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CLASSIFICATION OF LINEAR SKEW-PRODUCTS OF THE COMPLEX PLANE AND AN AFFINE ROUTE TO FRACTALIZATION

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Abstract. Linear skew products of the complex plane,

$$\left.\begin{array}{ccc}\theta & \mapsto & \theta+\omega,\\ z & \mapsto & a(\theta)z,\end{array}\right\}$$

where $\theta \in \mathbb{T}$, $z \in \mathbb{C}$, $\frac{\omega}{2\pi}$ is irrational, and $\theta \mapsto a(\theta) \in \mathbb{C}\setminus\{0\}$ is a smooth map, appear naturally when linearizing dynamics around an invariant curve of a quasi-periodically forced complex map. In this paper we study linear and topological equivalence classes of such maps through conjugacies which preserve the skewed structure, relating them to the Lyapunov exponent and the winding number of $\theta \mapsto a(\theta)$. We analyze the transition between these classes by considering one parameter families of linear skew products. Finally, we show that, under suitable conditions, an affine variation of the maps above has a non-reducible invariant curve that undergoes a fractalization process when the parameter goes to a critical value. This phenomenon of fractalization of invariant curves is known to happen in nonlinear skew products, but it is remarkable that it also occurs in simple systems as the ones we present.

1. Introduction.

In this work we are concerned with the dynamics of linear skew products of the complex plane. These are maps $F_\omega : \mathbb{T}\times\mathbb{C} \to \mathbb{T}\times\mathbb{C}$ of the form

$$\left.\begin{array}{ccc}\theta & \mapsto & \theta+\omega,\\ z & \mapsto & a(\theta)z,\end{array}\right\}$$

where ω is Diophantine (see Section 2) and $\theta \in \mathbb{T} \mapsto a(\theta) \in \mathbb{C}\setminus\{0\}$ is a smooth map. They appear in a natural way when linearizing the dynamics around an invariant curve of a quasi-periodically forced complex map. As in many differentiable systems, the linear part of the dynamics determines, under certain conditions, the stability and the local behaviour of the system around the invariant curve ([21]).

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