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DEFORMATION OF ENTIRE FUNCTIONS WITH BAKER DOMAINS

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Abstract. We consider entire transcendental functions f with an invariant (or periodic) Baker domain U. First, we classify these domains into three types (hyperbolic, simply parabolic and doubly parabolic) according to the surface they induce when we take the quotient by the dynamics. Second, we study the space of quasiconformal deformations of an entire map with such a Baker domain by studying its Teichmüller space. More precisely, we show that the dimension of this set is infinite if the Baker domain is hyperbolic or simply parabolic, and from this we deduce that the quasiconformal deformation space of f is infinite dimensional. Finally, we prove that the function $f(z) = z + e^{-z}$, which possesses infinitely many invariant Baker domains, is rigid, i.e., any quasiconformal deformation of f is affinely conjugate to f.

1. **Introduction.** Let $f : S \to S$ be a holomorphic endomorphism of a Riemann surface S. Then f partitions S into two sets: the Fatou set $\Omega(f)$, which is the maximal open set where the iterates $f^n, n = 0, 1, \ldots$ form a normal sequence; and the Julia set $J(f) = S \setminus \Omega(f)$ which is the complement.

If $S = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then f is a rational map, and every component of $\Omega(f)$ is eventually periodic by the non-wandering domains theorem in [25]. There is a classification of the periodic components of the Fatou set: such a component can either be a cycle of rotation domains or the basin of attraction of an attracting or indifferent periodic point.

If $S = \mathbb{C}$ and f does not extend to $\widehat{\mathbb{C}}$ then f is an entire transcendental mapping (i.e., infinity is an essential singularity) and there are more possibilities. For example a component of $\Omega(f)$ may be wandering, that is, it will never be iterated to a periodic component. Like for rational mappings there is a classification of the periodic components of $\Omega(f)$ (see [5]) and compared to rational mappings, entire

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ones allow for one more possibility: A period p periodic component U is called a Baker domain, if for all $z \in U$ we have $f^n(z) \to \infty$, as $n \to \infty$. The first example of an entire function with a Baker domain was given by Fatou in [13], who considered the function $f(z) = z + 1 + \mathrm{e}^{-z}$ and showed that the right half-plane is contained in an invariant Baker domain. Since then, many other examples have been considered, showing various properties that are possible for this type of Fatou components (see for example [11], [6], [3], [22], [23], [15] and also [4]). It follows from [2] that a Baker domain of an entire function is simply connected.

Taking an iterate of the map if necessary we consider only the cases of invariant Baker domains. We remark that in a Baker domain, orbits tend to infinity at a slow rate. More precisely, if γ is an unbounded invariant curve in a Baker domain (and hence all its points tend to infinity under iteration), then there exists a constant $A > 1$ such that $|f(z)| \leq A|z|$ for all $z \in \gamma$ [5]. This is in contrast to the fact that points in $\mathbb{C}$ that tend to infinity exponentially fast belong to the Julia set of f.

There is another important difference between rational and entire transcendental mappings which concerns the singularities of the inverse map f^{-1} or *singular values*. In the rational case, the points for which some branch of f^{-1} fails to be well defined are precisely the *critical values*, i.e., the images of the zeros of f'. In the transcendental case, one more possibility is allowed, namely the *asymptotic values*, which are points $a \in \mathbb{C}$ for which there exists a curve $\gamma(t) \to \infty$ as $t \to \infty$ satisfying $f(\gamma(t)) \to a$ as $t \to \infty$. It follows from a theorem of Denjoy, Carleman and Ahlfors that entire functions of finite order may have only a finite number of asymptotic values (see e.g. [20] or [26] Theorem 4.11), but in the other extreme there exists an entire map for which every value is an asymptotic value.

As is in the case with basins of attraction and rotation domains, there is also a relation between Baker domains and the singularities of the inverse map. In particular, it is shown in [12] that Baker domains do not exist for a map such that the set $\mathrm{Sing}(f^{-1})$ is bounded, where $\mathrm{Sing}(f^{-1})$ denotes the closure in $\mathbb{C}$ of the set of singular values. The actual relationship between this set and a Baker domain U is related to the distance of the singular orbits to the boundary of U (see [8] for a precise statement). We remark that it is not necessary, however, that any of the singular values be inside the Baker domain. Indeed, there are examples of Baker domains with an arbitrary number of singular values (including none) inside.

Our first goal in this paper is to give a classification of Baker domains. Our result is an extension of previous classifications of certain classes of Baker domains. Indeed, when the map f restricted to the Baker domain U is proper, we call U a *proper Baker domain*. In particular the degree of f restricted to U is finite. In the special case where this degree is one we call the domain U *univalent*. In [4] there is a classification of univalent Baker domains in terms of the map they induce in the unit disk via the Riemann map.

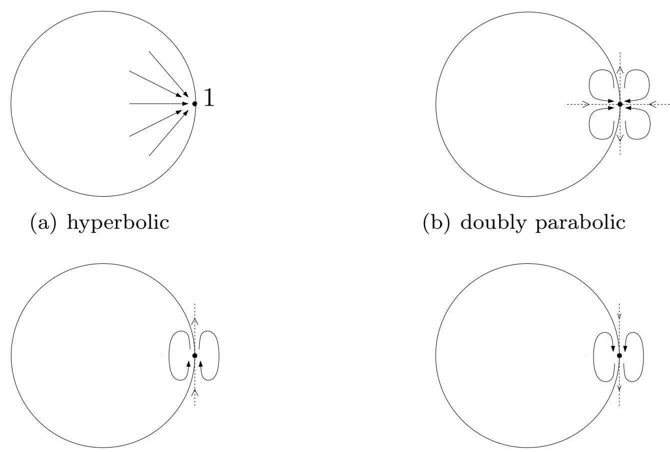
In [8] the classification is extended to accomodate a larger class of Baker domain, namely the *regular* Baker domains. More precisely let $\varphi : U \to \mathbb{D}$ denote a Riemann map, mapping U to the unit disk. Such a map conjugates f to a self-mapping of $\mathbb{D}$ that we denote by B_U. The map B_U is called the *inner function* associated to U. If B_U is proper then this mapping is a (finite) Blaschke product. It follows from the Denjoy-Wolff theorem (see e.g. [19], Thm. 5.4), that there exists a point $z_0 \in \partial\mathbb{D}$ such that B_U^n converges towards the constant mapping z_0 locally uniformly in $\mathbb{D}$ as n tends towards infinity. We call this point the *Denjoy-Wolff point* of B_U. If B_U

extends analytically to a neighborhood of z_0 we call U a *regular* Baker domain. In particular, proper Baker domains are a subclass of the regular Baker domains.

Our classification is as follows.

Proposition 1. *Let f be entire and U a Baker domain. Then U/f is a Riemann surface conformally isomorphic to exactly one of the following cylinders:*
(1) $\{-s < \mathrm{Im}(z) < s\}/\mathbb{Z}$ *for some* $s > 0$ *and we call* U hyperbolic*;*
(2) $\{\mathrm{Im}(z) > 0\}/\mathbb{Z}$ *and we call* U simply parabolic*;*
(3) $\mathbb{C}/\mathbb{Z}$ *and we call* U doubly parabolic. *In this case* $f : U \to U$ *is not proper or has degree at least* 2.

In the special case of regular Baker domains, the dynamics of the three types are shown in figure 1.



(a) hyperbolic (b) doubly parabolic (c) simply parabolic (d) simply parabolic

FIGURE 1. The three possibilities for the dynamics of B_U when U is a regular Baker domain. Having normalized so that the Denjoy-Wolff point is $z_0 = 1$, we have that $0 < B_U(1) < 1$ in the hyperbolic case (a); $B_U(z) = z - a(z-1)^3 + \mathcal{O}((z-1)^4)$ for some $a > 0$, in the doubly parabolic case (b); $B_U(z) = z + ia(z-1)^2 + \mathcal{O}((z-1)^3)$, where either $a > 0$ (c) or $a < 0$ (d). By the symmetry of the map, $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ must belong to the basin of attraction of 1 and hence the Julia set must be a subset of the unit circle.

It is a natural question to ask wether examples of Baker domains of all three types exist. They do, as we show in Section 4. However, our examples for hyperbolic and simply parabolic domains are univalent and, to our knowledge, no concrete examples are known of such maps with degree larger than one.

Our second goal in this paper is to study the possible quasiconformal deformations of entire maps with a Baker domain. We can consider the space of entire mappings with a fixed Baker domain as a subset of the space of entire mappings modulo conjugacy with affine mappings. It is natural to ask how this set looks. It is easy to see it cannot be open, since any entire map with a Baker domain can be approximated by polynomials, and no polynomial possesses a Baker domain. Lifting maps with Herman rings (see Example 1 in section 4) for different rotation numbers converging to a rational p/q, shows that the set is not closed. Can it have components that are reduced to points? By considering the space of quasiconformal deformations we will see that if such a point exists, the corresponding mapping can only have Baker domains which are doubly parabolic.

More precisely we will consider the Teichmüller space of an entire mapping f with a fixed Baker domain, using the general framework given by [18] (see Section 5). We will see that the dimension of this set is infinite if the Baker domain is hyperbolic or simply parabolic, and from this we will deduce that the quasiconformal deformation space of f is infinite dimensional. The precise statement is as follows.

Main Theorem. *Let U be a fixed Baker domain of the entire function f and $\mathcal{U}$ its grand orbit. Denote by S the set of singular points of f in $\mathcal{U}$, and by $\widehat{\mathcal{S}}$ the closure of the grand orbit of S taken in $\mathcal{U}$. Then $\mathcal{T}(f,\mathcal{U})$ is infinite dimensional except if U is doubly parabolic and the cardinality of $\widehat{\mathcal{S}}/f$ is finite. In that case the dimension of $\mathcal{T}(f,\mathcal{U})$ equals $\#\widehat{\mathcal{S}}/f-1$.*

Furthermore we show that the lowest dimension is possible, that is we give an example of a rigid map with a proper Baker domain. Using the Main Theorem we can show the following (see Section 6).

Proposition 2. *The map $f(z)=z+e^{-z}$ is rigid, i.e., if $\widetilde{f}$ is a holomorphic map which is quasiconformally conjugate to f, then $\widetilde{f}$ is affinely conjugate to f.*

2. **Preliminaries – quasiconformal mappings.** In this section we recall shortly the relevant definitions and results relative to quasiconformal mappings, to be used in Section 5. The standard references are [1] and [16]. In this section, $V, V' \subset \mathbb{C}$ are open subsets of the complex plane or more generally, one dimensional complex manifolds.

Definition 1. Given a measurable function $\mu : V \to \mathbb{C}$, we say that μ is a $k-$*Beltrami coefficient* of V if $|\mu(z)| \le k < 1$ almost everywhere in V. Two Beltrami coefficients of V are equivalent if they coincide almost everywhere in V.

Definition 2. A homeomorphism $\phi : V \to V'$ is said to be $k-$*quasiconformal* if it has locally square integrable weak derivatives and

$$\mu_\phi(z) = \frac{\frac{\partial\phi}{\partial\bar{z}}(z)}{\frac{\partial\phi}{\partial z}(z)} = \frac{\bar{\partial}\phi(z)}{\partial\phi(z)}$$

is a $k-$Beltrami coefficient. In this case, we say that μ_ϕ is the *complex dilatation* or the *Beltrami coefficient* of ϕ.

With the same definition, but skipping the hypothesis on ϕ to be a homeomorphism, ϕ is called a $k-$*quasiregular map.*

Definition 3. Given a Beltrami coefficient μ of V and a quasiregular map $f : V \to V'$, we define the *pull-back* of μ by f as the Beltrami coefficient of V defined by:

$$f^*\mu = \frac{\frac{\partial f}{\partial\bar{z}} + (\mu\circ f)\overline{\frac{\partial f}{\partial z}}}{\frac{\partial f}{\partial z} + (\mu\circ f)\overline{\frac{\partial f}{\partial\bar{z}}}}.$$

We say that μ is f–invariant if $f^*\mu = \mu$. If $\mu = \mu_g$ for some quasiregular map g, then $f^* = \mu_{g\circ f}$.

It follows from Weyl's Lemma that f is holomorphic if and only if $f^*\mu_0 = \mu_0$, where $\mu_0 \equiv 0$.

Definition 4. Given a Beltrami coefficient μ, the partial differential equation

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z)\frac{\partial \phi}{\partial z} \tag{1}$$

is called *the Beltrami equation.* By *integration* of μ we mean the construction of a quasiconformal map ϕ solving this equation almost everywhere, or equivalently, such that $\mu_\phi = \mu$ almost everywhere.

The famous *Measurable Riemann Mapping Theorem* by Morrey, Bojarski, Ahlfors and Bers states that every k-Beltrami coefficient is integrable.

Theorem 1 (Measurable Riemann Mapping Theorem, [1])**.** *Let μ be a Beltrami coefficient of $\mathbb{C}$. Then, there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\mu_\phi = \mu$. Moreover, ϕ is unique up to post-composition with affine maps.*

We end this section with a lemma that will be important in Section 5. Since we are unable to give a reference, we include its proof here.

Lemma 1. *Let $\mathcal{A}$ denote the set of K-quasiconformal homeomorphims $\omega : \mathbb{D} \to \mathbb{D}$ that extend continuously to the boundary as the identity. Then there exists a constant $C = C(K)$ such that for all $\omega \in \mathcal{A}$ and all $z \in \mathbb{D}$ we have that the hyperbolic distance $d_\mathbb{D}$ in $\mathbb{D}$ satisfies*

$$d_\mathbb{D}(z, \omega(z)) \leq C.$$

Proof. This is a standard compactness argument. Let $\mathcal{B}$ denote the set of K-quasiconformal homeomorphisms of the sphere that fix $-1, 1$ and ∞. We endow $\mathcal{A}$ and $\mathcal{B}$ with the topologies corresponding to uniform convergence. A map $\omega \in \mathcal{A}$ can be extended to the sphere, by letting it coincide with the identity outside $\mathbb{D}$. This defines an injection $\mathcal{A} \to \mathcal{B}$ which can be seen to be a homeomorhism onto its image. It is easy to see that the image of $\mathcal{A}$ in $\mathcal{B}$ is closed. Now, it is well-known that $\mathcal{B}$ is sequentially compact (cf. [17]), and it follows that $\mathcal{A}$ is sequentially compact. Then, take a sequence of maps $\omega_n \in \mathcal{A}$ and points $z_n \in \mathbb{D}$ and suppose that $d_\mathbb{D}(z_n, \omega_n(z_n)) \to \infty$. Let $\hat{\omega}_n$ be the map we obtain by conjugating ω_n with a Möbius transformation that sends $\mathbb{D}$ to itself and z_n to 0. Now, $\hat{\omega}_n$ is a sequence of maps in $\mathcal{A}$ with $|\omega_n(0)| \to 1$. This is in contradiction with the fact that $\mathcal{A}$ is sequentially compact. □

3. **Classification of Baker Domains. Proof of Proposition 1.** Let U be an open subset of the complex plane or, more generally, a one dimensional complex manifold. For an endomorphism f of the space U, the *grand orbit* of $y \in U$ is the set $\{x \in U \mid f^n(x) = f^m(y) \text{ for some } n, m > 0\}$. The grand orbit of a set is the union of the grand orbits of its elements. The *grand orbit relation* is the equivalence relation such that $x \sim y$ if and only if they have the same grand orbit. We denote by U/f the quotient space obtained from U by identifying points under the grand orbit relation of f.

Let f be an entire transcendental map and U an invariant Baker domain of f. We recall the statement of Proposition 1.

Proposition 1. *Let f be entire and U a Baker domain. Then U/f is a Riemann surface conformally isomorphic to one of the following cylinders:*

(1) $\{-s < \mathrm{Im}(z) < s\}/\mathbb{Z}$ *for some* $s > 0$ *and we call* U hyperbolic*, or*
(2) $\{\mathrm{Im}(z) > 0\}/\mathbb{Z}$ *and we call* U simply parabolic*, or*
(3) $\mathbb{C}/\mathbb{Z}$ *and we call* U doubly parabolic*. In this case* $f : U \to U$ *is not proper or has degree at least* 2*.*

The proposition is a direct consequence of the work of Cowen. In [9] he defines the notion of a *fundamental set* for an endomorphism ψ of a domain Ω as an open, simply connected and forward invariant subdomain $V \subset \Omega$, such that for any compact set $K \subset \Omega$ there exists $n > 0$ so that $\psi^n(K) \subset V$. Cowen shows the following theorem.

Theorem 2 ([9])**.** *Let* $\phi : \mathbb{D} \to \mathbb{D}$ *be analytic and without fixed points. Then there exist a fundamental set* V *for* ϕ *on* $\mathbb{D}$ *and an analytic mapping* $\sigma : \mathbb{D} \to \Omega$*, with* $\Omega = \mathbb{C}$ *or* $\Omega = \mathbb{D}$*, and a Möbius transformation* Φ *mapping* Ω *onto itself such that:*
(a) ϕ *and* σ *are univalent on* V*;*
(b) $\sigma(V)$ *is a fundamental set for* Φ *on* Ω*;*
(c) σ *(semi)conjugates* ϕ *to* Φ*, i.e.,* $\sigma \circ \phi = \Phi \circ \sigma$*.*
Moreover, Φ *is unique up to conjugation by a conformal isomorphism of* Ω*, and* Φ *and* σ *depend only on* ϕ*, not on the particular fundamental set* V*.*

Proof. We prove how Proposition 1 follows from Cowen's theorem. Let $\phi = B_U$ be the inner function associated to $f : U \to U$, as defined in the introduction. Since f has no fixed points in U, ϕ has no fixed points and we can invoke Cowen's theorem to obtain mappings σ and Φ as well as a fundamental domain $V \subset \mathbb{D}$. Since V is a fundamental domain for ϕ and $\sigma(V)$ is a fundamental domain for Φ we get

$$U/f \simeq \mathbb{D}/\phi \simeq V/\phi \simeq \sigma(V)/\Phi \simeq \Omega/\Phi.$$

First suppose $\Omega = \mathbb{C}$. The two fixed points of Φ must then coincide at infinity, and $\Phi : \mathbb{C} \to \mathbb{C}$ is conjugate to translation by one on the plane. This is the doubly parabolic case of the proposition. Now if $\phi : \mathbb{D} \to \mathbb{D}$ were proper and of degree one, it would be conjugate to the translation by one (or minus one) on the upper half plane, which is impossible since a doubly infinite cylinder is not conformally equivalent to a one sided infinite cylinder. So if $f : U \to U$ is proper it is not of degree one.

Now suppose that $\Omega = \mathbb{D}$. By the symmetry of Φ, the fixed points of Φ must be on the unit circle. If the two fixed points coincide, we conjugate Φ with a map sending $\mathbb{D}$ onto the upper half plane and the fixed point to infinity. By the invariance of the half plane, the mapping we obtain must be of the form $z \mapsto z + a$ for a real and non-zero. This is the simply parabolic case of the proposition. The last possibility is that the two fixed points of Φ are distinct. Denote the multiplier of the attracting fixed point by λ (the repelling fixed point then has multiplier $1/\lambda$). We conjugate Φ with a Möbius transformation that sends the attracting fixed point to infinity, the repelling fixed point to 0 and the unit disc to the right half plane to obtain a Möbius transformation $\tilde{\Phi}$ mapping the right half plane onto itself. This mapping must be $z \mapsto \frac{1}{\lambda} z$ and by invariance of the right half plane, $0 < \lambda < 1$. Finally conjugating by $z \mapsto \frac{1}{\lambda}\log(z)$, $\tilde{\Phi}$ on the right half plane is conjugated to translation by one on the strip $\{-\pi/\lambda < \mathrm{Im}(z) < \pi/\lambda\}$. This is the hyperbolic case of the proposition. □

4. **Examples.** Examples of hyperbolic and simply parabolic univalent Baker domains were already given in [4], but we include them here for completeness. Additionally we present examples of degree two and three doubly parabolic domains.

Up to this date, we do not know of any example of a hyperbolic or simply parabolic proper Baker domain with degree larger than one. We summarize this in the following table.

	Univalent	$1 <$ degree
Hyperbolic	Example 1	?
Simply parabolic	Example 2	?
Doubly parabolic	$\times\times\times$	Examples 3 and 4

Example 1. **(univalent, hyperbolic)**
Let $f(z) = z + \alpha + \beta \sin(z)$ for $0 < \alpha < 2\pi$ and $0 < \beta < 1$. Projecting f by $w = e^{iz}$, we obtain the map

$$F(w) = e^{i\alpha} w e^{\frac{\beta}{2}(w - 1/w)}$$

which is a holomorphic self-map of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is easy to check that F restricted to the unit circle $\mathbb{S}^1$ is the well-known standard family of circle maps.

For appropriately chosen values of the parameters α and β, the map F has a Herman ring V symmetric with respect to $\mathbb{S}^1$. It is easy to check that lifting V by e^{iz} we obtain a Fatou component U of f, which is an invariant Baker domain, symmetric with respect to the real axis. See Figure 2. Since V is a rotation domain, the map F is univalent in V. Using the fact $f(z+2k\pi) = f(z) + 2k\pi$ one can easily show that $f|_U$ must also be univalent.

One can check that U is conformally equivalent to a horizontal band B of finite height and that f in U is conjugate to a horizontal translation in B. It follows easily that U is hyperbolic.

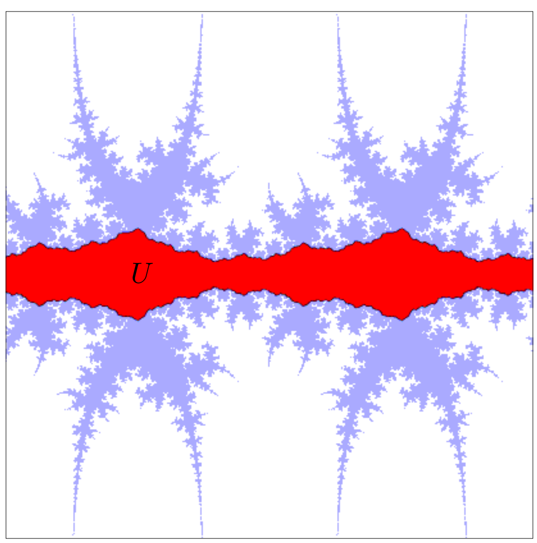


FIGURE 2. The dynamical plane of $f(z) = z + \alpha + \beta \sin(z)$ for certain values of α and β such that f has a univalent hyperbolic Baker domain.

Example 2. (univalent, simply parabolic) Let $g(w) = \lambda w \exp w$. Then 0 is a fixed point of g with multiplier λ. The map g has only one critical point at $z = -1$. Observe that g is semiconjugate to the map $f(z) = z + \log(\lambda) + e^z$ by $w = e^z$.

Let $\lambda = e^{2\pi i\theta}$ where θ is chosen so that g has a Siegel disk Δ around 0 (we can choose θ to be any Brjuno number). Then Δ lifts under e^z to a domain U which contains a left half plane. See Figure 3. The invariant closed curves in Δ lift to invariant almost vertical curves in U, the points of which move upwards towards infinity. Hence U is a Baker domain which is easily seen to be univalent. We can lift the linearizing map $\phi : \Delta \to \mathbb{D}$ by the exponential to get a mapping $\Phi : U \to \{\mathrm{Re}(z) < 0\}$, that conjugates $f : U \to U$ to the translation by $2i\pi\theta$ on the right half plane. It follows that U/f is a one-sided infinite cylinder.

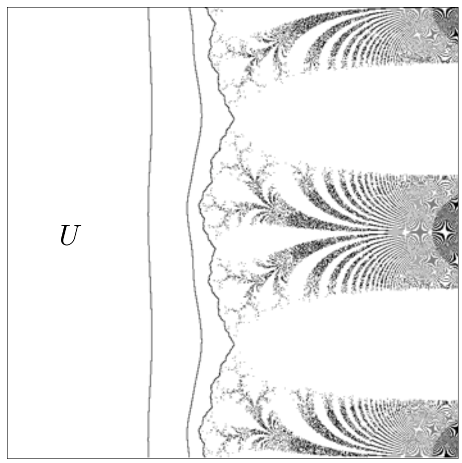


FIGURE 3. The dynamical plane of $f(z) = z + \alpha + e^z$ with $\alpha = \frac{\sqrt{5}-1}{2}$, which contains a univalent simply parabolic Baker domain.

Example 3. (degree 2, doubly parabolic)

In this section we study the example

$$f(z) = z + e^{-z},$$

which was also investigated in [3], showing the existence of infinitely many invariant Baker domains for f. We start by proving the same fact using different arguments and then proceed to show that the domains are doubly parabolic.

To study the dynamics of f it is convenient to work with the map $g(w) = we^{-w}$ that is semiconjugate to f by $w = e^{-z}$. Observe that $w = 0$ is a fixed point of g of multiplier 1, and $g(w) = w - w^2 + \mathcal{O}(w^3)$ near 0. The attracting and repelling direction of 0 are the positive and negative real axis respectively. There exists an attracting petal $\mathcal{P}$ of f at 0 which determines a basin of attraction $\mathcal{A}$. Let $\mathcal{A}^0$ denote the immediate basin of attraction, i.e., the connected component of $\mathcal{A}$ that contains $\mathcal{P}$. Then, $\mathcal{A}^0$ also contains the unique critical point $w = 1$.

We now lift this picture back to the dynamical plane of f (see Figures 4 and 5). Observe that the preimages of $\mathbb{R}^-$ under e^{-z} are the horizontal lines $\{\mathrm{Im} z = (2k+1)\pi, k \in \mathbb{Z}\}$. Hence all of them are invariant by f and their points have orbits whose real part tends to $-\infty$ exponentially fast. This implies that all of them lie in the Julia set of f.

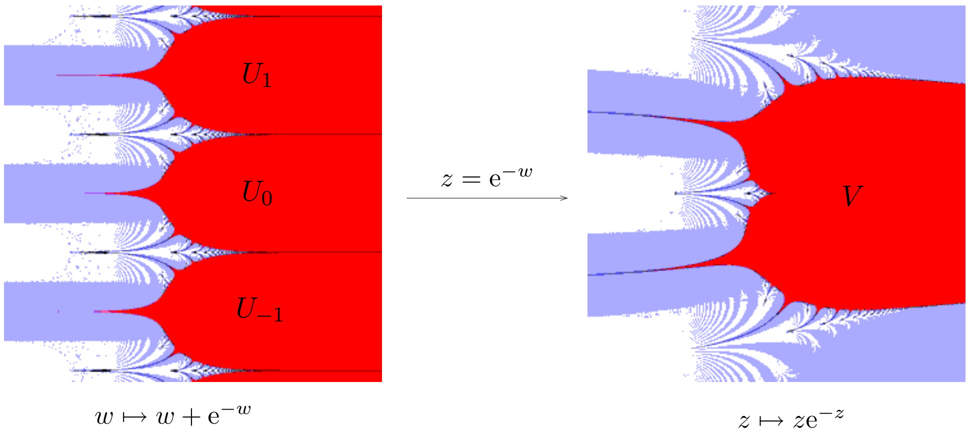


FIGURE 4. To the left is an illustration of the dynamics of $f : w \mapsto w + \mathrm{e}^{-w}$. This map possesses a sequence of fixed doubly parabolic proper Baker domains $\ldots, U_{-1}, U_0, U_1, \ldots$. The map f is semiconjugate to $g : z \mapsto z\mathrm{e}^{-z}$ by $z = \mathrm{e}^{-w}$. The Baker domains of f correspond to the immediate parabolic basin of attraction of the parabolic fixed point 0 of g.

The horizontal strips that lie in between these preimages are mapped to the whole dynamical plane of g in a one-to-one fashion and, therefore, they each contain a preimage of $\mathcal{A}^0$. Let us denote these preimages by $\ldots, U_{-1}, U_0, U_1, \ldots$, and observe that each U_k contains the invariant horizontal line $\mathrm{Im} z = 2k\pi$, since these are mapped to the positive real axis by e^{-z}. Hence, for all $k \in \mathbb{Z}$, the set U_k is invariant and its points tend to infinity under iteration of f (since this is the preimage of 0 under the conjugation). Therefore each of these sets is an invariant Baker domain.

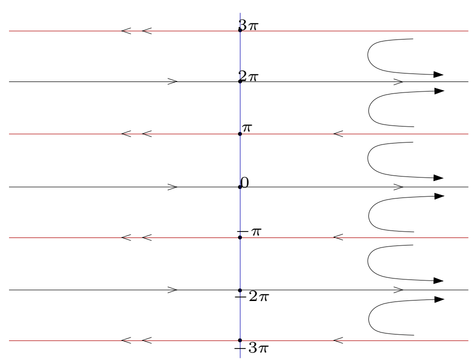


FIGURE 5. Sketch of the dynamical plane of f. There is an invariant Baker domain in every strip $\mathrm{Im} z \in ((2k+1)\pi, (2k+3)\pi)$, $k \in \mathbb{Z}$.

We now proceed to check that each U_k is doubly parabolic. Indeed, $U_k/f \simeq \mathcal{A}^0/g$, which can be seen to be equivalent to a double infinite cylinder, for instance by using the Fatou coordinates.

Example 4. (degree 3, doubly parabolic) This example is similar to the last, except that we lift a mapping with a parabolic fixed point and a double critical point in the basin. More precisely, consider the map $g(w) = exp(w^2/2 - 2w)$. It is easy to check that $w = 0$ is parabolic with multiplier 1 and that the immediate basin contains a double critical point. The negative real axis, as in the example above, is contained in the Julia set of g.

Lifting by the exponential function we obtain infinitely many Baker domains separated by the horizontal lines $\{\mathrm{Im}(z) = (2k+1)\pi\}$. Arguing like the previous example we see that these are doubly parabolic Baker domains of degree 3.

Note that, in a similar fashion, it is easy to construct examples of doubly parabolic proper Baker domains of any degree.

Example 5. (Fatou's example: infinite degree, non-regular, doubly parabolic)

Let $f(z) = z+1+e^{-z}$. It is well known that f has a Baker domain that contains the right half plane, and that all the infinitely many critical points belong to the Baker domain. Hence the degree is infinite, and since the critical points accumulate at the Denjoy-Wolf point, the inner map cannot be extended to a neighborhood of this point. Hence the Baker domain is not regular. Since the map near infinity is basically a translation by one, it is clear that the quotient by f is a doubly infinite cylinder. Hence the Baker domain is doubly parabolic.

5. **Deformations. Proof of the Main Theorem.** In this section we consider the Teichmüller space of an entire mapping f with a fixed Baker domain, using the general framework given by [18]. We will see that the dimension of this set is infinite if the Baker domain is hyperbolic or simply parabolic, and from this we will deduce that the quasiconformal deformation space of f is infinite dimensional. For some preliminaries on quasiconformal mappings see Section 2.

Let V be an open subset of the complex plane or more generally a one dimensional complex manifold and f a holomorphic endomorphism of V. Define an equivalence relation $\sim$ on the set of quasiconformal homeomorphisms on V by identifying $\phi : V \to V'$ with $\psi : V \to V''$ if there exists a conformal isomorphism $c : V' \to V''$ such that $c \circ \phi = \psi$, i.e. the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V' \\ & \underset{\psi}{\searrow} & \downarrow{\scriptstyle c} \\ & & V'' \end{array}$$

It then follows that $\phi \circ f \circ \phi^{-1}$ and $\psi \circ f \circ \psi^{-1}$ are conformally conjugate (although the converse is not true in general). Then the deformation space of f on V is

$$\mathrm{Def}(f, V) = \{\phi : V \to V' \text{ quasi conformal} \mid \mu_\phi \text{ is } f\text{-invariant } \}/\sim .$$

As a consequence of the Measurable Riemann Mapping Theorem (see [1] or Theorem 1) one obtains a bijection between $\mathrm{Def}(f, V)$ and

$$\mathcal{B}_1(f, V) = \{f - \text{invariant Beltrami forms } \mu \in L^\infty \text{ with } ||\mu||_\infty < 1\},$$

and this is used to endow $\mathrm{Def}(f, V)$ with the structure of a complex manifold. Indeed, $\mathcal{B}_1(f, V)$ is the unit ball in the Banach space of f-invariant Beltrami forms equipped with the infinity norm.

We denote by $\mathrm{QC}(f, V)$ the set of quasiconformal automorphisms of V that commute with f. A family of q.c. mappings is called uniformly K-q.c. if each element of the family is K-q.c.

A hyperbolic Riemann surface V is covered by the unit disk; in fact V is isomorphic to $\mathbb{D}/\Gamma$ where Γ is a Fuchsian group. Let $\Omega \subseteq \mathbb{S}^1$ denote the complement of the limit set of Γ. Then $(\mathbb{D} \cup \Omega)/\Gamma$ is a bordered surface and Ω/Γ is called the ideal boundary of V. A homotopy $\omega_t : V \to V, 0 \leq t \leq 1$ is called the *rel ideal boundary* if there exists a lift $\hat{\omega}_t : \mathbb{D} \to \mathbb{D}$ that extends continously to Ω as the identity. If V is not hyperbolic then the ideal boundary is defined to be the empty set.

We denote by $\mathrm{QC}_0(f, V) \subseteq \mathrm{QC}(f, V)$ the subgroup of automorphisms which are homotopic to the identity the rel ideal boundary of V through a uniformly K-q.c. subset of $\mathrm{QC}(f, V)$.

Earle and McMullen [10] prove the following result for hyperbolic subdomains of the Riemann sphere.

Theorem 3. *Suppose $V \subseteq \widehat{\mathbb{C}}$ is a hyperbolic subdomain of the Riemann sphere. Then a uniformly quasiconformal homotopy $\omega_t : V \to V, 0 \leq t \leq 1$ can be extended to a uniformly quasiconformal homotopy of $\widehat{\mathbb{C}}$ by letting $\omega_t = \mathrm{Id}$ on the complement of V. Conversely, a uniformly quasiconformal homotopy $\omega_t : V \to V$ such that each ω_t extends continuously as the identity to the topological boundary $\partial V \subseteq \widehat{\mathbb{C}}$ is a homotopy rel ideal boundary.*

Proof. The proof can be found in [10]: Proposition 2.3 and the proof of Corollary 2.4 imply the first statement. Theorem 2.2 implies the second. □

The group $\mathrm{QC}(f, V)$ acts on $\mathrm{Def}(f, V)$ by $\omega_* \phi = \phi \circ \omega^{-1}$. Indeed if ϕ and ψ represent the same element in $\mathrm{Def}(f, V)$ then $\omega_* \phi = \omega_* \psi$ as elements of $\mathrm{Def}(f, V)$.

Definition 5. The Teichmüller space $\mathcal{T}(f, V)$ is the deformation space $\mathrm{Def}(f, V)$ modulo the action of $\mathrm{QC}_0(f, V)$, i.e. $\mathcal{T}(f, V) = \mathrm{Def}(f, V)/\mathrm{QC}_0(f, V)$. If V is a one dimensional complex manifold we denote by $\mathcal{T}(V)$ the Teichmüller space $\mathcal{T}(\mathrm{Id}, V)$.

Teichmüller space can be equipped with the structure of a complex manifold and a (pre)metric (we refer to [18]).

Let us give a rough idea of Teichmüller space and the motivation for studying it. In holomorpic dynamics one is often interested in studying the set $\mathcal{F}$ of holomorphic mappings that are quasiconformally conjugate to a given holomorhic map $f : V \to V$ modulo conjugacy by conformal isomorphisms. Such a mapping can be written as $\phi \circ f \circ \phi^{-1}$ for a $\phi \in \mathrm{Def}(f, V)$. Now $\phi \circ f \circ \phi^{-1}$ and $\psi \circ f \circ \psi^{-1}$ are conformally conjugate exactly when they represent the same element in $\mathrm{Def}(f, V)/\mathrm{QC}(f, V)$. So we can study $\mathcal{F}$ by looking at $\mathrm{Def}(f, V)/\mathrm{QC}(f, V)$. Clearly the Teichmüller space is related to this space, and it can be shown to be, at least morally, a covering of it. Because of the nice properties of Teichmüller space, this space is often more convenient to study than $\mathcal{F}$.

Sullivan and McMullen prove stronger versions of the following two theorems.

Theorem 4. *Let f be an entire mapping, and suppose that U_α is a family of pairwise disjoint completely invariant open subsets of $\mathbb{C}$. Then*

$$\mathcal{T}(f, \cup U_\alpha) \simeq \prod \mathcal{T}(f, U_\alpha).$$

Proof. This follows from Theorem 5.5 in [18]. □

Theorem 5. *Suppose every component of the one-dimensional manifold V is hyperbolic, $f : V \to V$ is a holomorphic covering map, and the grand orbit relation of f is discrete. If V/f is connected then V/f is a Riemann surface and*

$$\mathcal{T}(f, V) \simeq \mathcal{T}(V/f).$$

Proof. This is a consequence of Theorem 6.1 in [18]. □

After these general definitions, we return to the case where f is an entire mapping with a Baker domain U. By definition $f : U \to U$ is conjugate to its inner function $B_U : \mathbb{D} \to \mathbb{D}$, with a non-repelling fixed point at 1.

We now show that the grand orbit of the set of singular values is formed by dynamically distinguished points. More precisely we have the following proposition.

Proposition 3. *Let f be an entire mapping and $\mathcal{U}$ a totally invariant open set whose connected components are simply connected and hyperbolic. Denote by S the set of singular values of f in $\mathcal{U}$. Then any $\omega \in \mathrm{QC}_0(f,\mathcal{U})$ restricts to the identity on the closure of the grand orbit of S in $\mathcal{U}$.*

To prove the proposition we need the following lemma.

Lemma 2. *Let V be a simply connected hyperbolic subset of $\mathbb{C}$, and $f : V \to \mathbb{C} \setminus \{a,b\}$ be a holomorphic map into the thrice punctured sphere. Suppose $\gamma : [0,+\infty] \to \widehat{\mathbb{C}}$ is a curve such that*

1. $\gamma([0,+\infty)) \subset V$,
2. $\gamma(+\infty) \in \partial V \cup \{\infty\}$, *and*
3. $\lim_{\tau\to+\infty} f \circ \gamma(\tau) = x_0 \in \widehat{\mathbb{C}}$.

Let $(z_n) \subset V$ be a sequence converging to the boundary of V in $\widehat{\mathbb{C}}$ and satisfying that $d_V(z_n,\gamma) \le C$ for some C. Then, if $f(z_n)$ converges to a point in $\widehat{\mathbb{C}}$ this point must be x_0.

Proof. Set $x_1 = \lim f(z_n)$. We must show that $x_1 = x_0$. Let $\phi : \mathbb{H} \to V$ be a Riemann mapping that sends the upper half plane $\mathbb{H}$ conformally onto V. By [21] (Proposition 2.14) the curve $\tilde{\gamma} = \phi^{-1} \circ \gamma|_{[0,+\infty)}$ extends continuously to a curve $\tilde{\gamma} : [0,+\infty] \to \overline{\mathbb{H}} \cup \{\infty\}$, with $\tilde{\gamma}(+\infty) \in \partial H \cup \{\infty\}$. By replacing ϕ with another Riemann mapping we can suppose $\tilde{\gamma}(\infty) = 0$. Let $w_n = \phi^{-1}(z_n)$ and let $\tau_n \ge 0$ be a sequence such that $d_{\mathbb{D}}(\tilde{\gamma}(\tau_n), w_n) \le C$; we must have $t_n \to +\infty$. Let L_n denote the affine mapping that maps $\mathbb{H}$ onto itself and sends $\gamma(\tau_n)$ to i. Set $g_n = f \circ \phi \circ L_n^{-1} : \mathbb{H} \to \mathbb{C} \setminus \{a,b\}$. By Montel's theorem g_n is a normal sequence and by passing to a subsequence we suppose that g_n converges to a map $g_\infty : \mathbb{H} \to \widehat{\mathbb{C}}$, locally uniformly in $\mathbb{H}$. Clearly $g_\infty(i) = x_0$.

We claim that g_∞ is the constant mapping. Let $r \in (0,1)$ be arbitrary. Take $\tau_n' > \tau_n$ such that

$$|L_n \circ \tilde{\gamma}(\tau_n') - L_n \circ \tilde{\gamma}(\tau_n)| = |L_n \circ \tilde{\gamma}(\tau_n') - i| = r.$$

Such τ_n' exists since $\mathrm{Im}(L_n \circ \tilde{\gamma}(\tau)) \to 0$ as $\tau \to +\infty$. Now, $g_n \circ L_n(\tilde{\gamma}(\tau_n')) = \gamma(\tau_n') \to x_0$ and it follows that there exists a point on the circle with center i and radius r that g_∞ maps to x_0 (any accumulation point of $\{L_n(\tilde{\gamma}(\tau_n'))\}$ will do). Since r was arbitrary, the Identity Theorem implies that g_∞ is the constant map $w \mapsto x_0$.

Note that $L_n(w_n)$ is contained in the closure of the hyperbolic disk $\{d_{\mathbb{H}}(\zeta, i) < C\}$. On the one hand $g_n(L_n(w_n)) \to x_0$. On the other hand $g_n(L_n(w_n)) = f(z_n) \to x_1$. We conclude that $x_0 = x_1$. □

Proof of Proposition 3. We first show that ω restricts to the identity on the set of critical values, then that it restricts to the identity on the set of asymptotic values, and finally to the closure of the grand orbit of these two sets.

Let ω_t be a path in $\mathrm{QC}(f,\mathcal{U})$ that connects $\omega_0 = \mathrm{Id}$ to $\omega_1 = \omega$. Since ω_t commutes with f, the set of critical points is ω_t invariant. So, if $c \in \mathcal{U}$ is a critical point, the path $t \mapsto \omega_t(c)$ is a subset of the critical points. Since this set is discrete $\omega_t(c) = c$ for all t. Since ω_t commutes with f we immediately get that every ω_t fixes the critical values.

Now let $x_0 \in \mathcal{U}$ be an asymptotic value, and $\gamma : [0, +\infty) \to \mathcal{U}$ a corresponding asymptotic path; i.e. a curve with the property that $\lim_{\tau\to\infty} \gamma(\tau) = \infty$ and $\lim_{\tau\to\infty} f \circ \gamma(\tau) = x_0$. Let V denote the component of $\mathcal{U}$ that contains γ. Then V is simply connected and hyperbolic, [12]. By assumption $\omega_t|_V : V \to V$ fixes the ideal boundary of V. Set

$$x_t = \omega_t(x_0) = \lim_{\tau\to\infty} f \circ \omega_t \circ \gamma(\tau)$$

We must show that $x_t = x_0$. Let $\tau_n > 0$ be a sequence tending towards $+\infty$ and set $z_n = \omega_t \circ \gamma(\tau_n)$ By Lemma 1 there exists a constant C such that the hyperbolic distance in V satisfies $d_V(\gamma(\tau_n), z_n) \le C$. Since $f(V)$ is contained in a component of $\mathcal{U}$ which is hyperbolic, we can apply Lemma 2 and we get that $x_t = \lim_{n\to+\infty} f(z_n) = x_0$. So ω_t fixes the asymptotic values of f in $\mathcal{U}$.

Since every singular value is in the closure of the set of asymptotic and critical values, we get by continuity that ω_t fixes the singular values of f in $\mathcal{U}$. Since ω_t commutes with f we get that ω_t restricts to the identity on the forward orbit of this set. Now suppose $\omega_t(y) = y$ for all t and that $f^n(x) = y$. Then $\omega_t(x)$ must map into $f^{-n}\{y\}$. Since this set is discrete we get that $\omega_t(x) = x$ for all x. It follows that ω_t restricts to the identity on the grand orbit of S for all t and by continuity this is also true on the closure. □

We can now prove our main theorem whose statement we recall here.

Main Theorem. *Let U be a proper fixed Baker domain of the entire function f and $\mathcal{U}$ its grand orbit. Denote by S the set of singular values of f in $\mathcal{U}$, and by $\widehat{\mathcal{S}}$ the closure of the grand orbit of S taken in $\mathcal{U}$. Then $\mathcal{T}(f,\mathcal{U})$ is infinite dimensional except if U is doubly parabolic and the cardinality of $\widehat{\mathcal{S}}/f$ is finite. In that case the dimension of $\mathcal{T}(f,\mathcal{U})$ equals $\#\widehat{\mathcal{S}}/f - 1$.*

Proof. By Lemma 3 every element of $\mathrm{QC}_0(f,\mathcal{U})$ restricts to the identity on $\widehat{\mathcal{S}}$. Hence

$$\begin{aligned} \mathcal{T}(f,\mathcal{U}) &\simeq \mathcal{B}_1(f,\mathcal{U})/\mathrm{QC}_0(f,\mathcal{U}) \simeq \left(\mathcal{B}_1(f,\widehat{\mathcal{S}}) \times \mathcal{B}_1(f,\mathcal{U}-\widehat{\mathcal{S}})\right)/\mathrm{QC}_0(f,\mathcal{U}) \\ &\simeq \mathcal{B}_1(f,\widehat{\mathcal{S}}) \times \left(\mathcal{B}_1(f,\mathcal{U}-\widehat{\mathcal{S}})/\mathrm{QC}_0'(f,\mathcal{U})\right), \end{aligned}$$

where we denote by $\mathrm{QC}_0'(f,\mathcal{U})$ the group formed by the restriction of each element in $\mathrm{QC}_0(f,\mathcal{U})$ to $\mathcal{U} - \widehat{\mathcal{S}}$. Since the elements in $\mathrm{QC}_0(f,\mathcal{U})$ are the identity on $\widehat{\mathcal{S}}$, it follows from Theorem 3 that

$$\mathrm{QC}_0'(f,\mathcal{U}) = \mathrm{QC}_0(f,\mathcal{U}-\widehat{\mathcal{S}}).$$

Therefore,

$$\mathcal{T}(f,\mathcal{U}) \simeq \mathcal{T}(f,\mathcal{U}-\widehat{\mathcal{S}}) \times \mathcal{B}_1(f,\widehat{\mathcal{S}}).$$

By Proposition 1, $W = \mathcal{U}/f$ is an annulus of finite modulus when U is hyperbolic, one-sided infinite modulus when U is simply parabolic and two-sided infinite modulus when U is doubly parabolic. The subset $T = \widehat{\mathcal{S}}/f \subset W$ is relatively closed in W, so $W - T$ is an open set. We denote the components of W by V_i. Then each $V_i = \mathcal{V}_i/f$ for a completely invariant open subset $\mathcal{V}_i$ of $\mathbb{C}$ and $\cup\mathcal{V}_i = \mathcal{U} - \widehat{\mathcal{S}}$. By Theorem 4 we have

$$\mathcal{T}(f,\mathcal{U}-\widehat{\mathcal{S}}) \simeq \prod_i \mathcal{T}(f,\mathcal{V}_i),$$

and by Theorem 5 we have

$$\prod_i \mathcal{T}(f,\mathcal{V}_i) \simeq \prod_i \mathcal{T}(V_i).$$

If T contains interior points, then $\mathcal{B}_1(f,\widehat{\mathcal{S}})$ is infinite dimensional, so we can suppose it does not. Then T is a proper subset of W. If T has infinitely many components then a component V_i of $W-T$ is either of infinite connectivity or has ideal boundary (or both). In both cases the Teichmüller space is infinite (see [14]). So we can assume that T has only finitely many components. If one of these components is not a point then the presence of ideal boundary forces the dimension of the Teichmüller space to be infinite. Consequently we can assume that T is a finite set. Then $W-T$ has only one component; it is an annulus with finitely many punctures. If W is of finite or one-sided infinite modulus, again the presence of ideal boundary will force the dimension to be infinite. So we can suppose that W is an annulus of doubly infinite modulus and U is a doubly parabolic Baker domain.

Since T is finite $\mathcal{B}_1(f,\widehat{\mathcal{S}})$ is trivial and

$$\mathcal{T}(f,\mathcal{U}) \simeq \mathcal{T}(W-T).$$

Finally $W-T$ is conformally equivalent to the sphere with $2+\#T$ punctures. It is well known that the dimension of Teichmüller space of the sphere with n punctures is $n-3$. So the dimension of $\mathcal{T}(f,\mathcal{U})$ equals $2+\#T-3=\#T-1$. The proof is finished recalling that $\#T=\#\widehat{\mathcal{S}}/f$. □

We conclude this section by remarking that the dimension of the Teichmüller space of f on the grand orbit of a Baker domain gives a lower bound of the Teichmüller space of f. Indeed, with $\mathcal{U}$ denoting the grand orbit of a Baker domain, $J(f)$ the boundary of $\mathcal{U}$ and $\mathcal{V}$ the complement of $\mathcal{U}\cup J(f)$ we get

$$\mathcal{T}(f,\mathbb{C}) \simeq \mathcal{T}(f,\mathcal{U}) \times \mathcal{B}_1(f,J(f)) \times \mathcal{T}(f,\mathcal{V}).$$

So in general we expect $\mathcal{T}(f,\mathbb{C})$ to be high dimensional. It may then come as a surprise, that we can give an example of an entire function f with fixed proper Baker domains which is rigid, in the sense that the Teichmüller space $\mathcal{T}(f,\mathbb{C})$ is trivial. We will exhibit such an example in the next section.

6. **A rigid example; proof of Proposition 2.**

In this section we shall show that the doubly parabolic example

$$f(z) = z + e^{-z},$$

is rigid. More precisely, we show the following.

Proposition 1. *The map $f(z)=z+e^{-z}$ is rigid, i.e., if $\widetilde{f}$ is a holomorphic map which is quasiconformally conjugate to f, then $\widetilde{f}$ is conjugate to f by an affine map.*

We need the following preliminary lemma which follows easily from work by Eremenko and Lyubich.

Lemma 3. *Let $f(z)=z+e^{-z}$. The Julia set $J(f)$ has measure zero.*

Proof. The mapping $z\mapsto e^{-z}$ semi conjugates f to $g(z)=ze^{-z}$, and $J(f)$ is the preimage of $J(g)$ under $z\mapsto e^{-z}$ (see [7]). So to show $J(f)$ has measure zero, it will suffice to show that $J(g)$ has measure zero. The entire function g has exactly one critical point $\omega=1$ and exactly one asymptotic value $a=0$. Since the asymptotic value is absorbed by the parabolic fixed point at the origin, and the critical point is being attracted by the parabolic fixed point, Proposition 4 and Theorem 8 in [12] imply that $J(g)$ has zero measure. □

Proof of Proposition 2. Let U_j, $j \in \mathbb{Z}$, denote the Baker domains of f, and $\mathcal{U}_j$ their grand orbit. The boundary of each open set $\mathcal{U}_j$ coincides with with the Julia set $J(f)$. Since the Julia set is contained in the closure of dynamically distinguished points (periodic points for example), and since f has no other Fatou components we get:

$$\mathcal{T}(f,\mathbb{C}) \simeq \mathcal{T}(f,\cup\mathcal{U}_j) \times \mathcal{B}_1(f,J(f)).$$

By Theorem 4

$$\mathcal{T}(f,\cup\mathcal{U}_j) = \prod_j \mathcal{T}(f,\mathcal{U}_j).$$

Since f has no asymptotic values, and each doubly parabolic Baker domain $\mathcal{U}_j$ contains exactly one critical point, we get from the Main Theorem that each $\mathcal{T}(f,\mathcal{U}_j)$ is trivial. So

$$\mathcal{T}(f,\mathbb{C}) \simeq \mathcal{B}_1(f,J(f)).$$

In view of Lemma 3 $J(f)$ has measure 0 and so $\mathcal{B}_1(J(f))$ is trivial. In other words $\mathcal{T}(f,\mathbb{C}) \simeq \mathrm{Def}(f,\mathbb{C})/\mathrm{QC}_0(f,\mathbb{C})$ is formed by one point. Since $\mathrm{QC}_0(f,\mathbb{C})$ is a subgroup of $\mathrm{QC}(f,\mathbb{C})$ also $\mathrm{Def}(f,\mathbb{C})/\mathrm{QC}(f,\mathbb{C})$ has cardinality one. Finally this set is in one to one correspondance with the set of entire mappings quasiconformally conjugate to f modulo conjugacy by affine isomorphisms, and the proposition follows. □

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