TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 10, Pages 4711–4730
S 0002-9947(00)02608-8
Article electronically published on June 8, 2000

PERIODIC POINTS OF HOLOMORPHIC MAPS VIA LEFSCHETZ NUMBERS

NÚRIA FAGELLA AND JAUME LLIBRE

ABSTRACT. In this paper we study the set of periods of holomorphic maps on compact manifolds, using the periodic Lefschetz numbers introduced by Dold and Llibre, which can be computed from the homology class of the map. We show that these numbers contain information about the existence of periodic points of a given period; and, if we assume the map to be transversal, then they give us the exact number of such periodic orbits. We apply this result to the complex projective space of dimension n and to some special type of Hopf surfaces, partially characterizing their set of periods. In the first case we also show that any holomorphic map of $\mathbb{C}P(n)$ of degree greater than one has infinitely many distinct periodic orbits, hence generalizing a theorem of Fornaess and Sibony. We then characterize the set of periods of a holomorphic map on the Riemann sphere, hence giving an alternative proof of Baker's theorem.

1. Introduction

In dynamical systems and, in particular, in the study of iteration of self maps of a given manifold, periodic orbits play an important role.

Given a continuous map $f : X \to X$, a point $x \in X$ is called *periodic* if there exists $k \in \mathbb{N}$ such that $f^k(x) = x$. The minimum of such k is called the *period* of x, and the iterates $\{x, f(x), \dots, f^{k-1}(x)\}$ form a *periodic orbit*. For such a map, it is natural to ask how many periodic orbits it has or what are the possible periods that may appear. To deal with these problems, differential topological methods have often proved to be very useful, since it is clear that the topology of the manifold in question plays an essential role.

The Lefschetz Fixed Point Theorem was one of the main results in this direction. Knowing the homology class of the map, one can compute its Lefschetz number $L(f)$ and, if the result is nonzero, conclude the existence of a fixed point. Clearly, the same process, applied to the k^{th} iterate of the function, f^k, would give the existence of a periodic orbit of period k, or a divisor of k. We have gone a long way from this theorem, and there is plenty of literature on its generalizations and applications (see [2, 5, 8, 16]).

To deal with the problem of existence of periodic orbits with a given period the *periodic Lefschetz number of period k*, denoted by $l(f^k)$, was introduced in

Received by the editors October 28, 1998.
2000 *Mathematics Subject Classification.* Primary 55M20; Secondary 32H50, 37B99.
Key words and phrases. Set of periods, periodic points, holomorphic maps, Lefschetz fixed point theory.
Both authors are partially supported by DGICYT grant number PB96-1153.