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LIMITING DYNAMICS FOR THE COMPLEX STANDARD FAMILY

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The complexification of the standard family of circle maps $\mathbf{F}_{\alpha\beta}(\theta) = \theta + \alpha + \beta\ \sin(\theta) \mod (2\pi)$ is given by $F_{\alpha\beta}(\omega) = \omega e^{i\alpha} e^{(\beta/2)(\omega - 1/\omega)}$ and its lift $f_{\alpha\beta}(z) = z + \alpha + \beta\ \sin(z)$. We investigate the three-dimensional parameter space for $F_{\alpha\beta}$ that results from considering α complex and β real. In particular, we study the two-dimensional cross-sections $\beta =$ constant as β tends to zero. As the functions tend to the rigid rotation $F_{\alpha,0}$, their dynamics tend to the dynamics of the family $G_\lambda(z) = \lambda z e^z$ where $\lambda = e^{-i\alpha}$. This new family exhibits behavior typical of the exponential family together with characteristic features of quadratic polynomials. For example, we show that the λ-plane contains infinitely many curves for which the Julia set of the corresponding maps is the whole plane. We also prove the existence of infinitely many sets of λ values homeomorphic to the Mandelbrot set.

1. Introduction

There has been much interest in the two parameter family of maps of the circle given by

$$\mathbf{F}_{\alpha\beta}(\theta) = \theta + \alpha + \beta\ \sin(\theta) (\text{mod } 2\pi)\,, \quad \theta \in \mathbf{R} \quad (1)$$

where α and β are real parameters and $0 \leq \theta \leq 2\pi$.

This family is known as the "standard family" and its parameter space contains the well known Arnold Tongues [Arnold, 1961] (see Fig. 1). These tongues correspond to parameters for which there exists a periodic cycle of a given period and a given rate of rotation around the circle.

In order to better understand this picture, we will investigate the standard family in the complex plane allowing not only the variable but also the parameters to be complex.

In the complex plane, the standard family is represented by

$$F_{\alpha\beta}(\omega) = \omega\, e^{i\alpha}\, e^{\frac{\beta}{2}(\omega - \frac{1}{\omega})}\,,$$

where ω is now a complex variable. It is easy to check that for α, $\beta \in \mathbf{R}$, $F_{\alpha\beta}$ preserves the unit circle and that, on this circle, $F_{\alpha\beta}$ coincides with the standard family $\mathbf{F}_{\alpha\beta}$. For each α and β, $F_{\alpha\beta}$ is an entire function of $\mathbf{C}^*$ ($\mathbf{C} - \{0\}$) with two critical points and essential singularities at 0 and ∞. Such maps have been studied by Keen [1988] and Kotus [1987, 1990].

When considering α, $\beta \in \mathbf{C}$, the parameter space for this family is four-dimensional and includes the Arnold tongues picture as a two-dimensional slice. In this paper, we will analyze the three-dimensional slice that corresponds to $\beta \in \mathbf{R}$ and $\alpha \in \mathbf{C}$. More precisely, we will

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