Research Article

Mathematical Methods in the Applied Sciences

Received 14 August 2014

Published online in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.3365
MOS subject classification: 37G15; 37G10; 34C07

Zero-Hopf bifurcation in the FitzHugh–Nagumo system

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Communicated by A. Miranville

We characterize the values of the parameters for which a zero-Hopf equilibrium point takes place at the singular points, namely, O (the origin), P_+, and P_- in the FitzHugh–Nagumo system.
We find two two-parameter families of the FitzHugh–Nagumo system for which the equilibrium point at the origin is a zero-Hopf equilibrium. For these two families, we prove the existence of a periodic orbit bifurcating from the zero-Hopf equilibrium point O.
We prove that there exist three two-parameter families of the FitzHugh–Nagumo system for which the equilibrium point at P_+ and at P_- is a zero-Hopf equilibrium point. For one of these families, we prove the existence of one, two, or three periodic orbits starting at P_+ and P_-.

Keywords: FitzHugh–Nagumo system; periodic orbit; averaging theory; zero-Hopf bifurcation

1. Introduction and statements of the main result

In this paper, we study the zero-Hopf equilibrium points and the zero-Hopf bifurcations of periodic orbits that take place at the equilibria in the FitzHugh–Nagumo system.

This system was introduced in articles of FitzHugh [1] and Nagumo, Arimoto, and Yoshizawa [2] as one of the simplest models describing the excitation of neural membranes and the propagation of nerve impulses along an axon. In the MathSciNet, you can find several hundreds of papers published on this system, or related with it, but in them the zero-Hopf bifurcation has not been studied.

We consider the following FitzHugh–Nagumo partial differential system:

$$u_t = u_{xx} - f(u) - v, \qquad v_t = \delta(u - \gamma v), \tag{1}$$

where $f(u) = u(u-1)(u-a)$ and $0 < a < 1/2$ are constants, and $\delta > 0$ and $\gamma > 0$ are parameters. A bounded solution $(u, v)(x, t)$ with $x, t \in \mathbb{R}$ is called a traveling wave if $(u, v)(x, t) = (u, v)(\xi)$, where $\xi = x + ct$ and c is the constant denoting the wave speed. Substituting $u = u(\xi)$, $v = v(\xi)$ into (1), we obtain the following ordinary differential system:

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= b(x - dy), \\ \dot{z} &= x(x-1)(x-a) + y + cz, \end{aligned} \tag{2}$$

by introducing a new variable $w = \dot{u}$, where the dot denotes the derivative with respect to ξ, $x = u$, $y = v$, $z = w$, $b = \varepsilon/c$ and $d = \gamma$, see for more details [3].

In this paper, the ordinary differential system (2) will be called the *FitzHugh–Nagumo differential system*. We shall study this system depending on the parameters $(a, b, c, d) \in \mathbb{R}^4$.

Here, a *zero-Hopf equilibrium* is an equilibrium point of a three-dimensional autonomous differential system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues.

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