The Fine Structure of Herman Rings

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Abstract

We study the geometric structure of the boundary of Herman rings in a model family of Blaschke products of degree 3. Shishikura's quasiconformal surgery relates the Herman ring to the Siegel disk of a quadratic polynomial. By studying the regularity properties of the maps involved, we can transfer McMullen's results on the fine local geometry of Siegel disks to the Herman ring setting.

1 Introduction

We consider the dynamical system induced by the iterates of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere or compactified complex plane. We use the notation $f^n := f \circ \overset{n)}{\cdots} \circ f$ to denote the n^{th} iterate of f. Under this dynamics, the Riemann sphere splits into two completely invariant sets: the *Fatou set*, formed by those points for which the sequence $\{f^n\}$ is normal in some neighborhood; and its complement, the *Julia set*. By definition the Fatou set is open and therefore the Julia set is a compact set of the sphere. Connected components of the Fatou set, also known as *Fatou components*, map onto one another and are eventually periodic [Sul85]. The Julia set is the common boundary between the different Fatou components and, consequently, the dynamics on this set is chaotic. For background on the dynamics of rational maps we refer for example to [CG93] and [Mil06].

An especially relevant particular case of rational maps are polynomials, which are exactly (up to Möbius conjugation) those rational maps for which infinity is a fixed point and has no preimages other than itself. In particular this implies that infinity is a *super-attracting fixed point*, and the dynamics are locally conjugate to $z \mapsto z^d$ around this point for some $d \geq 2$, the degree of the polynomial; it also means that the basin of attraction of infinity, that is the set of points attracted to infinity under iteration, is connected and completely invariant. Therefore its boundary is compact in $\mathbb{C}$ and coincides with the Julia set of the polynomial.

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