena, see for instance the limit cycle of van der Pol equation [18, 19], or the one of the Belousov-Zhavotinskii model [2, 21], etc.

The *extended* 16*th Hilbert problem*, that is, to find an upper bound for the maximum number of limit cycles that a given class of differential systems can exhibit, is in general an unsolved problem. Only for very few classes of differential system this problem has been solved. For the class of discontinuous piecewise differential systems here studied, we can obtain the solution by using the first integrals of the two linear centers which form the discontinuous piecewise differential system separated by a non-regular line.

The study of the piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1], and nowadays such systems still continue to receive the attention of many researchers. These differential systems are widely used to model processes appearing in electronics, mechanics, economy, etc., see for instance the books of di Bernardo et al. [3] and Simpson [20], the survey of Makarenkov and Lamb [17], as well as hundreds of references quoted in these last three works.

The simplest class of discontinuous piecewise differential systems are the planar ones formed by two pieces separated by a straight line having a linear differential system in each piece. Several authors have tried to determine the maximum number of limit cycles for this class of discontinuous piecewise differential systems. Thus, in one of the first papers dedicated to this problem, Giannakopoulos and Pliete [8]

in 2001, showed the existence of discontinuous piecewise linear differential systems with two limit cycles. Then, in 2010 Han and Zhang [9] found other discontinuous piecewise linear differential systems with two limit cycles and they conjectured that the maximum number of limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line is two. But in 2012 Huan and Yang [11] provided numerical evidence of the existence of three limit cycles in this class of discontinuous piecewise linear differential systems. In 2012, Llibre and Ponce [14] inspired by the numerical example of Huan and Yang, proved for the first time that there are discontinuous piecewise linear differential systems with two pieces separated by a straight line having three limit cycles. Later on, other authors obtained also three limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line, see Braga and Mello [4] in 2013, Buzzi, Pessoa and Torregrosa [5] in 2013, Liping Li [13] in 2014, Freire, Ponce and Torres [7] in 2014, and Llibre, Novaes and Teixeira [15] in 2015. But proving that discontinuous piecewise linear differential systems separated by a straight line have at most three limit cycles is an open problem.

We consider the discontinuous piecewise differential systems of the form

$$(1) \qquad (\dot{x}, \dot{y}) = \mathbf{F}(x, y) = \begin{cases} \mathbf{F}_1(x, y) = (f_1(x, y), g_1(x, y)) \text{ if } (x, y) \in \mathcal{S}_1, \\ \mathbf{F}_2(x, y) = (f_2(x, y), g_2(x, y) \text{ if } (x, y) \in \mathcal{S}_2, \end{cases}$$

where f_i, g_i are linear polynomials for $i = 1, 2$ and the regions $\mathcal{S}_1$ and $\mathcal{S}_2$ are defined by

$$\mathcal{S}_1 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > \alpha x\},$$
$$\mathcal{S}_2 = \{(x, y) \in \mathbb{R}^2 : x < 0, \text{ or } x > 0, y < \alpha x\},$$

being $\alpha \in \mathbb{R}$.

System (1) is bi-valued on the non-regular separation line

$$(2) \qquad \mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 = \{(0, y) : y \geq 0\} \cap \{(x, \alpha x) : x > 0\}.$$

As usual, system (1) is denoted by $\mathcal{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathcal{S})$ or simply by $\mathcal{F} = (X, Y)$, when the separation line $\mathcal{S}$ is well understood. In order to establish a definition for the trajectories of $\mathcal{F}$ and to investigate its behaviour we need a criterion for the transition of the orbits between $\mathcal{S}_1$ and $\mathcal{S}_2$ across $\mathcal{S}$. The gradient $\nabla \mathcal{S}(p)$ at the point p of the discontinuity line is given by the vector $(-1, 0)$ at the points $p = (0, y)$ with $y > 0$, and by the vector $(\alpha, -1)$ at the points $p = (x, \alpha x)$ with $x > 0$. The contact between the vector field $\mathbf{F}_1$ (or $\mathbf{F}_2$) and the discontinuity line $\mathcal{S}$, is characterized by the derivative of $\mathcal{S}$ in the direction of the vector field $\mathbf{F}_1$ (or $\mathbf{F}_2$), also known as the Lie derivative of $\mathcal{S}$ with respect to $\mathbf{F}_1$ (or $\mathbf{F}_2$), that is by the expression

$$\mathbf{F}_1 \mathcal{S}(p) = \langle \nabla \mathcal{S}(p), \mathbf{F}_1(p) \rangle,$$

and for $i \geq 2$ we define $\mathbf{F}_1^i \mathcal{S}(p) = \langle \nabla \mathbf{F}_1^{i-1} \mathcal{S}(p), \mathbf{F}_1(p) \rangle$, where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [6]. We can divide the discontinuity line $\mathcal{S}$ in the following three sets:

  (a) Crossing set $\mathcal{S}^c : \{p \in \mathcal{S} : \mathbf{F}_1 \mathcal{S}(p) \cdot \mathbf{F}_2 \mathcal{S}(p) > 0\}$;
  (b) Escaping set $\mathcal{S}^e : \{p \in \mathcal{S} : \mathbf{F}_1 \mathcal{S}(p) > 0 \text{ and } \mathbf{F}_1 \mathcal{S}(p) < 0\}$;
  (c) Sliding set $\mathcal{S}^s : \{p \in \mathcal{S} : \mathbf{F}_1 \mathcal{S}(p) < 0 \text{ and } \mathbf{F}_1 \mathcal{S}(p) > 0\}$.

The *escaping* $\mathcal{S}^e$ or *sliding* $\mathcal{S}^s$ regions are respectively defined on points of $\mathcal{S}$ where both vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$ simultaneously point outwards or inwards from $\mathcal{S}$, while the interior of its complement in $\mathcal{S}$ defines the *crossing region* $\mathcal{S}^c$. The complementary of the union of these regions is the set formed by the *tangency* points between $\mathbf{F}_1$ or $\mathbf{F}_2$ with $\mathcal{S}$.

Following the Filippov's convention [6] the discontinuous piecewise linear differential systems can have *sliding* or *crossing* limit cycles. A sliding limit cycle contains *sliding segments* on the line of discontinuity, whereas *crossing limit cycles* contain only *crossing points*. In this paper we work only with crossing limit cycles, or simply *limit cycles*.

In [12] the authors proved that any piecewise differential system of the form (1) can be transformed into a piecewise differential system with $\alpha = 0$ by means of an invertible linear transformation. Thus, it is not restrictive to consider $\alpha = 0$. In this paper, we study the maximum number of limit cycles of systems (1) with $\alpha = 0$ formed by two linear differential centers, namely

$$(3) \qquad \begin{cases} \dot{x} = -Ax - (A^2 + \Omega^2)y + B, \\ \dot{y} = x + Ay + C, \end{cases} \qquad \text{for } (x,y) \in \mathcal{R}_1,$$

and

$$(4) \qquad \begin{cases} \dot{x} = -ax - (a^2 + \omega^2)y + b, \\ \dot{y} = x + ay + c, \end{cases} \qquad \text{for } (x,y) \in \mathcal{R}_2,$$

with $\Omega, \omega > 0$, $A, B, C, a, b, c \in \mathbb{R}$, $A, a \neq 0$ and in the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are

$$\mathcal{R}_1 = \{(x,y) \in \mathbb{R}^2 : x \geq 0, \; y \geq 0\},$$

$$\mathcal{R}_2 = \{(x,y) \in \mathbb{R}^2 : x \leq 0, \;\; \text{or} \;\; x \geq 0, \; y \leq 0\}.$$

Each system (3) and (4) have, respectively, the first integrals

$$(5) \qquad \begin{aligned} H_1(x,y) &= (x + Ay)^2 + 2(Cx - By) + y^2\Omega^2, \\ H_2(x,y) &= (x + ay)^2 + 2(cx - by) + y^2\omega^2. \end{aligned}$$

Now, the discontinuity line is denoted by

$$\mathcal{R} = \mathcal{R}_y \cup \mathcal{R}_x = \{(0,y) : y \geq 0\} \cup \{(x,0) : x \geq 0\}.$$

We note that, differential systems (3) and (4) are the most general expressions of linear differential systems, see Lemma 1 of [16] for more details .

Following the notation of [12], we denote by I_2 a crossing limit cycle having two intersection points with either $\mathcal{R}_x$, or $\mathcal{R}_y$; by II_2 a crossing limit cycle having one intersection point with $\mathcal{R}_x$ and another point with $\mathcal{R}_y$ (see Figure 1(a)); and by II_4 a crossing limit cycle having two intersection points with $\mathcal{R}_x$ and two intersection points with $\mathcal{R}_y$ (see Figure 1(b)). The study of the existence of limit cycles of type I_2 is the study of limit cycles existing for two linear centers separated by a straight line, and it was proved in Theorem 3 of [16] that such piecewise differential systems have no limit cycles. Then, we restrict our analysis to limit cycles of II_2 or II_4 type.

The main result of the present paper is to study the maximum number of limit cycles of types II_2 and II_4 that the discontinuous piecewise linear differential systems (3)-(4) can exhibit.

(a) The pair of limit cycles of type II_2 for system (8)–(9).

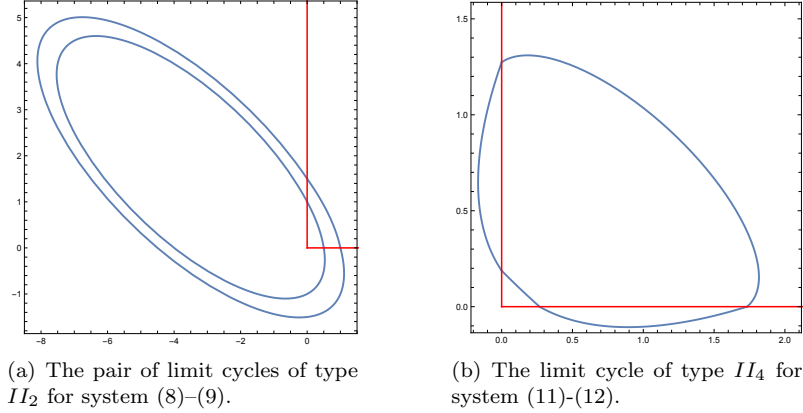(b) The limit cycle of type II_4 for system (11)-(12).

FIGURE 1

Theorem 1. *Consider discontinuous piecewise differential systems separated by the non-regular line $\mathcal{R}$ and formed by two arbitrary linear differential centers (3)-(4). The maximum number of limit cycles of these discontinuous piecewise linear differential systems are:*

(a) *two of type II_2 and there exists systems with exactly two limit cycles of this type (see Figure 1(a));*

(b) *one of type II_4 and there are systems with exactly one limit cycle of this type (see Figure 1(b)).*

The proof of Theorem 1 is given in the following section.

Other paper with several non-regular lines of discontinuity is [10].

2. Proof of Theorem 1

We separate the proof of both statements of Theorem 1.

Proof of statement (a) of Theorem 1. We consider the discontinuous planar linear differential system (3)-(4). If there exists a crossing limit cycle of type II_2, then it must intersect the non-regular separation curve $\mathcal{R}$ in two points of the form $(x, 0)$ and $(0, y)$, both different from the origin. Since the functions H_1 and H_2 defined in (5) are first integrals of the systems (3) and (4) respectively, these points must satisfy the equations

(6)
$$H_1(x, 0) - H_1(0, y) = 2Cx + x^2 + 2By - A^2y^2 - y^2\Omega^2 = 0,$$
$$H_2(x, 0) - H_2(0, y) = 2cx + x^2 + 2by - a^2y^2 - y^2\omega^2 = 0.$$

Assume that there exists a crossing limit cycle of type II_2 passing trough the points $(x_1, 0)$, $(0, y_1)$. Then, by solving equations in (6) with $x = x_1$ and $y = y_1$ with respect to C and c we get

$$C = \frac{-x_1^2 + y_1(-2B + y_1(A^2 + \Omega^2))}{2x_1}, \quad c = \frac{-x_1^2 + y_1(-2b + y_1(a^2 + \omega^2))}{2x_1}.$$

Now, assume that there exists a second crossing limit cycle of type II_2 passing through the points $(x_2, 0)$ and $(0, y_2)$ with $0 < x_1 < x_2$ and $0 < y_1 < y_2$.

Case 1: Assume $x_2 y_1 - x_1 y_2 \neq 0$. Using the obtained values of C and c we solve equations in (6) for $x = x_2$ and $y = y_2$ for the parameters B and b, i.e. we solve the system

$$
\begin{aligned}
H_1(x_2, 0) - H_1(0, y_2) &= x_2^2 + 2By_2 - A^2 y_2^2 - y_2^2 \Omega^2 \\
&\quad + \frac{x_2(-x_1^2 + y_1(-2B + y_1(A^2 + \Omega^2)))}{x_1} = 0,
\end{aligned}
$$

(7)

$$
\begin{aligned}
H_2(x_2, 0) - H_2(0, y_2) &= x_2^2 + 2by_2 - a^2 y_2^2 - y_2^2 \omega^2 \\
&\quad + \frac{x_2(-x_1^2 + y_1(-2b + y_1(a^2 + \omega^2)))}{x_1} = 0,
\end{aligned}
$$

and we get

$$
\begin{aligned}
B &= \frac{-x_1^2 x_2 + x_2 y_1^2 (A^2 + \Omega^2) + x_1(x_2^2 - y_2^2(A^2 + \Omega^2))}{2(x_2 y_1 - x_1 y_2)}, \\
b &= \frac{-x_1^2 x_2 + x_2 y_1^2 (a^2 + \omega^2) + x_1(x_2^2 - y_2^2(a^2 + \omega^2))}{2(x_2 y_1 - x_1 y_2)}.
\end{aligned}
$$

Now we substitute the obtained values of the parameters C, c, B, b in equations (6) and solving them with respect to x and y, obtaining the solutions

$$
(x, y) = (0, 0), \quad (x, y) = (x_1, y_1), \quad (x, y) = (x_2, y_2),
$$

$$
(x, y) = \left(\frac{(x_1 - x_2)(x_2 y_1 + x_1 y_2)}{x_1 y_2 - x_2 y_1}, \frac{(y_1 - y_2)(x_2 y_1 + x_1 y_2)}{x_2 y_1 - x_1 y_2} \right).
$$

Due to the conditions $0 < x_1 < x_2$ and $0 < y_1 < y_2$, the last solution has one negative component, so the maximum number of crossing limit cycles of type II_2 for system (3)-(4) is two.

Case 2: Assume $x_1 y_2 = x_2 y_1$. We solve equations (7) now for the parameters A and a, and we get

$$
A = \frac{\sqrt{(x_1 - y_1 \Omega)(x_1 + y_1 \Omega)}}{y_1}, \quad a = \frac{\sqrt{(x_1 - y_1 \omega)(x_1 + y_1 \omega)}}{y_1}.
$$

Substituting the values of C, c, A, a in (6) and taking into account that $x_1 y_2 = x_2 y_1$, we obtain that

$$
y = \frac{y_1}{x_1} x,
$$

and so, in this case there exists a continuum of periodic orbits, and consequently no limit cycles.

In summary, the maximum number of limit cycles of type II_2 that the discontinuous piecewise linear differential systems (3)-(4) can have is two.

Now we give a discontinuous piecewise linear differential system (3)-(4) having exactly two limit cycles of type II_2. In region $\mathcal{R}_1$ we consider the linear differential center

(8)
$$
\dot{x} = -\frac{3}{2} - 2x - 8y, \qquad \dot{y} = \frac{43}{4} + x + 2y,
$$

with the first integral

$$H_1(x, y) = 4y^2 + 2\left(\frac{43}{4}x + \frac{3}{2}y\right) + (x + 2y)^2;$$

and in region $\mathcal{R}_2$ we consider the linear differential center

(9) $$\dot{x} = -x - 2y, \qquad \dot{y} = \frac{7}{4} + x + y$$

with the first integral

$$H_2(x, y) = y^2 + \frac{7}{2}x + (x + y)^2.$$

In this case, the two solutions of equations (6) are

$$(x_1, y_1) = \left(\frac{1}{2}, 1\right), \qquad (x_2, y_2) = \left(1, \frac{3}{2}\right),$$

and the corresponding limit cycles are shown in Figure 1(a).   □

Proof of statement (b) *of Theorem* 1. We consider again the discontinuous piece-wise linear differential systems (3)-(4). If there exists a limit cycle of type II_4, then it has four intersection points on the discontinuity line $\mathcal{R}$ of the form $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$, satisfying $0 < x_1 < x_2$, $0 < y_1 < y_2$ and the equations

(10)
$$H_1(x_1, 0) - H_1(0, y_1) = 2Cx_1 + 2By_1 + x_1^2 - (A^2 + \Omega^2)y_1^2 = 0,$$
$$H_1(x_2, 0) - H_1(0, y_2) = 2Cx_2 + 2By_2 + x_2^2 - (A^2 + \Omega^2)y_2^2 = 0,$$
$$H_2(x_1, 0) - H_2(x_2, 0) = (x_1 - x_2)(2c + x_1 + x_2) = 0,$$
$$H_2(0, y_1) - H_2(0, y_2) = (y_1 - y_2)\left(2b - (y_1 + y_2)(a^2 + \omega^2)\right) = 0.$$

Since $x_1 < x_2$ and $y_1 < y_2$, we can obtain from the third and fourth equations that

$$x_2 + x_1 = -2c, \qquad y_2 + y_1 = \frac{2b}{a^2 + \omega^2}.$$

Using these last expressions, we can write the first and second equations of (10) in terms of x_1 and y_1, and we get

$$P(x_1, y_1) = 2Cx_1 + 2By_1 + x_1^2 - (A^2 + \Omega^2)y_1^2 = 0,$$
$$Q(x_1, y_1) = \frac{1}{(a^2 + \omega^2)^2}\big((2c + x_1)(2c - 2C + x_1)(a^2 + \omega^2)^2 - (2b$$
$$- y_1(a^2 + \omega^2))(-2B(a^2 + \omega^2) + (2b - y_1(a^2 + \omega^2))$$
$$(A^2 + \Omega^2))\big) = 0.$$

Now, we compute the resultant of the polynomials P and Q with respect to the variable y_1, and we obtain the following polynomial in the variable x_1:

$$
\begin{aligned}
R(x_1) =& 16(A^2 + \Omega^2)(A^2b^2 - (a^2 + \omega^2)(bB + c(c - C)(a^2 + \omega^2)) + b^2\Omega^2)(A^4b^2 \\
&+ 2B^2(a^2 + \omega^2)^2 - (a^2 + \omega^2)(3bB + c(c - C)(a^2 + \omega^2))\Omega^2 + b^2\Omega^4 + \\
&A^2(-(a^2 + \omega^2)(3bB + c(c - C)(a^2 + \omega^2)) + 2b^2\Omega^2)) \\
&+ 32c(a^2 + \omega^2)^2(A^2 + \Omega^2)(-A^4b^2 - B^2(a^2 + \omega^2)^2 \\
&+ (a^2 + \omega^2)(2bB + (c - C)^2(a^2 + \omega^2))\Omega^2 - b^2\Omega^4 + A^2((a^2 + \omega^2)(2bB \\
&+ (c - C)^2(a^2 + \omega^2)) - 2b^2\Omega^2))x_1 \\
&+ (a^2 + \omega^2)^2(A^2 + \Omega^2)(-A^4b^2 - B^2(a^2 + \omega^2)^2 \\
&+ (a^2 + \omega^2)(2bB + (c - C)^2(a^2 + \omega^2))\Omega^2 - b^2\Omega^4 + A^2((a^2 + \omega^2)(2bB \\
&+ (c - C)^2(a^2 + \omega^2)) - 2b^2\Omega^2))x_1^2.
\end{aligned}
$$

The polynomial $R(x_1)$ is quadratic in the variable x_1, whose solutions are x_1 and $-x_1 - 2c$. This fact implies that equations (10) have only a pair of solutions, that is $(x_1, x_2) = (x_1, -x_1 - 2c)$ (and analogously for (y_1, y_2)). Hence the maximum number of limit cycles of type II_4 that the discontinuous piecewise linear differential system (3)-(4) can have is one.

Now we give a discontinuous piecewise differential systems with the discontinuity line $\mathcal{R}$ formed by two linear differential centers having exactly one limit cycle of type II_4. In region $\mathcal{R}_1$ we consider the linear differential center

$$
(11) \qquad \dot{x} = \frac{17}{8} - x - 2y, \qquad \dot{y} = -\frac{191}{128} + x + y
$$

with the first integral

$$
H_1(x, y) = -2\left(\frac{191}{128}x + \frac{17}{8}y\right) + y^2 + (x + y)^2,
$$

and in region $\mathcal{R}_2$ we consider the linear differential center

$$
(12) \qquad \dot{x} = \frac{187}{128} + x - 2y, \qquad \dot{y} = -1 + x - y,
$$

with the first integral

$$
H_2(x, y) = -2(x + \frac{187}{128}y) + y^2 + (x - y)^2.
$$

For this case, the solutions of equations (10) satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$ are

$$
x_1 = 0.267515, \qquad x_2 = 1.73249, \qquad y_1 = 0.187568, \qquad y_2 = 1.27337,
$$

and the corresponding limit cycle is shown in Figure 1(b). $\qquad \square$

3. Conclusions

Every discontinuous piecewise differential system separated by a non-regular line of the form (2) and formed by two arbitrary linear differential centers can be written as the discontinuous piecewise differential system (2)–(3). For this class of discontinuous piecewise differential systems we provide an upper bound for the maximum

number of limit cycles that they can exhibit, i.e. we have solved the extended 16th
Hilbert problem to this class of discontinuous piecewise linear systems.

More precisely, this class of discontinuous piecewise differential systems can have
two kinds of limit cycles, denoted by II2 and II4, see Figures 1(a) and 1(b), respec-
tively. The maximum number of limit cycles of type II2 is two and of type II4 is
one. Moreover, we have shown that these upper bounds are reached, see Theorem
1.

Acknowlegments

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