The Complex Standard Family

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The complexification of the standard family of circle maps

$$\mathbf{F}_{\alpha\beta}(\theta) = \theta + \alpha + \beta \sin(\theta) \pmod{2\pi}$$

is given by $F_{\alpha\beta}(\omega) = \omega e^{i\alpha} e^{\frac{\beta}{2}(\omega - \frac{1}{\omega})}$ and its lift $f_{\alpha\beta}(z) = z + \alpha + \beta \sin(z)$.

We investigate the 3-dimensional parameter space for $F_{\alpha\beta}$ that results from considering α complex and β real. In particular, we study the 2-dimensional cross sections β =constant as β tends to 0. As the functions tend to the rigid rotation $F_{\alpha,0}$, their dynamics tend to the dynamics of the family $G_\lambda(z) = \lambda z e^z$ where $\lambda = e^{-i\alpha}$. This new family exhibits behavior typical of the exponential family together with characteristic features of quadratic polynomials. For example, we show that the λ-plane contains infinitely many curves for which the Julia set of the corresponding maps is the whole plane. We also prove the existence of infinitely many sets of λ values homeomorphic to the Mandelbrot set.

For real values of the parameter, we study the dynamics of the family $F_{\alpha\beta}$, which are conjugate to those of $f_{\alpha\beta}$. As in the exponential family, the Julia set contains infinitely many curves whose points tend exponentially fast to infinity under iteration. These curves are organized by some symbolic dynamics and they move continuously with the parameters. As α and β move through the Arnold tongues picture, these curves attach and detach from the unit circle, giving a clearer picture of the bifurcations that occur. We give a necessary and sufficient condition for the existence of parameter values for which these curves attach to the unit circle.